book.

Consider the vector format of the first Maurer Cartan structure equation given here in the notation of chapter one:

$$\underline{T}^a(\text{orb}) = -\underline{\nabla} q^a_0 - \frac{\partial \underline{q}^a}{\partial t} - \omega^a_{\ 0b}\underline{q}^b + q^b_{\ 0}\underline{\omega}^a_{\ b} \quad -(41)$$

and

$$\underline{T}^a(\text{spin}) = \underline{\nabla} \times \underline{q}^a - \underline{\omega}^a_{\ b} \times \underline{q}^b. \quad -(42)$$

The fundamental ECE hypothesis was devised for electromagnetism and defines the electromagnetic potential in terms of the tetrad:

$$A^a_\mu = A^{(0)} q^a_\mu. \quad -(43)$$

Now define the linear momentum tetrad:

$$p^a_\mu = p^{(0)} q^a_\mu \quad -(44)$$

in an analogous manner, using the minimal prescription:

$$p^a_\mu \rightarrow p^a_\mu + eA^a_\mu. \quad -(45)$$

It follows from Eqs. (41) and (44) that the orbital force of ECE theory is:

$$\underline{F}^a(\text{orb}) = -\underline{\nabla}\phi^a_0 - \frac{\partial \underline{p}^a}{\partial t} - \omega^a_{\ 0b}\underline{p}^b + \phi^b_{\ 0}\underline{\omega}^a_{\ b} \quad -(46)$$

and that the spin force is:

$$\underline{F}^a(\text{spin}) = \underline{\nabla} \times \underline{p}^a - \underline{\omega}^a{}_b \times \underline{p}^b \quad -(47)$$

In the simplified single polarization theory:

$$\underline{F}(\text{orb}) = -\underline{\nabla}\phi - \frac{\partial \underline{p}}{\partial t} - \omega_0 \underline{p} + \phi\,\underline{\omega} \quad -(48)$$

and:

$$\underline{F}(\text{spin}) = \underline{\nabla} \times \underline{p} - \underline{\omega} \times \underline{p}. \quad -(49)$$

In the non relativistic limit the spin connection vanishes and:

$$\underline{F}(\text{orb}) = -\underline{\nabla}\phi - \frac{\partial \underline{p}}{\partial t}. \quad -(50)$$

The famous equivalence of inertial and gravitational mass is recovered from Eq. (50) using the anti symmetry law of ECE theory described earlier in this book. So:

$$-\frac{\partial \underline{p}}{\partial t} = -\underline{\nabla}\phi \quad -(51)$$

and:

$$\phi = -\frac{mMG}{r} \quad -(52)$$

where ϕ is the gravitational potential. This is defined in direct analogy to the electromagnetic scalar potential ϕ_e as follows:

$$p_\mu^a = \left(\frac{\phi^a}{c}, -\underline{p}^a\right) \quad -(53)$$

and

$$A^a_\mu = \left(\frac{\phi^a_e}{c}, -\underline{A}^a \right). \quad -(54)$$

In Newtonian dynamics:

$$\phi = -\frac{mMG}{r} \quad -(55)$$

so the force is:

$$F = -\frac{mMG}{r^2} \quad -(56)$$

and the acceleration due to gravity is:

$$g = \frac{MG}{r^2}. \quad -(57)$$

This powerful and precise result of ECE theory was first inferred in UFT 141. The ECE theory is therefore precise to one part in ten to the power seventeen, the precision of the experimental proof of the equivalence of gravitational and inertial mass. The equivalence is due to Cartan geometry.

The calculation of light deflection due to gravitation proceeds by applying the ECE anti symmetry law to Eq. (48) to find that:

$$-\underline{\nabla}\phi + \underline{\omega}\phi = -\frac{d\underline{p}}{dt} - \omega_0 \underline{p} \quad -(58)$$

in which it has been assumed that:

$$\frac{d\underline{p}}{dt} = \frac{\partial \underline{p}}{\partial t} \quad -(59)$$

So the force is:

$$\underline{F} = 2\left(-\frac{d\underline{p}}{dt} - \omega_0 \underline{p} \right) = -2\left(\underline{\nabla}\phi - \underline{\omega}\phi \right) \quad -(60)$$

The factor two in Eq. (60) can be eliminated without affecting the physics by assuming that:

$$p^a_\mu = \frac{p^{(0)}}{2} q^a_\mu \quad -(61)$$

so the orbital force becomes:

$$\underline{F} = -\frac{d\underline{p}}{dt} - \omega_0 \underline{p} = -\underline{\nabla}\phi + \underline{\omega}\phi \quad -(62)$$

an equation which gives the equivalence principle (51) for vanishing spin connection.

