General reflection/refraction formula and cross-check

```
(%i1) kill(all);
(%o0) done
```

```
(%i1) N0: 1.5; N1: 1.0;
(%o1) 1.5
(%o2) 1.0
```

```
(%i3) defA(dummy) := (
      y:  h[bar]*omega[0]*f[T],
      y1: h[bar]*omega[1]*f[T],
      y2: h[bar]*omega[2]*f[T],
      A[0]: (1-y)/y,
      A[1]: (1-y1)/y1,
      A[2]: (1-y2)/y2,
      kill(y)
      )$
```

```
(%i4) /*th3(theta,n0,n1):= if n1>n0 then theta-asin((n0*sin(theta))/n1)
      else -(theta-asin((n0*sin(theta))/n1))$*/nil;
(%o4) nil
```

```
(%i5) th3(theta,n0,n1):= asin((n0*sin(theta))/n1)-theta;
```

(%o5) $\mathrm{th3}(\theta, n0, n1) := \mathrm{asin}\left(\frac{n0\,\sin(\theta)}{n1}\right) - \theta$

```
(%i6) th3(.3,1,1.5); th3(.3,1.5,1.);
(%o6) -0.10168924954136
(%o7) 0.15925487201513
```

1 Energy and momentum conservation

1.1 Energy

```
(%i8) E1: omega[0]*A[0] =
      A[1]*omega[1] + A[2]*omega[2];
```

(%o8) $\omega_0 A_0 = \omega_2 A_2 + \omega_1 A_1$

1.2 Refraction case: (bold kappa_1)^2 = (bold kappa_0 - bold kappa_2)^2

```
(%i9) E2: omega[1]^2*A[1]^2*n[1]^2 =
      omega[0]^2*A[0]^2*n[0]^2
       + omega[2]^2*A[2]^2*n[0]^2
       + 2*n[0]^2*A[0]*A[2]*omega[0]*omega[2]*cos(2*theta[0]);
```

(%o9) $n_1^2\,\omega_1^2\,A_1^2 = 2\,n_0^2\,\omega_0\,A_0\,\omega_2\,A_2\cos(2\,\theta_0) + n_0^2\,\omega_2^2\,A_2^2 + n_0^2\,\omega_0^2\,A_0^2$

1.3 Reflection case: (bold kappa_2)^2 = (bold kappa_0 - bold kappa_1)^2

```
(%i10) E3: omega[2]^2*A[2]^2*n[0]^2 =
       omega[0]^2*A[0]^2*n[0]^2
        + omega[1]^2*A[1]^2*n[1]^2
        - 2*n[0]*n[1]*A[0]*A[1]*omega[0]*omega[1]*cos(theta[3]);
```

(%o10) $n_0^2\,\omega_2^2\,A_2^2 = -2\,n_0\,\omega_0\,A_0\,n_1\,\omega_1\,A_1\cos(\theta_3) + n_1^2\,\omega_1^2\,A_1^2 + n_0^2\,\omega_0^2\,A_0^2$

1-photon theory

1 1-photon case, refraction-based, omega_1

```
(%i11) E11: ev(E1,[A[0]=1, A[1]=1, A[2]=1]);
```

(%o11) $\omega_0 = \omega_2 + \omega_1$

```
(%i12) E12: ev(E2,[A[0]=1, A[1]=1, A[2]=1]);
```

(%o12) $n_1^2\,\omega_1^2 = 2\,n_0^2\,\omega_0\,\omega_2\cos(2\,\theta_0) + n_0^2\,\omega_2^2 + n_0^2\,\omega_0^2$

```
(%i13) E13: ev(E3,[A[0]=1, A[1]=1, A[2]=1]);
```

(%o13) $n_0^2\,\omega_2^2 = -2\,n_0\,\omega_0\,n_1\,\omega_1\cos(\theta_3) + n_1^2\,\omega_1^2 + n_0^2\,\omega_0^2$

1.1 Refraction (correct)

```
(%i14) E12a: subst(omega[0]-omega[1], omega[2], E12);
```

(%o14) $n_1^2\,\omega_1^2 = 2\,n_0^2\,\omega_0\,(\omega_0 - \omega_1)\cos(2\,\theta_0) + n_0^2\,(\omega_0 - \omega_1)^2 + n_0^2\,\omega_0^2$

```
(%i15) E12b: solve(E12a, omega[1]);
```

(%o15) $$\left[\omega_1 = \frac{n_0\,\omega_0\sqrt{n_0^2\cos(2\,\theta_0)^2 + 2\,n_1^2\cos(2\,\theta_0) + 2\,n_1^2 - n_0^2} - n_0^2\,\omega_0\cos(2\,\theta_0) - n_0^2\,\omega_0}{n_1^2 - n_0^2},\ \omega_1 = -\frac{n_0\,\omega_0\sqrt{n_0^2\cos(2\,\theta_0)^2 + 2\,n_1^2\cos(2\,\theta_0) + 2\,n_1^2 - n_0^2} + n_0^2\,\omega_0\cos(2\,\theta_0) + n_0^2\,\omega_0}{n_1^2 - n_0^2}\right]$$

```
(%i16) o1a: factor(rhs(first(E12b))/omega[0]);
```

(%o16) $$\frac{n_0\left(\sqrt{n_0^2\cos(2\,\theta_0)^2 + 2\,n_1^2\cos(2\,\theta_0) + 2\,n_1^2 - n_0^2} - n_0\cos(2\,\theta_0) - n_0\right)}{(n_1 - n_0)(n_1 + n_0)}$$

```
(%i17) o1b: factor(rhs(second(E12b))/omega[0]);
```

(%o17) $$-\frac{n_0\left(\sqrt{n_0^2\cos(2\theta_0)^2+2n_1^2\cos(2\theta_0)+2n_1^2-n_0^2}+n_0\cos(2\theta_0)+n_0\right)}{(n_1-n_0)(n_1+n_0)}$$

1.2 Reflection

```
(%i18) E12c: subst(omega[0]-omega[2], omega[1], E12);
```

(%o18) $n_1^2(\omega_0-\omega_2)^2=2n_0^2\omega_0\omega_2\cos(2\theta_0)+n_0^2\omega_2^2+n_0^2\omega_0^2$

```
(%i19) E12d: solve(E12c, omega[2]);
```

