310(4): Evans Morris Effects for Photon with Mass and an Electron Beam Interacting with Matter.

From previous notes the flux density of a beam of photons with mass made up of one frequency ω is:

$$\Phi = \frac{cE}{V} = \frac{\hbar\omega}{3c^2\pi^2} \frac{(\omega^2 - \omega_r^2)^{3/2}}{e^y - 1} \quad -(1)$$

in watts per square metre. Here ω_r is the rest frequency:

$$\omega_r = \frac{mc^2}{\hbar} \quad -(2)$$

where m is the photon mass. Here:

$$y = \frac{\hbar\omega}{kT} = \frac{\gamma mc^2}{kT} \quad -(3)$$

as in the previous note.

The Beer Lambert law is:

$$\frac{\Phi}{\Phi_0} = \exp(-\alpha Z) \quad -(4)$$

in the usual notation, so the Evans / Morris type frequency shift is given by:

$$\frac{\Phi}{\Phi_0} = \exp(-\alpha Z) = \frac{\omega}{\omega_0} \left(\frac{\omega^2 - \omega_r^2}{\omega_0^2 - \omega_r^2}\right)^{3/2} \left(\frac{e^{y_0} - 1}{e^y - 1}\right) \quad -(5)$$

In the high temperature or low frequency approximation:

) $\frac{\hbar\omega_0}{kT} \ll 1 \; ; \; \frac{\hbar\omega}{kT} \ll 1$ — (6)

eq. (5) simplifies to:

$$\left(\frac{\omega^2-\omega_r^2}{\omega_0^2-\omega_r^2}\right)^{3/2} = \exp\left(-\alpha Z\right) \quad -(7)$$

i.e.

$$\boxed{\omega^2-\omega_r^2 = \left(\omega_0^2-\omega_r^2\right)\exp\left(-\frac{2\alpha Z}{3}\right)} \quad -(8)$$

For a polychromatic beam the Rayleigh Jeans density of states is needed, defined by:

$$\frac{dN}{V} = \frac{1}{3c^2\pi^2}\left(\left((\omega+d\omega)^2-\omega_r^2\right)^{3/2} - \left(\omega^2-\omega_r^2\right)^{3/2}\right) \quad -(9)$$

Then:

$$\frac{dE}{V} = \langle\hbar\omega\rangle\frac{dN}{V} \quad -(10)$$

Rayleigh assumed that only first order infinitesimals are needed for eq. (9) and that ω_r is zero, i.e. that the photon mass is zero. It is now known that both these assumptions are incorrect. In the original Rayleigh theory with two polarizations:

$$\frac{dE}{V} = \langle \hbar\omega \rangle \frac{dN}{V} \quad -(11)$$

So the density of states is:

$$\frac{1}{V}\frac{dN}{d\omega} = \frac{\omega^2}{c^3\pi^2} \quad -(12)$$

Therefore

$$\frac{1}{V}\frac{dE}{d\omega} = \frac{\omega^2}{c^3\pi^2}\langle \hbar\omega \rangle \quad -(13)$$

where

$$\langle \hbar\omega \rangle = \frac{\hbar\omega}{e^{y}-1} \quad -(14)$$

It follows that:

$$\frac{E}{V} = \int \frac{\hbar\omega^3}{e^{y}-1}\, d\omega \quad -(15)$$

is the total electromagnetic energy E in a volume of radiation V. For black body radiation

$$\frac{E}{V} = \int_0^\infty \frac{\hbar\omega^3}{e^{y}-1}\, d\omega \quad -(16)$$

The Rayleigh theory was based on the d'Alembert wave equation:

$$\Box \phi = 0 \quad -(17)$$

but as shown in the last note it should be

4) Based on the Proca equation:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \quad -(18)$$

which can be written as:

$$\left(\Box + \kappa_0^2\right)\phi = 0 \quad -(19)$$

where

$$\kappa_0 = \frac{mc}{\hbar} \quad -(20)$$

is the rest wavenumber.

For an electron beam interacting with matter the equivalent of eq. (1) is:

$$\Phi = \frac{cE}{V} = \frac{\hbar\omega}{3c^2\pi^2}\left(\frac{\left(\omega^2 - \omega_r^2\right)^{3/2}}{e^y + 1}\right) \quad -(21)$$

and this is the correctly relativistic expression for an electron gas. The underlying wave equation is the Dirac equation in wave format, eq. (19), rather than the Proca equation.

So the Evans / Morris effects for an electron beam interacting with matter are:

$$\frac{\Phi}{I_0} = \exp(-\alpha Z) = \frac{\omega}{\omega_0}\left(\frac{\omega^2 - \omega_r^2}{\omega_0^2 - \omega_r^2}\right)^{3/2}\left(\frac{e^{y_0} + 1}{e^y + 1}\right)$$

$-(22)$

5) and in the approximation (6) this again reduces to eq. (8).

So an electron beam interacting with material matter enters the sample with the frequency ω_0 and exits with the redshifted frequency ω.

