310(3) : Planck Distributions for Bosons and Fermions with Mass.

1) Monochromatic Distribution for Boson with Mass

This is:

$$\frac{E}{V} = \langle \hbar\omega \rangle \frac{N}{V} \quad -(1)$$

where $\langle \hbar\omega \rangle$ is the mean energy of the oscillator, and N/V is the number of oscillators in a volume V of bosons. Here $\hbar$ is the reduced Planck constant and ω the angular frequency. The mean energy of the oscillator is:

$$\langle \hbar\omega \rangle = \frac{\hbar\omega}{e^{y} - 1} \quad -(2)$$

where

$$\hbar\omega = \gamma m c^2 \quad -(3)$$

Here γ is the Lorentz factor:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(4)$$

where v is the boson velocity. In eq. (2):

$$y = \frac{\hbar\omega}{kT} \quad -(5)$$

where k is Boltzmann's constant and T the temperature of the boson gas.

2) As in the previous note:

$$\frac{N}{V} = \frac{1}{3c^3\pi^2}\left(\omega^2 - \omega_0^2\right)^{3/2} \quad -(6)$$

where

$$\omega_0 = \frac{mc^2}{\hbar} \quad -(7)$$

is the rest frequency of a boson of mass m, for example a photon.

So the Planck distribution is:

$$\boxed{\frac{E}{V} = \frac{\hbar\omega}{3c^3\pi^2}\frac{\left(\omega^2 - \omega_0^2\right)^{3/2}}{e^y - 1}} \quad -(8)$$

in joules per cubic metre. The flux density is:

$$\Phi = c\frac{E}{V} \quad -(9)$$

in watts per square metre.

2) Polychromatic Distribution for Boson with Mass

In this case:

$$\frac{dE}{V} = \langle \hbar\omega \rangle \frac{dN}{V} \quad -(10)$$

3) where:

$$\frac{dN}{V} = \frac{1}{3c^3\pi^2}\left(\left((\omega + d\omega)^2 - \omega_0^2\right)^{3/2} - \left(\omega^2 - \omega_0^2\right)^{3/2}\right) \quad -(11)$$

So:

$$\frac{dE}{V} = \frac{\hbar\omega}{3c^3\pi^2(e^y - 1)}\left(\left((\omega+d\omega)^2 - \omega_0^2\right)^{3/2} - \left(\omega^2 - \omega_0^2\right)^{3/2}\right) \quad -(12)$$

and
$$\frac{E}{V} = \frac{1}{V}\int_{\omega_1}^{\omega_2} dE \quad -(13)$$

As in previous work ~~the~~ higher order infinitesimals must be taken into account.
The flux density is again:

$$\Phi = c\frac{E}{V} = \frac{c}{V}\int_{\omega_1}^{\omega_2} dE \quad -(14)$$

For black body radiation:

$$\Phi = c\frac{E}{V} = \frac{c}{V}\int_0^{\infty} dE \quad -(15)$$

3) Monochromatic Planck Distribution for a Fermion with Mass (for example an electron).

In this case:

$$\frac{E}{V} = \langle \hbar\omega \rangle \frac{N}{V} \quad -(16)$$

and $\langle \hbar\omega \rangle$ is calculated with the Pauli exclusion principle and Fermi Dirac statistics:

$$\langle \hbar\omega \rangle = \frac{\hbar\omega}{e^{y} + 1} \quad -(17)$$

where

$$y = \frac{\hbar\omega}{kT} \quad -(18)$$

So:

$$\boxed{\frac{E}{V} = \frac{\hbar\omega}{3c^{3}\pi^{2}} \frac{(\omega^{2} - \omega_{0}^{2})^{3/2}}{e^{y} + 1}} \quad -(19)$$

and

$$\Phi = c\frac{E}{V} \quad -(20)$$

is the flux density of electrons in an electron gas or an electron beam.

) In eq. (19):

$$y = \frac{\hbar\omega}{kT} \quad -(21)$$

and

$$\omega_0 = \frac{mc^2}{\hbar} \quad -(22)$$

is the rest frequency of the fermion from the de Broglie equation, where m is the mass of the fermion. By the de Broglie Einstein equations:

$$E = \hbar\omega = \gamma mc^2 \quad -(23)$$

and

$$\underline{p} = \hbar\underline{\kappa} = \gamma m\underline{v} \quad -(24)$$

is the relativistic momentum.

In eqs (6) and (19):

$$\hbar\omega = \gamma mc^2 \quad -(25)$$

and

$$\omega^2 - \omega_0^2 = \left(\gamma^2 - 1\right)\left(\frac{mc^2}{\hbar}\right)^2 \quad -(26)$$

So:

$$\hbar\omega\left(\omega^2 - \omega_0^2\right)^{3/2}$$

$$= \gamma mc^2\left(\gamma^2 - 1\right)^{3/2}\left(\frac{mc^2}{\hbar}\right)^3 \quad -(27)$$

6)

$$= \frac{(mc^2)^4}{\hbar^3} \gamma (\gamma^2 - 1)^{3/2}$$

So eq. (8) is:

$$\boxed{\frac{E}{V} = \frac{(mc^2)^4}{3c^3\pi^2\hbar^3} \frac{\gamma(\gamma^2 - 1)^{3/2}}{e^y - 1}} \quad - (28)$$

where

$$y = \frac{\hbar\omega}{kT} = \frac{\gamma mc^2}{kT} \quad - (29)$$

Therefore the Planck distribution can be worked out in terms of photon velocity. This can be compared with the experimental Planck distribution for monochromatic radiation to find the photon velocity v and the photon mass m.

Similarly, for a fermion gas, eq. (19) is:

$$\frac{E}{V} = \frac{(mc^2)^4 \gamma (\gamma^2 - 1)^{3/2}}{3c^3\pi^2\hbar^3 (e^y + 1)} \quad - (30)$$