(%o19) $$\left[\omega_2=\frac{n_0\omega_0\sqrt{n_0^2\cos(2\theta_0)^2+2n_1^2\cos(2\theta_0)+2n_1^2-n_0^2}+n_0^2\omega_0\cos(2\theta_0)+\omega_0 n_1^2}{n_1^2-n_0^2},\ \omega_2=-\frac{n_0\omega_0\sqrt{n_0^2\cos(2\theta_0)^2+2n_1^2\cos(2\theta_0)+2n_1^2-n_0^2}-n_0^2\omega_0\cos(2\theta_0)-\omega_0 n_1^2}{n_1^2-n_0^2}\right]$$

```
(%i20) o2a: factor(rhs(first(E12d))/omega[0]);
```

(%o20) $$\frac{n_0\sqrt{n_0^2\cos(2\theta_0)^2+2n_1^2\cos(2\theta_0)+2n_1^2-n_0^2}+n_0^2\cos(2\theta_0)+n_1^2}{(n_1-n_0)(n_1+n_0)}$$

```
(%i21) o2b: factor(rhs(second(E12d))/omega[0]);
```

(%o21) $$-\frac{n_0\sqrt{n_0^2\cos(2\theta_0)^2+2n_1^2\cos(2\theta_0)+2n_1^2-n_0^2}-n_0^2\cos(2\theta_0)-n_1^2}{(n_1-n_0)(n_1+n_0)}$$

1.3 Graphics

```
(%i22) str: [n[0]=N0, n[1]=N1, theta[0]=theta];
```

(%o22) $[n_0=1.5,\ n_1=1.0,\ \theta_0=\theta]$

```
(%i23) po1a: ev(o1a, str);
       po1b: ev(o1b, str);
       po2a: ev(o2a, str);
       po2b: ev(o2b, str);
```

(%o23) $-1.2\left(\sqrt{2.25\cos(2\theta)^2+2.0\cos(2\theta)-0.25}-1.5\cos(2\theta)-1.5\right)$

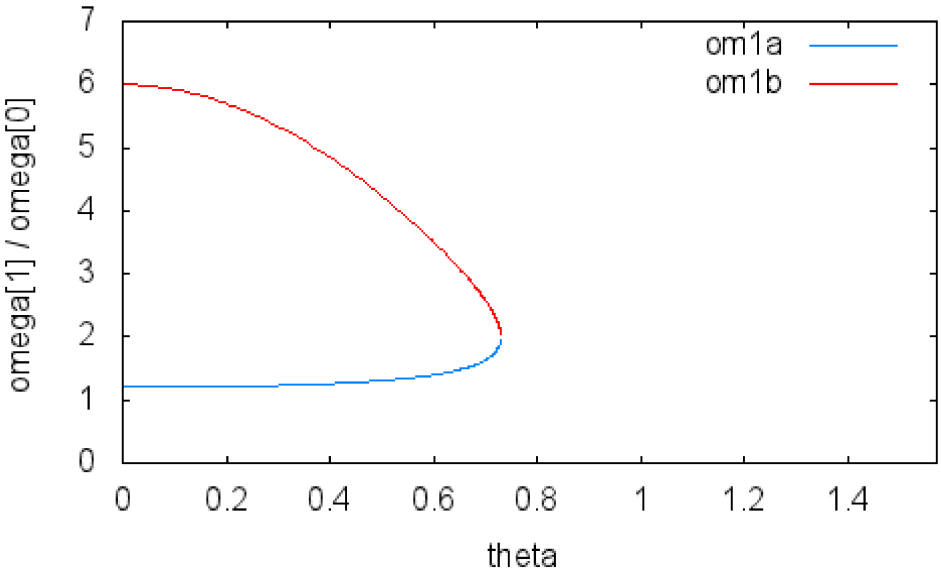
(%o24) $1.2\left(\sqrt{2.25\cos(2\theta)^2+2.0\cos(2\theta)-0.25}+1.5\cos(2\theta)+1.5\right)$

(%o25) $-0.8\left(1.5\sqrt{2.25\cos(2\theta)^2+2.0\cos(2\theta)-0.25}+2.25\cos(2\theta)+1.0\right)$

(%o26) $0.8\left(1.5\sqrt{2.25\cos(2\theta)^2+2.0\cos(2\theta)-0.25}-2.25\cos(2\theta)-1.0\right)$

```
(%i27) wxplot2d([po1a,po1b], [theta,0,%pi/2],
       [ylabel, "omega[1] / omega[0]"], [legend, "om1a", "om1b"])$
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
```

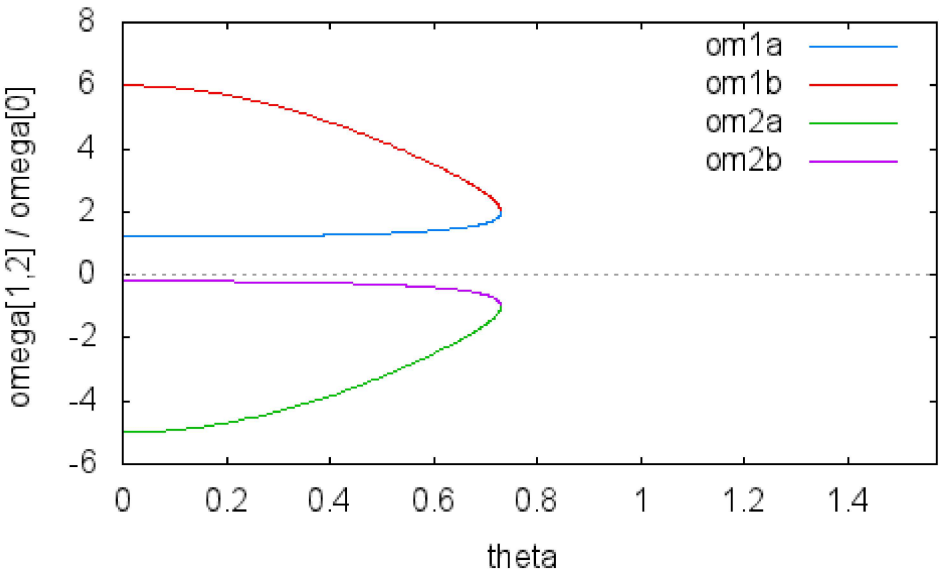
(%t27)




```
(%i28) plot2d([po1a,po1b], [theta,0,%pi/2],
       [ylabel, "omega[1] / omega[0]"], [legend, "om1a", "om1b"],
       [gnuplot_term, "png  linewidth 2.5 font 'Arial' 16 size 800,600"],
       [gnuplot_out_file, "D:/Doc/Artikel-Eck/ECE-Theorie/paper280/Fig6.png"
       )$
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
```

```
(%i29) wxplot2d([po1a,po1b,po2a,po2b], [theta,0,%pi/2],
       [ylabel, "omega[1,2] / omega[0]"], [legend, "om1a", "om1b", "om2a", "
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
```