Now define:

$$\underline{p} = p_r \underline{e}_r \quad -(63)$$

$$\underline{\omega} = \omega_r \underline{e}_r \quad -(64)$$

and compare Eqs. (20) and (62) to find that:

$$F = -\frac{\partial \phi}{\partial r} + \phi \omega_r = -\frac{kx^2}{r^2} - \frac{k(1-x^2)\alpha}{r^3} \quad -(63)$$

For small deviations from a Newtonian orbit as in planetary precession or any observable precession in astronomy:

$$-\frac{\partial \phi}{\partial r} = -\frac{kx^2}{r^2} \quad -(64)$$

i. e. :

$$x \sim 1 \quad -(65)$$

to an excellent approximation. From Eqs. (63) and (64) :

$$\phi \omega_r = -\frac{k\alpha}{r^3}(1-x^2) \quad -(66)$$

in an almost Newtonian approximation. In this approximation the gravitational potential is well known to be:

$$\phi = -\frac{k}{r} \quad -(67)$$

so the spin connection can be expressed in terms of x as follows:

$$\omega_r = (1-x^2)\frac{\alpha}{r^2} = (1-x^2)\frac{b^2}{ar^2}. \quad -(68)$$

Using Eq. (68), the correction needed to produce Eq. (40) form Eq. (39) is:

$$\frac{R_0 c^2}{mG} \rightarrow \frac{R_0 c^2}{mG} + \frac{\alpha}{R_0}\left(\frac{1-x^2}{x^2}\right). \quad -(69)$$

Using Eq. (32) it is found that:

$$2\alpha\phi = 2\frac{R_0 c^2}{mG} + 2(1+\epsilon)\left(\frac{1-x^2}{x^2}\right). \quad -(70)$$

Experimentally:

$$(1+\epsilon)\left(\frac{1-x^2}{x^2}\right) = \frac{R_0 c^2}{mG} \quad -(71)$$

and using Eq. (27):

$$\frac{1}{\epsilon} = \sin\left(\frac{\Delta\phi}{2}\right). \quad -(72)$$

For small deflections:

$$\frac{1}{\epsilon} \sim \frac{\Delta\phi}{2} \quad -(73)$$

so to an excellent approximation:

$$\left(1+\frac{2}{\Delta\phi}\right)\left(\frac{1-x^2}{x^2}\right)^{-1} = \frac{R_0 c^2}{MG} \quad -(74)$$

However by experiment:

$$\Delta\phi = \frac{4R_0c^2}{MG}, \quad -(75)$$

$$x \sim 1,$$

so using Eq. (68):

$$\omega_r = \frac{\Delta\phi}{4}\left(1+\frac{2}{\Delta\phi}\right)^{-1}\frac{\alpha}{r^2}. \quad -(76)$$

From Eq. (32):

$$\alpha = R_0(1+\epsilon) = R_0\left(1+\frac{2}{\Delta\phi}\right) \quad -(77)$$

and from Eqs. (76) and (77)

$$\omega_r = \frac{\Delta\phi}{4}\frac{R_0}{r^2}. \quad -(78)$$

This is a universal spin connection that describes all electromagnetic deflections from any relevant object M in the universe. This spin connection also describes planetary precession through its relation to x, Eq. (68). The procedure used to derive this result also gives the equivalence principle. Finally at distance of closest approach:

$$\omega_r = \frac{\Delta\phi}{4R_0} \quad -(79)$$

a very simple result that can be tabulated in astronomy for any relevant object of mass M.

8.3 THE VELOCITY CURVE OF A WHIRLPOOL GALAXY

Whirlpool galaxies are familiar objects in cosmology and are very complex in structure. However there is one feature that makes them useful for the study of the fundamental theories of cosmology such as those of Newton and Einstein, and ECE, and that is the velocity curve, the plot of the velocity of a star orbiting the centre of a galaxy versus the distance between the star and the centre. It was discovered experimentally in the late fifties that the velocity becomes constant as r goes to infinity. The first part of this section will give the basic kinematics of the orbit and will show that both the Newton and Einstein theories fail completely to describe the velocity curve. The second part will describe how ECE theory gives a plausible explanation of the velocity curve without the use of random empiricism such as dark matter. It appears that the theory of dark matter has been refuted experimentally, leaving ECE cosmology as the only explanation.

