258(8): Some Suggestions for Animation

1) General Solution

This is given by Reed, eq. (33):

$$\underline{\nabla} \cdot \underline{B} = 0 \qquad - (1)$$

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} \qquad - (2)$$

and

$$\underline{B} = B_0 \left(\kappa \underline{\nabla} \times (\phi \underline{a}) + \underline{\nabla} \times \left(\underline{\nabla} \times (\phi \underline{a}) \right) \right) \qquad - (3)$$

where $\underline{a}$ is an arbitrary constant vector and ϕ a scalar function. Here:

$$\underline{P} = \underline{\nabla} \times (\phi \underline{a}) \qquad - (4)$$

is the poloidal solution, and

$$\underline{T} = \underline{\nabla} \times \underline{P} \qquad - (5)$$

is the toroidal solution.

If:

$$\left.\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} &= \underline{0} \\ \underline{\nabla} \cdot \underline{E} &= 0 \\ \underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} &= \underline{0} \end{aligned}\right\} \qquad - (6)$$

then the Beltrami solution of the free field is given by:

$$\underline{B} = e^{i\omega t} B_0 \left(\kappa \underline{\nabla} \times (\phi \underline{a}) + \underline{\nabla} \times (\underline{\nabla} \times (\phi \underline{a})) \right)$$

The plane wave solution is obtained from: – (7)

$$\phi = \frac{1}{\kappa^2} e^{-i\kappa Z} \quad - (8)$$

$$\underline{a} = \underline{i} - i\underline{j} \quad - (9)$$

$$B_0 = \frac{B^{(0)}}{2\sqrt{2}} \quad - (10)$$

so:

$$\underline{B} = \frac{1}{2\kappa} \underline{\nabla} \times \underline{B} + \frac{1}{2\kappa^2} \underline{\nabla} \times (\underline{\nabla} \times \underline{B}) \quad - (11)$$

where

$$\underline{B} = \frac{B^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i(\omega t - \kappa Z)} \quad - (12)$$

so:

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} \quad - (13)$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{B}) = \kappa^2 \underline{B} \quad - (14)$$

However, in the general solution (3), $\underline{a}$ is defined as an arbitrary constant vector, and ϕ is defined as:

$$\nabla^2 \phi + \kappa^2 \phi = 0 \quad - (15)$$

3) which is the Helmholtz equation. So ϕ is always a solution of the Helmholtz wave equation, and:

$$\underline{a} = a_X \underline{i} + a_Y \underline{j} + a_Z \underline{k} \quad -(16)$$

in general, where $\underline{a}$ can be complex.

If $\underline{a}$ is constant:

$$\underline{\nabla} \times (\phi \underline{a}) = \phi \underline{\nabla} \times \underline{a} + \underline{\nabla} \phi \times \underline{a}$$

$$= \underline{\nabla} \phi \times \underline{a} \quad -(17)$$

because

$$\underline{\nabla} \times \underline{a} = \underline{0} \quad -(18)$$

So eq. (3) becomes:

$$\boxed{\underline{B} = B_0 \left(\kappa \underline{\nabla} \phi \times \underline{a} + \underline{\nabla} \times \left(\underline{\nabla} \phi \times \underline{a} \right) \right)} \quad -(19)$$

The choice (8) to (10) gives the plane wave solution.

More generally $\underline{a}$ is given by eq. (16).

2) The Lundquist Solution and Generalizations

Those solutions are given by Marsh in his equations (3.16), (3.17) and (3.18).

4) They are solutions of

$$\underline{\nabla} \cdot \underline{B} = 0 \quad - (20)$$

and of

$$\underline{\nabla} \times \underline{B} = \kappa \underline{B} \quad - (21)$$

in cylindrical polar coordinates:

$$\frac{1}{r} \frac{\partial (r B_r)}{\partial r} + \frac{\partial B_Z}{\partial Z} = 0 \quad - (22)$$

and

$$\frac{1}{r} \frac{\partial (r B_\theta)}{\partial r} = \kappa B_Z \quad - (23)$$

$$-\frac{1}{r} \frac{\partial (r B_\theta)}{\partial Z} = \kappa B_r \quad - (24)$$

$$\frac{\partial B_r}{\partial Z} - \frac{\partial B_Z}{\partial r} = \kappa B_\theta \quad - (25)$$

The solution is claimed to be:

$$B_Z(r, Z) = (\kappa^2 + \lambda^2)^{1/2} e^{-\lambda Z} J_0\left((\kappa^2 + \lambda^2)^{1/2} r\right) \quad - (26)$$

$$B_\theta(r, Z) = \kappa e^{-\lambda Z} J_1\left((\kappa^2 + \lambda^2)^{1/2} r\right) \quad - (27)$$

$$B_r(r, Z) = \lambda e^{-\lambda Z} J_1\left((\kappa^2 + \lambda^2)^{1/2} r\right) \quad - (28)$$

It need to be checked by computer algebra that eqns. (26) to (28) are indeed solutions of eqs (20) to (25), and then the flow can be analyzed and animated.

3) Chaotic Solutions

The chaotic solution is generated by the equations:

$$\frac{dX}{dt} = B_X = A \sin Z + C \cos Y \quad - (29)$$

$$\frac{dY}{dt} = B_Y = B \sin X + A \cos Z \quad - (30)$$

$$\frac{dZ}{dt} = B_Z = C \sin Y + B \cos X \quad - (31)$$

Marsh on his page 62 claims that these equations satisfy:

$$\nabla \times \underline{B} = \underline{B} \quad - (32)$$

It would be interesting to graph and animate eqs. (29) to (31).