(%t29)



2 1-photon case, reflection-based, omega_2

(%i30) E11;

(%o30) $\omega_0 = \omega_2 + \omega_1$

(%i31) E13;

(%o31) $n_0^2\,\omega_2^2 = -2\,n_0\,\omega_0\,n_1\,\omega_1\cos(\theta_3) + n_1^2\,\omega_1^2 + n_0^2\,\omega_0^2$

2.1 Refraction

(%i32) E13a: subst(omega[0]-omega[1], omega[2], E13);

(%o32) $n_0^2(\omega_0 - \omega_1)^2 = -2\,n_0\,\omega_0\,n_1\,\omega_1\cos(\theta_3) + n_1^2\,\omega_1^2 + n_0^2\,\omega_0^2$

(%i33) E13b: solve(E13a, omega[1]);

(%o33) $\left[\omega_1 = \frac{2\,n_0\,\omega_0\,n_1\cos(\theta_3) - 2\,n_0^2\,\omega_0}{n_1^2 - n_0^2},\ \omega_1 = 0\right]$

(%i34) o1a: factor(rhs(first(E13b))/omega[0]);

(%o34) $\frac{2\,n_0(n_1\cos(\theta_3) - n_0)}{(n_1 - n_0)(n_1 + n_0)}$

(%i35) o1b: factor(rhs(second(E13b))/omega[0]);

(%o35) 0

2.2 Reflection (correct)

(%i36) E13c: subst(omega[0]-omega[2], omega[1], E13);

(%o36) $n_0^2\,\omega_2^2 = -2\,n_0\,\omega_0\,n_1(\omega_0 - \omega_2)\cos(\theta_3) + n_1^2(\omega_0 - \omega_2)^2 + n_0^2\,\omega_0^2$

(%i37) E13d: solve(E13c, omega[2]);

(%o37) $\left[\omega_2 = -\frac{2\,n_0\,\omega_0\,n_1\cos(\theta_3) - \omega_0\,n_1^2 - n_0^2\,\omega_0}{n_1^2 - n_0^2},\ \omega_2 = \omega_0\right]$

(%i38) o2a: factor(rhs(first(E13d))/omega[0]);

(%o38) $-\frac{2\,n_0\,n_1\cos(\theta_3) - n_1^2 - n_0^2}{(n_1 - n_0)(n_1 + n_0)}$

(%i39) o2b: factor(rhs(second(E13d))/omega[0]);

(%o39) 1

2.3 Graphics

```
(%i40) str: [n[0]=N0, n[1]=N1, theta[0]=theta];
```

(%o40) $[n_0=1.5, n_1=1.0, \theta_0=\theta]$

```
(%i41) theta[3]: ev(th3(theta[0], n[0], n[1]), str);
```

(%o41) $\operatorname{asin}(1.5\sin(\theta))-\theta$

```
(%i42) o1a;
```

(%o42) $$\frac{2\,n_0\,(n_1\cos(\theta_3)-n_0)}{(n_1-n_0)(n_1+n_0)}$$

```
(%i43) po1a: ev(o1a, str);
       po1b: ev(o1b, str);
       po2a: ev(o2a, str);
       po2b: ev(o2b, str);
```

(%o43) $-2.4\left(1.0\cos(\operatorname{asin}(1.5\sin(\theta))-\theta)-1.5\right)$

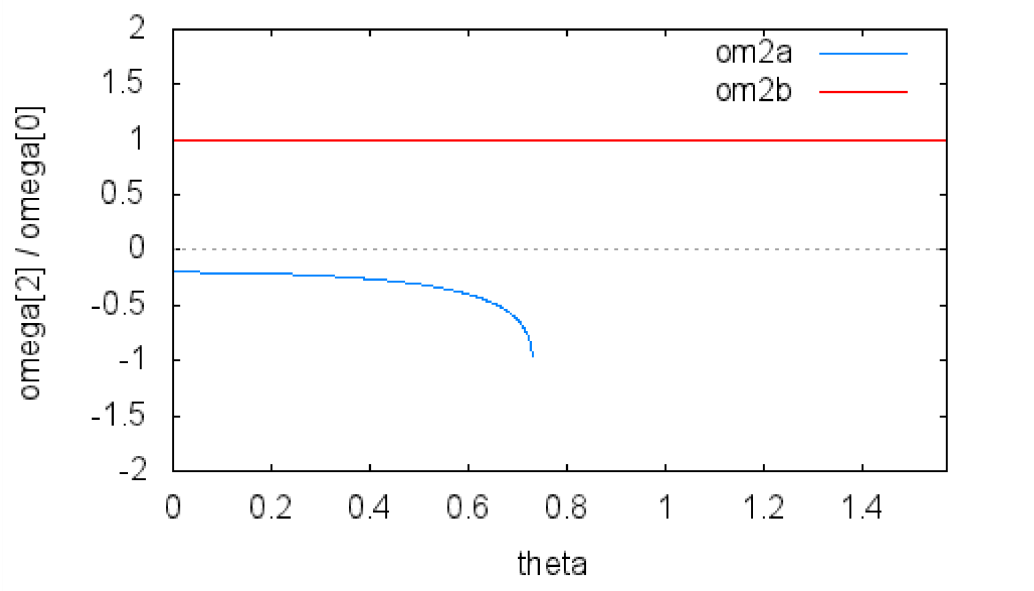
(%o44) 0

(%o45) $0.8\left(3.0\cos(\operatorname{asin}(1.5\sin(\theta))-\theta)-3.25\right)$

(%o46) 1

```
(%i47) wxplot2d([po2a,po2b], [theta,0,%pi/2], [y,-2.,2.],
       [ylabel, "omega[2] / omega[0]"], [legend, "om2a", "om2b"])$
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
```

