262(2): Spin Connection of a Whirlpool Galaxy from Fundamental Kinematics.

Consider the radial vector in plane polar coordinates:

$$\underset{\sim}{r} = r\,\underline{e}_r \quad -(1)$$

where the radial unit vector is defined as:

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta. \quad -(2)$$

The radial unit vector depends on θ and rotates with θ. Therefore a spin connection is generated by the rotation of $\underline{e}_r$ and the unit vector:

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta \quad -(3)$$

of the plane polar coordinate system (r, θ).

Therefore $\underline{e}_r$ and $\underline{e}_\theta$ depend on time.

Therefore:

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt} \quad -(4)$$

From eqs. (2) and (3):

$$\frac{d\underline{e}_r}{dt} = \frac{d\theta}{dt}\underline{e}_\theta = \omega\,\underline{e}_\theta \quad -(5)$$

where ω is the angular velocity:

2)

$$\omega = \frac{d\theta}{dt} \quad - (6)$$

The angular velocity is a type of spin connection because it governs the rotation of $\underline{e}_r$ and $\underline{e}_\theta$. The velocity can be expressed as:

$$\underline{v} = \frac{dr}{dt} \underline{e}_r + \omega r \underline{e}_\theta \quad - (7)$$

$$= \frac{dr}{dt} \underline{e}_r + \underline{\omega} \times \underline{r}$$

$$= \frac{D \underline{r}}{dt}$$

where D denotes a covariant derivative. The spin connection term is $\underline{\omega} \times$, so:

$$\frac{D \underline{r}}{dt} = \frac{dr}{dt} \underline{e}_r + \underline{\omega} \times \underline{r} \quad - (8)$$

which is an example of:

$$D_\mu r^a = \partial_\mu r^a + \omega^a_{\mu b} r^b \quad - (9)$$

Using the chain rule:

$$\frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} \quad - (10)$$

it is found that the velocity for any orbit is given by

3)

$$v^2 = \omega^2 \left(\left(\frac{dr}{d\theta} \right)^2 + r^2 \right) \quad -(11)$$

The angular momentum is a constant of motion:

$$L = m r^2 \omega \quad -(12)$$

so the magnitude of the spin connection is given by:

$$\omega = \frac{L}{m r^2} \quad -(13)$$

and

$$v^2 = \left(\frac{L}{mr} \right)^2 + \left(\frac{L}{mr^2} \left(\frac{dr}{d\theta} \right) \right)^2 \quad -(14)$$

It follows that:

$$\left(\frac{dr}{d\theta} \right)^2 = \frac{v^2}{\omega^2} - r^2 \quad -(15)$$

and

$$\omega^2 = v^2 \left(r^2 + \left(\frac{dr}{d\theta} \right)^2 \right)^{-1} \quad -(16)$$

In a whirlpool galaxy:

$$v \xrightarrow[r \to \infty]{} = v_\infty \quad -(17)$$

which is a constant. From eq. (14), as:

$$r \to \infty \quad -(18)$$

$$v \xrightarrow[r \to \infty]{} \frac{L}{mr^2} \frac{dr}{d\theta} = v_\infty \quad -(19)$$

Therefore:

$$\frac{d\theta}{dr} \xrightarrow[r\to\infty]{} \left(\frac{L}{mV_\infty}\right)\frac{1}{r^2} \quad -(20)$$

and

$$\boxed{\theta \xrightarrow[r\to\infty]{} \frac{L}{mV_\infty}\int\frac{dr}{dr^2} = -\left(\frac{L}{mV_\infty}\right)\frac{1}{r}} \quad -(21)$$

This a <u>hyperbolic spiral</u>.

In the <u>limit</u> $r\to\infty$ <u>any</u> <u>orbit</u> in a <u>plane</u> <u>becomes</u> a <u>hyperbolic spiral</u> <u>if</u> V_∞ is <u>constant</u>.

From eqns (15) and (20), as $r\to\infty$:

$$\left(\frac{mV_\infty}{L}\right)^2 r^4 = \frac{V_\infty^2}{\omega^2} - r^2 \quad -(22)$$

so:

$$\boxed{\omega \xrightarrow[r\to\infty]{} \frac{V_\infty}{r}\left(1+\left(\frac{mV_\infty}{L}\right)^2 r^2\right)^{-1/2}} \quad -(23)$$

However:

$$L = mr^2\omega \quad -(24)$$

so:

$$\omega^2 \xrightarrow[r\to\infty]{} \left(\frac{V_\infty}{r}\right)^2\left(1+\left(\frac{mV_\infty}{mr^2\omega}\right)^2 r^2\right)^{-1} \quad -(25)$$

5)

So:

$$\omega^2 \xrightarrow[r\to\infty]{} \left(\frac{V_\infty}{r}\right)^2\left(1+\left(\frac{V_\infty}{\omega r}\right)^2\right)^{-1} \quad -(26)$$

i.e

$$\omega^2\left(1+\left(\frac{V_\infty}{\omega r}\right)^2\right) \longrightarrow \left(\frac{V_\infty}{r}\right)^2 \quad -(27)$$

or

$$\boxed{\omega \xrightarrow[r\to\infty]{} 0} \quad -(28)$$

The spin connection goes to zero for any planar orbit whose orbital velocity is constant as $r \to \infty$.

This means that:

$$\boxed{\underline{V} \longrightarrow \underline{V}_\infty = \frac{dr}{dt}\,\underline{e}_r} \quad -(29)$$

and the orbit becomes a straight line.

This is as observed in whirlpool galaxies.