Consider the radial vector in the plane of any orbit:

$$\underline{r} = r\,\underline{e}_r \qquad -(80)$$

where $\underline{e}_r$ is the radial unit vector. The velocity of an object of mass m in orbit is defined as:

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{dr}{dt}\,\underline{e}_r + r\,\frac{d\underline{e}_r}{dt} \qquad -(81)$$

because in plane polar coordinates the unit vector $\underline{e}_r$ is a function of time so the Leibnitz theorem applies. In the Cartesian system the unit vectors $\underline{i}$ and $\underline{j}$ are not functions of time. The unit vectors of the plane polar system are defined by:

$$\underline{e}_r = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \qquad -(82)$$

$$\underline{e}_\theta = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j} \qquad -(83)$$

and it follows that:

$$\frac{d\underline{e}_r}{dt} = \frac{d\theta}{dt}\,\underline{e}_\theta = \omega\,\underline{e}_\theta \qquad -(84)$$

as described in UFT 236. The velocity in a plane is therefore:

$$\underline{v} = \frac{dr}{dt}\underline{e}_r + \omega r \underline{e}_\theta = \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times \underline{r} \quad -(85)$$

in which the angular velocity vector:

$$\underline{\omega} = \frac{d\theta}{dt}\underline{k} \quad -(86)$$

is the Cartan spin connection as proven in UFT 235 on www.aias.us. Therefore this spin connection is related to the universal spin connection inferred in Section 8.2 giving a coherent cosmology for the solar system and whirlpool galaxies. As we shall prove, the Newton and Einstein theories fail completely to do so.

Using the chain rule:

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad -(87)$$

it is found that the velocity is defined for any orbit by:

$$v^2 = \omega^2\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) \quad -(88)$$

and is therefore defined by the angular velocity or spin connection magnitude:

$$\omega = \frac{d\theta}{dt}. \quad -(89)$$

The orbit itself is defined by $dr/d\theta$, because any planar orbit is defined by r as a function of θ. The angular momentum of any planar orbit is defined by:

$$\underline{L} = \underline{r} \times \underline{p} = m\underline{r} \times \underline{v} \quad -(90)$$

and its magnitude is:

$$L = mr^2\omega. \quad -(91)$$

Therefore for any planar orbit:

$$v^2 = \left(\frac{L}{mr}\right)^2 + \left(\frac{L}{mr^2}\left(\frac{dr}{d\theta}\right)\right)^2 \quad -(92)$$

and as r becomes infinite:

$$r \rightarrow \infty \quad -(93)$$

the velocity reaches the limit:

$$\frac{dr}{d\theta} = \left(\frac{mv_\infty}{L}\right) r^2 \quad -(94)$$

where v_∞ is the velocity for infinite r. In whirlpool galaxies v_∞ is a constant by experimental observation. Therefore:

$$\frac{d\theta}{dr} = \left(\frac{L}{mv_\infty}\right)\frac{1}{r^2} \quad -(95)$$

and

$$\theta = \frac{L}{mv_\infty}\int \frac{dr}{r^2} = -\left(\frac{L}{mv_\infty}\right)\frac{1}{r} \quad -(96)$$

which is the equation of a hyperbolic spiral orbit. In UFT 76 on www.aias.us this hyperbolic spiral orbit was compared with the observed M101 whirlpool galaxy. So the essentials of galactic dynamics can be understood from the simple first principles of kinematics, defining the angular velocity as the spin connection of ECE theory.

Newtonian dynamics fails completely to describe this result because it produces a static conical section:

$$r = \frac{\alpha}{1 + \epsilon\cos\theta} \quad -(97)$$

with an inverse square law of attraction. From Eq. (97):

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha} \sin\theta \quad -(98)$$

and using this result in Eq. (88):

$$v^2 = \omega^2 r^2 \left(1 + \left(\frac{\epsilon r}{\alpha}\right)^2 \sin^2\theta\right) \quad -(99)$$

where:

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2. \quad -(100)$$

So the Newtonian velocity is:

$$v^2 = \omega^2 r^2 \left(\frac{2\alpha}{r} - \left(\frac{r}{\alpha}\right)^2 \left(1 - \epsilon^2\right)\right). \quad -(101)$$