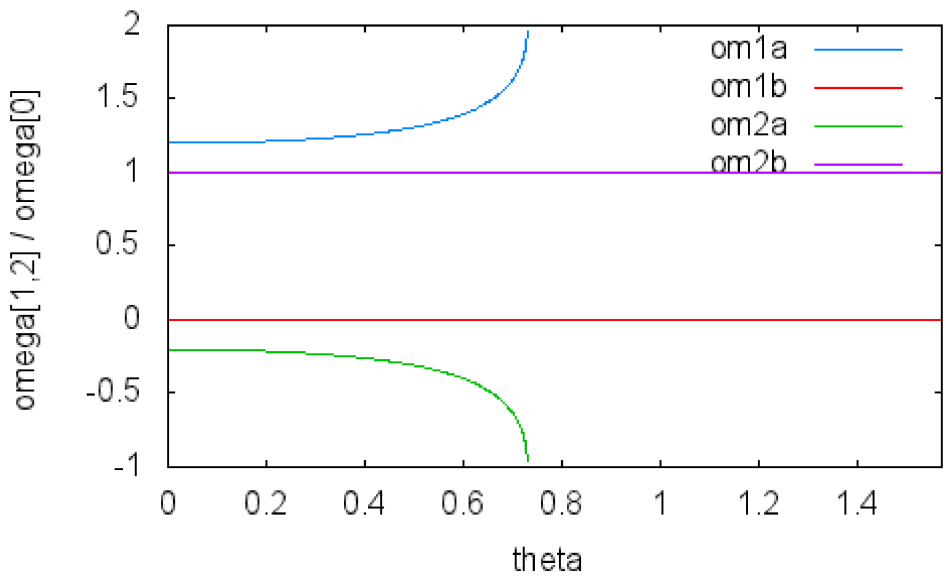
(%t47)




```
(%i48) plot2d([po2a,po2b], [theta,0,%pi/2], [y,-2.,2.],
       [ylabel, "omega[2] / omega[0]"], [legend, "om2a", "om2b"],
       [gnuplot_term, "png  linewidth 2.5 font 'Arial' 16 size 800,600"],
       [gnuplot_out_file, "D:/Doc/Artikel-Eck/ECE-Theorie/paper280/Fig7.png"
       )$
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
```

```
(%i49) wxplot2d([po1a,po1b,po2a,po2b], [theta,0,%pi/2],
       [ylabel, "omega[1,2] / omega[0]"], [legend, "om1a", "om1b", "om2a", "
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
```

(%t49)



Mutliple Planck oscillator theory

1 Planck oscillator theory, refraction-based, ome

```
(%i50) kill(A,theta);
```

(%o50) *done*

```
(%i51) E1; E2;
```

(%o51) $\omega_0 A_0 = \omega_2 A_2 + \omega_1 A_1$

(%o52) $n_1^2 \omega_1^2 A_1^2 = 2 n_0^2 \omega_0 A_0 \omega_2 A_2 \cos(2\, theta_0) + n_0^2 \omega_2^2 A_2^2 + n_0^2 \omega_0^2 A_0^2$

1.1 Refraction (correct)

```
(%i53) E12a: ratsubst(A[0]*omega[0]-A[1]*omega[1], A[2]*omega[2], E2);
```

(%o53) $n_1^2 \omega_1^2 A_1^2 = (2 n_0^2 \omega_0^2 A_0^2 - 2 n_0^2 \omega_0 A_0 \omega_1 A_1) \cos(2\, theta_0) + n_0^2 \omega_1^2 A_1^2 - 2 n_0^2 \omega_0 A_0 \omega_1 A_1 + 2 n_0^2 \omega_0^2 A_0^2$

```
(%i54) defA(0);
```

(%o54) *done*

```
(%i55) E12b: ev(E12a);
```

(%o55) $$\frac{n_1^2(1-\omega_1 h_{bar} f_T)^2}{h_{bar}^2 f_T^2}=\frac{n_0^2(1-\omega_1 h_{bar} f_T)^2}{h_{bar}^2 f_T^2}+\cos(2\,theta_0)\left(\frac{2\,n_0^2(1-\omega_0 h_{bar} f_T)^2}{h_{bar}^2 f_T^2}-\frac{2\,n_0^2(1-\omega_0 h_{bar} f_T)(1-\omega_1 h_{bar} f_T)}{h_{bar}^2 f_T^2}\right)+\frac{2\,n_0^2(1-\omega_0 h_{bar} f_T)^2}{h_{bar}^2 f_T^2}-\frac{2\,n_0^2(1-\omega_0 h_{bar} f_T)(1-\omega_1 h_{bar} f_T)}{h_{bar}^2 f_T^2}$$

```
(%i56) E12c: solve(E12b, omega[1])$
```

```
(%i57) o1a: factor(rhs(first(E12c)));
```

(%o57) $$(n_0\,\omega_0\sqrt{n_0^2\cos(2\,theta_0)^2+2\,n_1^2\cos(2\,theta_0)+2\,n_1^2-n_0^2}\,h_{bar}\,f_T-n_0^2\,\omega_0\cos(2\,theta_0)h_{bar}\,f_T-n_0^2\,\omega_0\,h_{bar}\,f_T-n_0\sqrt{n_0^2\cos(2\,theta_0)^2+2\,n_1^2\cos(2\,theta_0)+2\,n_1^2-n_0^2}+n_0^2\cos(2\,theta_0)+n_1^2)/((n_1-n_0)(n_1+n_0)h_{bar}\,f_T)$$

```
(%i58) o1b: factor(rhs(second(E12c)));
```

(%o58) $$-(n_0\,\omega_0\sqrt{n_0^2\cos(2\,theta_0)^2+2\,n_1^2\cos(2\,theta_0)+2\,n_1^2-n_0^2}\,h_{bar}\,f_T+n_0^2\,\omega_0\cos(2\,theta_0)h_{bar}\,f_T+n_0^2\,\omega_0\,h_{bar}\,f_T-n_0\sqrt{n_0^2\cos(2\,theta_0)^2+2\,n_1^2\cos(2\,theta_0)+2\,n_1^2-n_0^2}-n_0^2\cos(2\,theta_0)-n_1^2)/((n_1-n_0)(n_1+n_0)h_{bar}\,f_T)$$