Usually an electron gas is described by the well known Fermi theory which is a particle in the box theory based on the non-relativistic Schrodinger equation:

$$-\frac{\hbar^2\nabla^2}{2m}\phi = E\phi \quad -(21)$$

with boundary conditions:

$$\phi(0) = \phi(L) = 0 \quad -(22)$$

There are close similarities between the Rayleigh theory and Fermi theory. They become identical if the Rayleigh theory is extended to photons with mass and if the Fermi theory is made relativistic.

Once this is realized they are both based on the ECE wave equation, eq. (19), which is a limit of. The occupation number in the Rayleigh theory is defined by Bose Einstein statistics and is:

6)

$$f(\text{Rayleigh}) = \left(\exp\left(\frac{\hbar\omega}{kT}\right) - 1\right)^{-1} \quad -(23)$$

and the occupation number in the Fermi theory is based on Fermi Dirac statistics and is:

$$f(\text{Fermi}) = \left(\exp\left(\frac{\hbar\omega}{kT}\right) + 1\right)^{-1} \quad -(24)$$

In the Rayleigh theory without photon mass: $-(25)$

$$\phi = A\sin\left(\frac{\pi}{L}n_1 X\right)\sin\left(\frac{\pi}{L}n_2 Y\right)\sin\left(\frac{\pi}{L}n_3 Z\right),$$

where n_1, n_2 and n_3 are integers. In the non-relativistic Fermi theory the wave function is the same, but n_1, n_2 and n_3 become principal quantum numbers.

In the Rayleigh theory:

$$\omega^2 = \frac{c^2\pi^2}{L^2}\left(n_1^2 + n_2^2 + n_3^2\right) \quad -(26)$$

In the Fermi theory:

$$E = \frac{\hbar^2\pi^2}{2mL^2}\left(n_1^2 + n_2^2 + n_3^2\right) \quad -(27)$$

$$= \frac{\hbar^2}{2m}\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2\right)$$

so:

$$\kappa_i = \frac{n\pi}{L_i}, \quad i = 1, 2, 3 \quad -(28)$$

7) Eqs. (26) and (27) are the same if for i = 1, 2, 3:

$$E = \hbar\omega = \hbar\kappa c \quad -(29)$$

Eq. (29) is true for a massless photon and for the non-relativistic electron with:

$$E = \frac{p^2}{2m} = \frac{\hbar^2\kappa^2}{2m} \quad -(30)$$

Note carefully that the correct Rayleigh and Fermi theories must both be based on eq. (19).

In both the Rayleigh and Fermi theories:

$$V = L^3 \quad -(31)$$

In the non-relativistic Fermi theory:

$$\frac{N}{V} = \frac{\kappa_F^3}{3\pi^2} \quad -(32)$$

where κ_F is the Fermi wavenumber. In the original Rayleigh theory:

$$\frac{N}{V} = \frac{1}{3\pi^2}\left(\frac{\omega}{c}\right)^3 \quad -(33)$$

for two polarizations. The reason for the close similarity between the theories is that they are approximations to eq. (19).

In the Fermi theory the total energy of

) the electron gas is :

$$U = \int \frac{dN}{dE} \cdot \left(\frac{E}{e^{y} + 1} \right) dE \quad -(34)$$

where

$$E_n = \frac{\hbar^2}{2m} \left(\frac{N\pi}{L} \right)^2 - (35)$$

In general:

$$N = \left(\frac{V}{3\pi^2} \right) k^3 = \left(\frac{V}{3\pi^2} \right) \left(\frac{2mE}{\hbar^2} \right)^{3/2} \quad -(36)$$

so

$$\frac{dN}{dE} = \left(\frac{V}{3\pi^2} \right) \left(\frac{3}{2} \left(\frac{2m}{\hbar^2} \right)^{3/2} E^{1/2} \right) \quad -(37)$$

$$= \frac{3N}{2E}$$

The number of electrons is :

$$N = \int_0^\infty \frac{dN}{dE} f(E) \, dE \quad -(38)$$

Here:

$$f(E) = \left(\exp\left(\frac{E-\mu}{kT} \right) + 1 \right)^{-1} - (39)$$

where μ is the chemical potential. These expressions are approximated by :

$$kT \ll E_F , \quad \mu \sim E_F \quad -(40)$$

9) In the Rayleigh theory:

$$\frac{U}{V} = \frac{1}{V}\int \frac{dN}{d\omega}\left(f(\omega)E\right)d\omega \quad -(41)$$

and in the Fermi theory:

$$U = \int \frac{dN}{dE}\left(f(E)E\right)dE \quad -(42)$$

where:

$$f(\omega) = \left(\exp\left(\frac{\hbar\omega}{kT}\right)-1\right)^{-1} \quad -(43)$$

and

$$f(E) = \left(\exp\left(\frac{E-E_f}{kT}\right)+1\right)^{-1} \quad -(44)$$

The next step is to evaluate the Beer Lambert law for the absorption of an electron beam. Note carefully that the current flux in the Fermi theory is:

$$\Phi(\text{Fermi}) = \frac{cE}{V} = \frac{\hbar\omega}{3c^2\pi^2}\left(\frac{(\omega^2-\omega_r^2)^{3/2}}{\exp\left(\frac{E-\mu}{kT}\right)+1}\right) \quad -(45)$$

where

$$\omega_r = \frac{mc^2}{\hbar} \quad -(46)$$

is the rest frequency of the electron.