The semi major axis of an elliptical orbit is defined by:

$$a = \frac{\alpha}{1-\epsilon^2} \quad -(102)$$

so Newtonian dynamics produces:

$$v^2 = \frac{1}{\alpha}\left(\frac{L}{m}\right)^2 \left(\frac{2}{r} - \frac{1}{a}\right). \quad -(103)$$

Using the Newtonian half right latitude:

$$\alpha = \frac{L^2}{m^2 MG} \quad -(104)$$

gives:

$$v^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(105)$$

Note that:

$$\frac{1}{a} = \frac{1-\epsilon^2}{\alpha} = \frac{1}{r}\left(1+\epsilon\cos\theta\right)\left(1-\epsilon^2\right) \quad -(106)$$

so the Newtonian velocity is:

$$v^2(\text{Newton}) = \frac{MG}{r}\left(2-\left(1-\epsilon^2\right)\left(1+\epsilon\cos\theta\right)\right) \quad -(107)$$

It follows that:

$$v(\text{Newton}) \xrightarrow[r\to\infty]{} 0 \quad -(108)$$

so the theory fails completely to describe the velocity curve of a whirlpool galaxy.

The Einstein theory does no better because it produces a precessing ellipse , Eq. (3), from which:

$$\frac{dr}{d\theta} = \frac{x\epsilon r^2}{\alpha}\sin(x\theta). \quad -(109)$$

Using Eq. (109) in Eq. (88) gives:

$$v^2 = \left(\frac{L}{mr}\right)^2\left(1+\left(\frac{x\epsilon\sin(x\theta)}{1+\epsilon\cos(x\theta)}\right)^2\right) \quad -(110)$$

and again it is found that:

$$v(\text{Einstein}) \xrightarrow[r\to\infty]{} 0 \quad -(111)$$

and the Einstein theory fails completely to describe the dynamics of a whirlpool galaxy. This leaves ECE theory as the only correct and general theory of cosmology. The latter can be developed by considering again the acceleration in plane polar coordinates

$$\underline{a} = \frac{d\underline{v}}{dt} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta . \quad -(112)$$

As shown in UFT 235 this can be expressed as:

$$\left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r = \frac{d^2 r}{dt^2}\underline{e}_r + \underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right) \quad -(113)$$

and

$$\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta = \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \underline{\dot{r}} \quad -(114)$$

Eq. (114) is the Coriolis acceleration and $\underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right)$ is the centrifugal acceleration. In the UFT papers it is shown that the Coriolis acceleration vanishes for all planar orbits (see Eq. (11)). Using the chain rule it can be shown as in the UFT papers that:

$$\frac{d^2 r}{dt^2} = \left(\frac{L}{mr}\right)^2 \left(\frac{dr}{d\theta}\right) \frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) . \quad -(115)$$

The centrifugal acceleration is defined by:

$$\underline{\omega} \times \left(\underline{\omega} \times \underline{r}\right) = -\omega^2 r\,\underline{e}_r = -\frac{L^2}{m^2 r^3}\underline{e}_r \quad -(116)$$

so the total acceleration is defined by:

$$\underline{a} = \left(\frac{L}{mr}\right)^2 \left[\left(\frac{dr}{d\theta}\right)\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{1}{r}\right]\underline{e}_r \quad -(117)$$

for all planar orbits.

In this equation:

$$\frac{d}{dr}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) = \frac{d\theta}{dr}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) \quad -(118)$$

so:

$$\underline{a} = \left(\frac{L}{mr}\right)^2 \left[\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{1}{r}\right] \underline{e}_r \quad -(119)$$

Now note that:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right)\frac{dr}{d\theta} \quad -(120)$$

so:

$$\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) = \frac{1}{r^2}\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right) = -\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right). \quad -(121)$$

Therefore the acceleration is:

$$\underline{a} = -\left(\frac{L}{mr}\right)^2 \left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\underline{e}_r \quad -(122)$$

and using the definition of force:

$$\underline{F} = m\,\underline{a} \quad -(123)$$

which is Eq (19) derived from lagrangian dynamics. This analysis of any planar orbit is therefore rigorously self consistent.