1.2 Reflection

```
(%i59) kill(A,theta);
```

(%o59) *done*

```
(%i60) E1; E2;
```

(%o60) $\omega_0\,A_0=\omega_2\,A_2+\omega_1\,A_1$

(%o61) $n_1^2\,\omega_1^2\,A_1^2=2\,n_0^2\,\omega_0\,A_0\,\omega_2\,A_2\cos(2\,theta_0)+n_0^2\,\omega_2^2\,A_2^2+n_0^2\,\omega_0^2\,A_0^2$

```
(%i62) E12d: ratsubst(A[0]*omega[0]-A[2]*omega[2], A[1]*omega[1], E2);
```

(%o62) $n_1^2(\omega_2^2\,A_2^2-2\,\omega_0\,A_0\,\omega_2\,A_2+\omega_0^2\,A_0^2)=2\,n_0^2\,\omega_0\,A_0\,\omega_2\,A_2\cos(2\,theta_0)+n_0^2\,\omega_2^2\,A_2^2+n_0^2\,\omega_0^2\,A_0^2$

```
(%i63) defA(0);
```

(%o63) *done*

```
(%i64) E12e: ev(E12d);
```

(%o64) $$n_1^2\left(\frac{(1-\omega_2\,h_{bar}\,f_T)^2}{h_{bar}^2\,f_T^2}+\frac{(1-\omega_0\,h_{bar}\,f_T)^2}{h_{bar}^2\,f_T^2}-\frac{2(1-\omega_0\,h_{bar}\,f_T)(1-\omega_2\,h_{bar}\,f_T)}{h_{bar}^2\,f_T^2}\right)=\frac{n_0^2(1-\omega_2\,h_{bar}\,f_T)^2}{h_{bar}^2\,f_T^2}+\frac{n_0^2(1-\omega_0\,h_{bar}\,f_T)^2}{h_{bar}^2\,f_T^2}+\frac{2\,n_0^2\cos(2\,theta_0)(1-\omega_0\,h_{bar}\,f_T)(1-\omega_2\,h_{bar}\,f_T)}{h_{bar}^2\,f_T^2}$$

```
(%i65) E12f: solve(E12e, omega[2])$
```

```
(%i66) o2a: factor(rhs(first(E12f)));
```

(%o66) $$(n_0\,\omega_0\sqrt{n_0^2\cos(2\,theta_0)^2+2\,n_1^2\cos(2\,theta_0)+2\,n_1^2-n_0^2}\,h_{bar}\,f_T+n_0^2\,\omega_0\cos(2\,theta_0)h_{bar}\,f_T+\omega_0\,n_1^2\,h_{bar}\,f_T-n_0\sqrt{n_0^2\cos(2\,theta_0)^2+2\,n_1^2\cos(2\,theta_0)+2\,n_1^2-n_0^2}-n_0^2\cos(2\,theta_0)-n_0^2)/((n_1-n_0)(n_1+n_0)h_{bar}\,f_T)$$

```
(%i67) o2b: factor(rhs(second(E12f)));
```

(%o67) $$-(n_0\,\omega_0\sqrt{n_0^2\cos(2\,theta_0)^2+2\,n_1^2\cos(2\,theta_0)+2\,n_1^2-n_0^2}\,h_{bar}\,f_T-n_0^2\,\omega_0\cos(2\,theta_0)h_{bar}\,f_T-\omega_0\,n_1^2\,h_{bar}\,f_T-n_0\sqrt{n_0^2\cos(2\,theta_0)^2+2\,n_1^2\cos(2\,theta_0)+2\,n_1^2-n_0^2}+n_0^2\cos(2\,theta_0)+n_0^2)/((n_1-n_0)(n_1+n_0)h_{bar}\,f_T)$$

1.3 Graphics

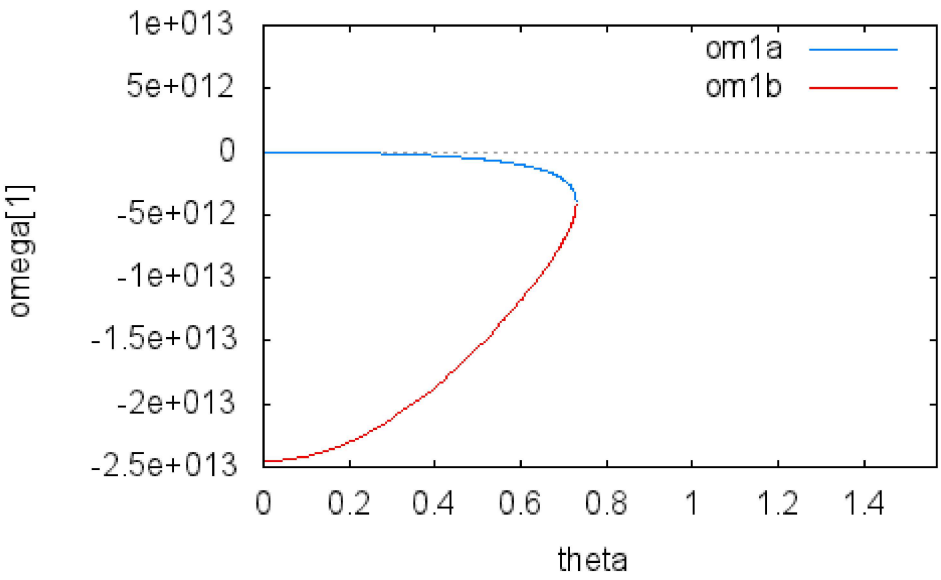
```
(%i68) str: [n[0]=N0, n[1]=N1, h[bar]=6.62618e-34, f[T]=1/(1.38066e-23*293),
       theta[0]=theta, omega[0]=1.e12];
```

(%o68) $[n_0=1.5,\ n_1=1.0,\ h_{bar}=6.6261800000000001\ 10^{-34},\ f_T=2.4719838941350153\ 10^{20},\ theta_0=theta,\ \omega_0=1.0\ 10^{12}]$

```
(%i69) po1a: ev(o1a, str)$
       po1b: ev(o1b, str)$
       po2a: ev(o2a, str)$
       po2b: ev(o2b, str)$
```

```
(%i73) wxplot2d([po1a,po1b], [theta,0,%pi/2],
       [ylabel, "omega[1]"], [legend, "om1a", "om1b"])$
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
```

