273(3): Calculation of Orbital Linear Velocity with 3-D Theory.

The orbital linear velocity in 3-D theory is defined by

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta + r\dot{\phi}\sin\theta\,\underline{e}_\phi \quad -(1)$$

so

$$v^2 = \dot{r}^2 + r^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) \quad -(2)$$

In the 2-D theory:

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2 \quad -(3)$$

$$= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2.$$

In the 2-D theory:

$$\omega = \frac{d\phi}{dt} = \frac{L_Z}{mr^2} \quad -(4)$$

and

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} \quad -(5)$$

So

$$v^2 = \frac{L_Z^2}{m^2r^2} + \left(\frac{dr}{d\phi}\right)^2\left(\frac{d\phi}{dt}\right)^2$$

$$= \frac{L_Z^2}{m^2r^2}\left(1 + \frac{1}{r^2}\left(\frac{dr}{d\phi}\right)^2\right) \quad -(6)$$

In this expression:

2)

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\epsilon^2 r^4}{\alpha^2}\left(1 - \cos^2\phi\right) \quad -(7)$$

where
$$\cos^2\phi = \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2 \quad -(8)$$

So:
$$v^2 = \frac{L_z^2}{m^2}\left(\frac{1}{r^2} + \frac{\epsilon^2}{\alpha^2}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)\right)$$

$$= \frac{L_z^2}{m^2\alpha}\left(\frac{\epsilon^2 - 1}{\alpha} + \frac{2}{r}\right)$$

$$= \frac{L_z^2}{m^2\alpha}\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(9)$$

$$= M_s G\left(\frac{2}{r} - \frac{1}{a}\right).$$

In the 3-D theory:
$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\beta}{dt}\right)^2 \quad -(10)$$

and
$$\omega = \frac{d\beta}{dt} = \frac{L}{mr^2} \quad -(11)$$

A direct comparison of eqs. (3) and (10) can be made, and graphed.

2-D Theory

In this case:

$$\dot{r} = \left(\frac{2}{m} \left(E - \frac{L_z^2}{2mr^2} + \frac{k}{r} \right) \right)^{1/2} \quad -(12)$$

where

$$r = \frac{\alpha}{1 + \epsilon \cos \phi} \quad -(13)$$

and

$$\dot{\phi} = \frac{L_z}{mr^2} \quad -(14).$$

So v can be graphed as a function of ϕ.

In this case:

$$v^2 = \dot{r}^2 + r^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) \quad -(15)$$

where:

$$\dot{r} = \left(\frac{2}{m} \left(E - \frac{L^2}{2mr^2} + \frac{k}{r} \right) \right)^{1/2}, \quad -(16)$$

$$\dot{\phi} = \frac{L_z}{mr^2 \sin^2 \theta} \quad -(17)$$

$$\dot{\theta} = \frac{1}{mr^2} \left(L^2 - \frac{L_z^2}{\sin^2 \theta} \right)^{1/2} \quad -(18)$$

$$r = \frac{\alpha}{1 + \epsilon \cos \beta} \quad -(19)$$

4)

$$\cos\beta = \frac{\cos\phi}{\left(\cos^2\phi + \left(\frac{L_z}{L}\right)^2 \sin^2\phi\right)^{1/2}} \quad -(20)$$

and

$$\sin^2\theta = \left(\frac{L_z}{L}\right)^2 + \left(1 - \left(\frac{L_z}{L}\right)^2\right)\left(\frac{\cos^2\phi}{\cos^2\phi + \left(\frac{L_z}{L}\right)^2 \sin^2\phi}\right) \quad -(21)$$

So v can again be expressed as a function of ϕ using the chain of equations (15) to (21). So a direct comparison of the 2-D and 3-D theories can be made.

Another type of direct comparison can be made with:

$$v^2 = \frac{L^2}{m^2 r^2}\left(1 + \frac{1}{r^2}\left(\frac{dr}{d\beta}\right)^2\right). \quad -(22)$$

Now use:

$$\frac{dr}{d\beta} = \frac{dr}{d\phi}\frac{d\phi}{d\beta} \quad -(23)$$

where:

$$\frac{d\phi}{d\beta} = \frac{L_Z}{L\sin^2\theta} \quad -(24)$$

This means that:

$$v^2 = \frac{L^2}{m^2 r^2}\left(1 + \frac{y^2}{r^2}\left(\frac{dr}{d\phi}\right)^2\right) \quad -(25)$$

where

$$y = \frac{L_Z}{L\sin^2\theta} \quad -(26)$$

Comparison of Eqs. (6) and (25) gives a direct comparison of the 2-D and 3-D theories. It is seen that in the 3-D theory L_Z in eq. (6) is replaced by L, and $dr/d\phi$ is replaced by $y\, dr/d\phi$. The easiest way of making this comparison is to use eq. (22) and eq. (19). It follows that:

$$\left(\frac{dr}{d\beta}\right)^2 = \frac{\epsilon^2 r^4}{\alpha^2}\left(1 - \cos^2\beta\right) \quad -(27)$$

where

$$\cos^2\beta = \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right) \quad -(28)$$

6) So in direct analogy to eqs. (7) to (9):

$$v^2 = mG\left(\frac{2}{r} - \frac{1}{a}\right) \quad -(29)$$

from the 3-D theory. However:

$$\frac{1}{a} = \frac{1-\epsilon^2}{\alpha} \quad -(30)$$

$$= \frac{2E}{k}$$

So the difference between the 2-D and 3-D theories comes down to the difference in the total energy E in 2-D and 3-D.

The difference is made clear only by comparing the graph of v against ϕ. In 2-D use eqs. (10) to (14), and in 3-D use eqs. (15) to (21).