The Lagrangian method of deriving Eq. (123) sets up the Lagrangian:

$$\mathcal{L} = \frac{1}{2}mv^2 - U \quad -(124)$$

in which the velocity is defined by:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2. \quad -(125)$$

The force is derived from the potential energy as follows:

$$F = -\frac{\partial U}{\partial r} \quad -(126)$$

The two Euler Lagrange equations are:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right), \quad \frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(127)$$

and the angular momentum is defined by the lagrangian to be a constant of motion:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2 \frac{d\theta}{dt} = \text{constant}. \quad -(128)$$

Eq. (122) is the result of pure kinematics in a plane, and is also an equation of Cartan geometry. It is the result of the fundamental expression for acceleration in a plane. Eq. (122) is also an equation of Cartan geometry because the spin connection is the angular velocity.

The covariant derivative of Cartan may be defined for use in classical kinematics in three dimensional space. For any vector $\underline{V}$ the covariant derivative is:

$$\frac{D\underline{V}}{dt} = \left(\frac{d\underline{V}}{dt}\right)_{\text{axes fixed}} + \underline{\omega} \times \underline{V} \quad -(129)$$

where the spin connection vector is the angular velocity ω . In plane polar coordinates define:

$$\underline{V} = V\underline{e}_r \quad -(130)$$

for simplicity of development. The velocity is then defined by:

$$\underline{v} = \frac{D\underline{r}}{dt} = \frac{d\underline{r}}{dt} + \underline{\omega} \times \underline{r} \quad -(131)$$

where:

$$\frac{d\underline{r}}{dt} = \left(\frac{d\underline{r}}{dt}\right)_{\text{axes fixed}} \quad -(132)$$

By definition:

$$\frac{D\underline{r}}{dt} = \frac{D}{dt}\left(r\underline{e}_r\right) = \frac{dr}{dt}\underline{e}_r + r\frac{d\underline{e}_r}{dt} \quad -(133)$$

so:

$$\left(\frac{d\underline{r}}{dt}\right)_{\text{axes fixed}} = \left(\frac{dr}{dt}\right)\underline{e}_r \quad -(134)$$

and

$$\underline{\omega} \times \underline{r} = r\frac{d\underline{e}_r}{dt}. \quad -(135)$$

The acceleration is defined by:

$$\underline{a} = \frac{D\underline{v}}{dt} = \frac{d\underline{v}}{dt} + \underline{\omega} \times \underline{v} \quad -(136)$$

where:

$$\frac{d\underline{v}}{dt} = \left(\frac{d\underline{v}}{dt}\right)_{\text{axes fixed}} \quad -(137)$$

From fundamental kinematics as described above:

$$\underline{a} = \frac{d\underline{v}}{dt} + \underline{\omega} \times \underline{v} = \left(\ddot{r} - \omega^2 r\right)\underline{e}_r + \left(r\frac{d\omega}{dt} + 2\frac{dr}{dt}\omega\right)\underline{e}_\theta \quad -(138)$$

where the unit vectors of the plane polar coordinates system are defined by:

$$\underline{e}_r \times \underline{e}_\theta = \underline{k} \quad -(139)$$

$$\underline{k} \times \underline{e}_r = \underline{e}_\theta \quad -(140)$$

$$\underline{e}_\theta \times \underline{k} = \underline{e}_r \quad -(141)$$

Therefore:

$$\frac{d\underline{v}}{dt} + \underline{\omega} \times \underline{v} = \frac{d^2 r}{dt^2}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\underline{\omega}}{dr} \times \underline{r} + 2\underline{\omega} \times \left(\frac{dr}{dt}\underline{e}_r\right). \quad -(142)$$

From Eq. (131)

$$\underline{v} = \frac{dr}{dt} + \underline{\omega} \times \underline{r} \quad -(143)$$

so in Eq. (136):

$$\underline{a} = \frac{d^2 r}{dt^2}\underline{e}_r + \frac{d\underline{\omega}}{dt} \times \underline{r} + \underline{\omega} \times \left(\frac{dr}{dt}\right)_{\text{axes fixed}} + \underline{\omega} \times \frac{dr}{dt}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(144)$$

In this equation:

$$\underline{\omega} \times \left(\frac{dr}{dt}\right)_{\text{axes fixed}} = \underline{\omega} \times \frac{dr}{dt}\underline{e}_r \quad -(145)$$

so:

$$\underline{a} = \frac{d^2 r}{dt^2}\underline{e}_r + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \frac{d\underline{\omega}}{dt} \times \underline{r} + 2\underline{\omega} \times \left(\frac{dr}{dt}\underline{e}_r\right) \quad -(146)$$

which is Eq. (142), QED.