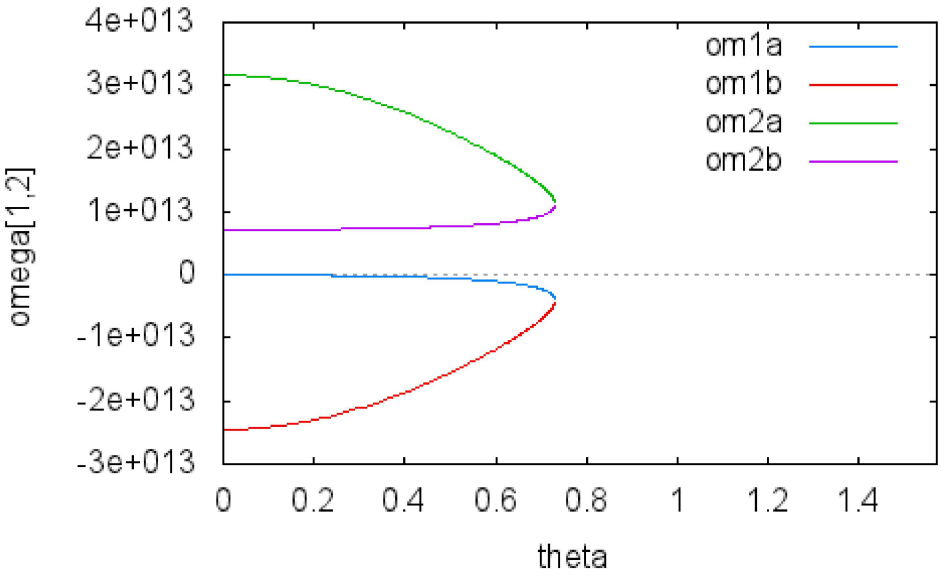
(%t73)




```
(%i74) plot2d([po1a,po1b], [theta,0,%pi/2],
       [ylabel, "omega[1]"], [legend, "om1a", "om1b"],
       [gnuplot_term, "png  linewidth 2.5 font 'Arial' 16 size 800,600"],
       [gnuplot_out_file, "D:/Doc/Artikel-Eck/ECE-Theorie/paper280/Fig8.png"
       )$
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
```

```
(%i75) wxplot2d([po1a,po1b,po2a,po2b], [theta,0,%pi/2],
       [ylabel, "omega[1,2]"], [legend, "om1a", "om1b", "om2a", "om2b"])$
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
```

(%t75)



2 Planck oscillator theory, reflection-based, ome

```
(%i76) kill(A,theta);
```
(%o76) *done*

```
(%i77) E1; E3;
```
(%o77) $\omega_0 A_0 = \omega_2 A_2 + \omega_1 A_1$

(%o78) $n_0^2 \omega_2^2 A_2^2 = -2 n_0 \omega_0 A_0 n_1 \omega_1 A_1 \cos(theta_3) + n_1^2 \omega_1^2 A_1^2 + n_0^2 \omega_0^2 A_0^2$

2.1 Refraction

```
(%i79) E13a: ratsubst(A[0]*omega[0]-A[1]*omega[1], A[2]*omega[2], E3);
```
(%o79) $n_0^2 (\omega_1^2 A_1^2 - 2 \omega_0 A_0 \omega_1 A_1 + \omega_0^2 A_0^2) = -2 n_0 \omega_0 A_0 n_1 \omega_1 A_1 \cos(theta_3) + n_1^2 \omega_1^2 A_1^2 + n_0^2 \omega_0^2 A_0^2$

```
(%i80) defA(0);
```
(%o80) *done*

```
(%i81) E13b: ev(E13a);
```
(%o81) $$n_0^2 \left(\frac{(1-\omega_1 h_{bar} f_T)^2}{h_{bar}^2 f_T^2} + \frac{(1-\omega_0 h_{bar} f_T)^2}{h_{bar}^2 f_T^2} - \frac{2(1-\omega_0 h_{bar} f_T)(1-\omega_1 h_{bar} f_T)}{h_{bar}^2 f_T^2} \right) = \frac{n_1^2 (1-\omega_1 h_{bar} f_T)^2}{h_{bar}^2 f_T^2} + \frac{n_0^2 (1-\omega_0 h_{bar} f_T)^2}{h_{bar}^2 f_T^2} - \frac{2 n_0 n_1 \cos(theta_3)(1-\omega_0 h_{bar} f_T)(1-\omega_1 h_{bar} f_T)}{h_{bar}^2 f_T^2}$$

```
(%i82) E13c: solve(E13b, omega[1])$
```

```
(%i83) o1a: factor(rhs(first(E13c)));
```
(%o83) $$\frac{2 n_0 \omega_0 n_1 \cos(theta_3) h_{bar} f_T - 2 n_0^2 \omega_0 h_{bar} f_T - 2 n_0 n_1 \cos(theta_3) + n_1^2 + n_0^2}{(n_1 - n_0)(n_1 + n_0) h_{bar} f_T}$$

```
(%i84) o1b: factor(rhs(second(E13c)));
```
(%o84) $$\frac{1}{h_{bar} f_T}$$

2.2 Reflection (correct)

```
(%i85) kill(A,theta);
```
(%o85) *done*

```
(%i86) E1; E3;
```
(%o86) $\omega_0 A_0 = \omega_2 A_2 + \omega_1 A_1$

(%o87) $n_0^2 \omega_2^2 A_2^2 = -2 n_0 \omega_0 A_0 n_1 \omega_1 A_1 \cos(theta_3) + n_1^2 \omega_1^2 A_1^2 + n_0^2 \omega_0^2 A_0^2$

```
(%i88) E13d: ratsubst(A[0]*omega[0]-A[2]*omega[2], A[1]*omega[1], E3);
```

(%o88) $n_0^2\,\omega_2^2\,A_2^2 = n_1\,(2\,n_0\,\omega_0\,A_0\,\omega_2\,A_2 - 2\,n_0\,\omega_0^2\,A_0^2)\cos(theta_3) + n_1^2\,(\omega_2^2\,A_2^2 - 2\,\omega_0\,A_0\,\omega_2\,A_2 + \omega_0^2\,A_0^2) + n_0^2\,\omega_0^2\,A_0^2$

```
(%i89) defA(0);
```

(%o89) *done*

```
(%i90) E13e: ev(E13d);
```

(%o90) $$\frac{n_0^2\,(1-\omega_2\,h_{bar}\,f_T)^2}{h_{bar}^2\,f_T^2} = n_1^2\left(\frac{(1-\omega_2\,h_{bar}\,f_T)^2}{h_{bar}^2\,f_T^2} + \frac{(1-\omega_0\,h_{bar}\,f_T)^2}{h_{bar}^2\,f_T^2} - \frac{2\,(1-\omega_0\,h_{bar}\,f_T)(1-\omega_2\,h_{bar}\,f_T)}{h_{bar}^2\,f_T^2}\right) + n_1\cos(theta_3)\left(\frac{2\,n_0\,(1-\omega_0\,h_{bar}\,f_T)(1-\omega_2\,h_{bar}\,f_T)}{h_{bar}^2\,f_T^2} - \frac{2\,n_0\,(1-\omega_0\,h_{bar}\,f_T)^2}{h_{bar}^2\,f_T^2}\right) + \frac{n_0^2\,(1-\omega_0\,h_{bar}\,f_T)^2}{h_{bar}^2\,f_T^2}$$

```
(%i91) E13f: solve(E13e, omega[2])$
```

```
(%i92) o2a: factor(rhs(first(E13f)));
```

(%o92) $$-\frac{2\,n_0\,\omega_0\,n_1\cos(theta_3)\,h_{bar}\,f_T - \omega_0\,n_1^2\,h_{bar}\,f_T - n_0^2\,\omega_0\,h_{bar}\,f_T - 2\,n_0\,n_1\cos(theta_3) + 2\,n_0^2}{(n_1-n_0)(n_1+n_0)\,h_{bar}\,f_T}$$

```
(%i93) o2b: factor(rhs(second(E13f)));
```

(%o93) ω_0

2.3 Graphics

```
(%i94) str: [n[0]=N0, n[1]=N1, h[bar]=6.62618e-34, f[T]=1/(1.38066e-23*293),
       theta[0]=theta, omega[0]=1.e12];
```

(%o94) $[n_0 = 1.5,\ n_1 = 1.0,\ h_{bar} = 6.6261800000000001\ 10^{-34},\ f_T = 2.4719838941350153\ 10^{20},\ theta_0 = theta,\ \omega_0 = 1.0\ 10^{12}]$

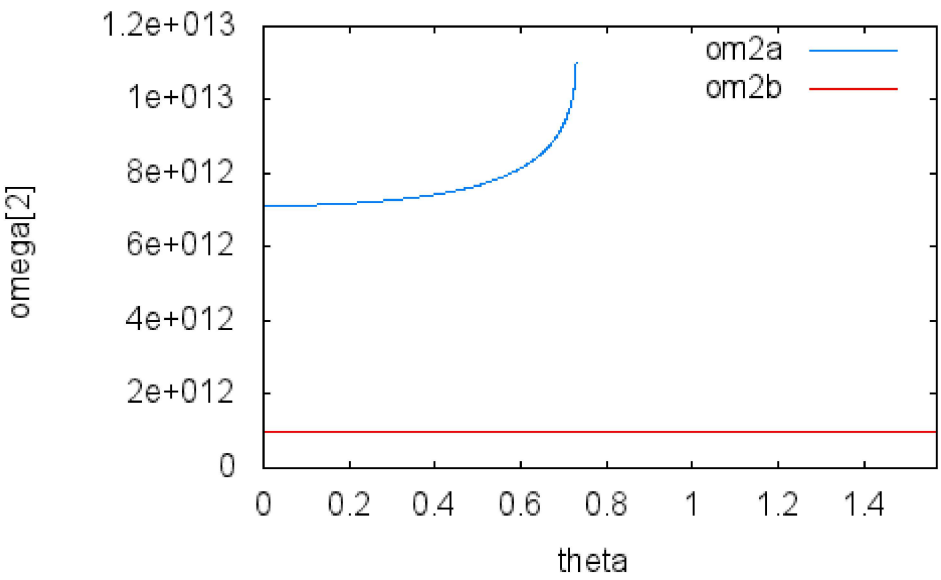
```
(%i95) theta[3]: ev(th3(theta[0], n[0], n[1]), str);
```

(%o95) $\operatorname{asin}(1.5\sin(theta)) - theta$

```
(%i96) po1a: ev(o1a, str)$
       po1b: ev(o1b, str)$
       po2a: ev(o2a, str)$
       po2b: ev(o2b, str)$
```

```
(%i100) wxplot2d([po2a,po2b], [theta,0,%pi/2],
        [ylabel, "omega[2]"], [legend, "om2a", "om2b"])$
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
```

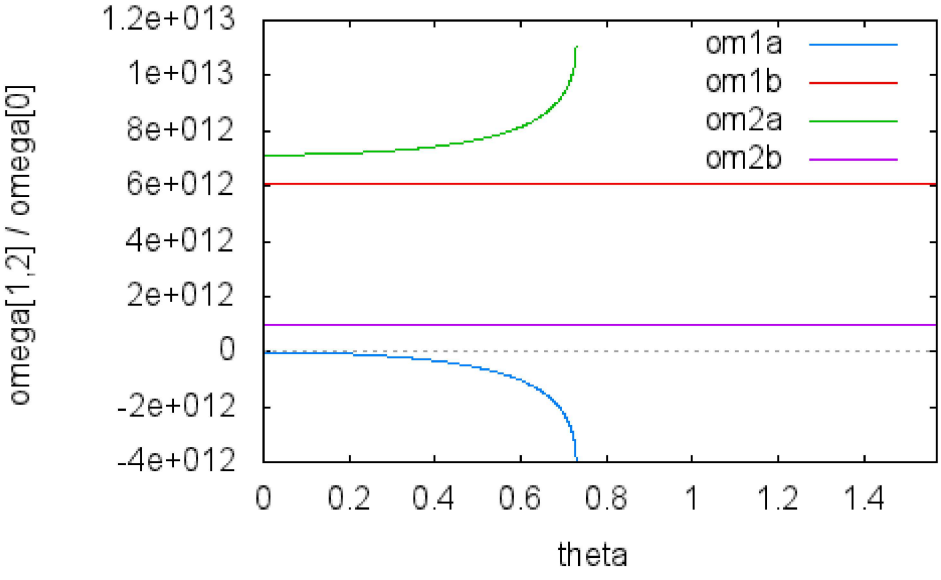
(%t100)




```
(%i101) plot2d([po2a,po2b], [theta,0,%pi/2],
        [ylabel, "omega[2]"], [legend, "om2a", "om2b"],
        [gnuplot_term, "png  linewidth 2.5 font 'Arial' 16 size 800,600"],
        [gnuplot_out_file, "D:/Doc/Artikel-Eck/ECE-Theorie/paper280/Fig9.png"
        )$
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
```

```
(%i102) wxplot2d([po1a,po1b,po2a,po2b], [theta,0,%pi/2],
        [ylabel, "omega[1,2] / omega[0]"], [legend, "om1a", "om1b", "om2a", "
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
plot2d: expression evaluates to non-numeric value somewhere in plotting rang
```