The covariant derivatives used in these calculations are examples of the Cartan covariant derivative:

$$D_\mu V^a = \partial_\mu V^a + \omega^a_{\ \mu b} V^b. \quad -(147)$$

The well known centripetal acceleration:

$$\underline{a} = \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(148)$$

and the Coriolis acceleration:

$$\underline{a} = \frac{d\omega}{dt} \times \underline{r} + 2\underline{\omega} \times \left(\frac{dr}{dt} \underline{e}_r \right) \quad -(149)$$

are produced by the plane polar system of coordinates. These accelerations do not exist in the Cartesian system and depend entirely on the existence of the spin connection of Cartan.

As shown already the Coriolis acceleration vanishes for all closed planar orbits and the acceleration simplifies to:

$$\underline{a} = \left(\ddot{r} - \omega^2 r \right) \underline{e}_r = \frac{d^2 r}{dt^2} \underline{e}_r + \underline{\omega} \times \left(\underline{\omega} \times \underline{r} \right) . \quad -(150)$$

For example the acceleration due to gravity is:

$$\underline{g} = \frac{d^2 r}{dt^2} \underline{e}_r + \underline{\omega} \times \left(\underline{\omega} \times \underline{r} \right) \quad -(151)$$

and includes the centripetal acceleration:

$$\underline{\omega} \times \left(\underline{\omega} \times \underline{r} \right) = - \omega^2 r \underline{e}_r . \quad -(152)$$

The acceleration due to gravity in the plane polar system is the sum of $\underline{g}$ in the Cartesian system:

$$\underline{g}\,(\text{Cartesian}) = \frac{d^2 r}{dt^2} \underline{e}_r \quad -(153)$$

and the centripetal acceleration. To make this point clearer consider the acceleration of an elliptical orbit or closed elliptical trajectory in the plane polar system. It is:

$$\underline{a} = - \frac{L^2}{m^2 r^2 \alpha} \underline{e}_r \quad -(154)$$

where the angular momentum is a constant of motion and defined by:

$$L = |\underline{L}| = |\underline{r} \times \underline{p}| = m r^2 \omega . \quad -(155)$$

The acceleration due to gravity of the elliptical motion of a mass m is:

$$\underline{g} = -\frac{L^2}{m^2 r^2 \alpha} \underline{e}_r \quad -(156)$$

in plane polar coordinates. The Newtonian result is recovered using the half right latitude:

$$\alpha = \frac{L^2}{m^2 M G} \quad -(157)$$

so:

$$\underline{g} = -\frac{MG}{r^2} \underline{e}_r . \quad -(158)$$

The only force present in the plane polar system of coordinates is:

$$\underline{F} = m\underline{g} = -\frac{mMG}{r^2} \underline{e}_r \quad -(159)$$

which is the equivalence principle, Q. E. D.

The acceleration in the Cartesian system of coordinates from Eq. (151) is:

$$\underline{a}\,(\text{Cartesian}) = \underline{g} - \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad -(160)$$

in which the centrifugal acceleration is:

$$-\underline{\omega} \times (\underline{\omega} \times \underline{r}) = \omega^2 r \, \underline{e}_r . \quad -(161)$$

Therefore in the Cartesian system the acceleration produced by the same elliptical trajectory

is:

$$\left(\frac{d^2 r}{dt^2}\right)_{\text{Cartesian}} \underline{e}_r = \left(-\frac{L^2}{m^2 r^2 \alpha} + \omega^2 r\right) \underline{e}_r \quad -(162)$$

It generalizes the Newtonian theory to give:

$$\left(\frac{d^2 r}{dt^2}\right)_{\text{Cartesian}} \underline{e}_r = \left(-\frac{MG}{r^2} + \frac{L^2}{m^2 r^3}\right) \underline{e}_r \quad -(163)$$

and the familiar force:

$$\underline{F} = m\left(\frac{d^2 r}{dt^2}\right)_{\text{Cartesian}} \underline{e}_r = \left(-\frac{mMG}{r^2} + \frac{L^2}{m r^3}\right) \underline{e}_r \quad -(164)$$

of the textbooks. From a comparison of Eqs. (159) and (164) the forces in the plane polar and Cartesian systems are different. If the frame of reference is static with respect to the observer the force is Eq. (164). If the frame of reference is rotating with respect to the observer the force is defined by Eq. (159).