(%t102)



3 Frequency shift of reflection at angle of total reflection

3.1 Reflection (correct)

```
(%i103) E13e;
```

(%o103) $$\frac{n_0^2 (1-\omega_2 h_{bar} f_T)^2}{h_{bar}^2 f_T^2} = n_1^2 \left(\frac{(1-\omega_2 h_{bar} f_T)^2}{h_{bar}^2 f_T^2} + \frac{(1-\omega_0 h_{bar} f_T)^2}{h_{bar}^2 f_T^2} - \frac{2(1-\omega_0 h_{bar} f_T)(1-\omega_2 h_{bar} f_T)}{h_{bar}^2 f_T^2}\right) + n_1 \cos(theta_3) \left(\frac{2 n_0 (1-\omega_0 h_{bar} f_T)(1-\omega_2 h_{bar} f_T)}{h_{bar}^2 f_T^2} - \frac{2 n_0 (1-\omega_0 h_{bar} f_T)^2}{h_{bar}^2 f_T^2}\right) + \frac{n_0^2 (1-\omega_0 h_{bar} f_T)^2}{h_{bar}^2 f_T^2}$$

```
(%i104) kill(theta);
```

(%o104) *done*

```
(%i105) E13h: ratsubst(n[1]/n[0], cos(theta[3]), E13e);
```

(%o105) $$\frac{n_0^2 \omega_2^2 h_{bar}^2 f_T^2 - 2 n_0^2 \omega_2 h_{bar} f_T + n_0^2}{h_{bar}^2 f_T^2} = \frac{(n_1^2 \omega_2^2 - \omega_0^2 n_1^2 + n_0^2 \omega_0^2) h_{bar}^2 f_T^2 + (-2 n_1^2 \omega_2 + 2 \omega_0 n_1^2 - 2 n_0^2 \omega_0) h_{bar} f_T + n_0^2}{h_{bar}^2 f_T^2}$$

```
(%i106) E13i: solve(E13h, omega[2]);
```

(%o106) $$[\omega_2 = -\frac{\omega_0 h_{bar} f_T - 2}{h_{bar} f_T}, \omega_2 = \omega_0]$$

```
(%i107) o2a: (rhs(first(E13i)));
```

(%o107) $$-\frac{\omega_0 h_{bar} f_T - 2}{h_{bar} f_T}$$

```
(%i108) o2b: (rhs(second(E13i)));
```

(%o108) ω_0

3.2 Graphics

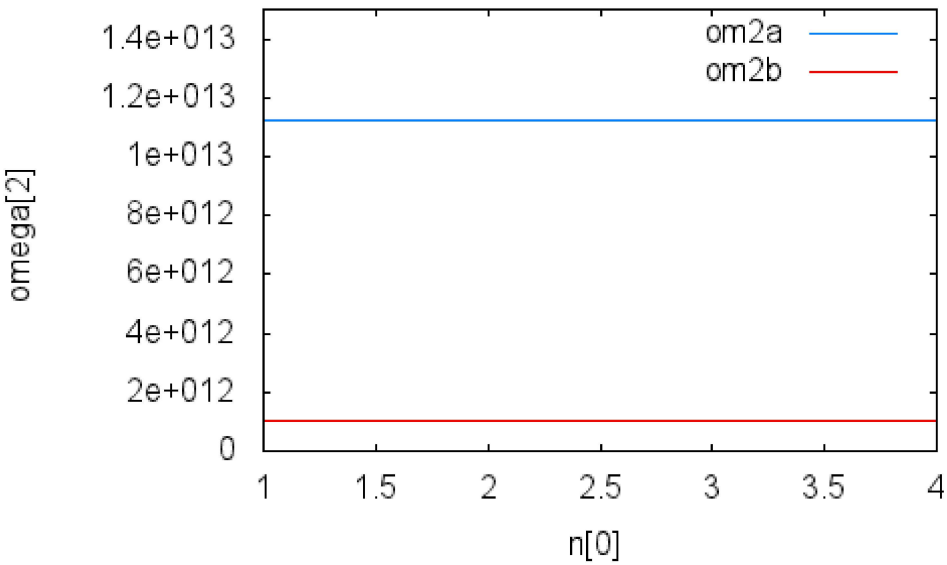
```
(%i109) str: [n[1]=N1, h[bar]=6.62618e-34, f[T]=1/(1.38066e-23*293),
        theta[0]=theta, omega[0]=1.e12];
```

(%o109) $[n_1 = 1.0, h_{bar} = 6.6261800000000001\ 10^{-34}, f_T = 2.4719838941350153\ 10^{20}, theta_0 = theta, \omega_0 = 1.0\ 10^{12}]$

```
(%i110) po2a: ev(o2a, str)$
        po2b: ev(o2b, str)$
```

```
(%i112) wxplot2d([po2a,po2b], [n[0],1,4], [y,0,1.5e13],
        [ylabel, "omega[2]"], [legend, "om2a", "om2b"])$
```

(%t112)




```
(%i113) plot2d([po2a,po2b], [n[0],1,4], [y,0,3e13], [y,0,1.5e13],
        [ylabel, "omega[2]"], [legend, "om2a", "om2b"],
        [gnuplot_term, "png  linewidth 2.5 font 'Arial' 16 size 800,600"],
        [gnuplot_out_file, "D:/Doc/Artikel-Eck/ECE-Theorie/paper280/Fig10.png
        )$
```