310(6): Correction to the Stefan Boltzmann Law due to Photon Mass

As in previous notes the Stefan Boltzmann law with photon mass is:

$$\frac{E}{V} = \frac{\hbar}{c^3 \pi^2} \int_0^\infty \frac{\omega^2 (\omega^2 - \omega_0^2)^{1/2}}{e^y - 1} d\omega \quad -(1)$$

Assume that: $\omega_0 << \omega \quad -(2)$

then

$$(\omega^2 - \omega_0^2)^{1/2} = \omega \left(1 - \left(\frac{\omega_0}{\omega} \right)^2 \right)^{1/2} \quad -(3)$$

$$\sim \omega \left(1 - \frac{1}{2} \left(\frac{\omega_0}{\omega} \right)^2 \right)$$

So:

$$\frac{E}{V} = \frac{\hbar}{c^3 \pi^2} \left[\int_0^\infty \frac{\omega^3}{e^y - 1} d\omega - \frac{\omega_0^2}{2} \int_0^\infty \frac{\omega}{e^y - 1} d\omega \right] \quad -(4)$$

Use: $$\int_0^\infty \frac{x^{2n-1}}{e^x - 1} dx = (2\pi)^{2n} \frac{B_n}{4n} \quad -(5)$$

where B_n are the Bernouilli numbers:

$$B_1 = \frac{1}{6}, \; B_2 = \frac{1}{30}, \ldots \quad -(6)$$

Use: $$y = \frac{\hbar \omega}{kT}, \quad dy = \frac{\hbar}{kT} d\omega \quad -(7)$$

and: $\omega^3 = \left(\frac{kT}{\hbar}\right)^3 y^3$; $\omega = \left(\frac{kT}{\hbar}\right) y$ — (8)

It follows that:

$$\int_0^\infty \frac{\omega^3}{e^y - 1} d\omega = \frac{k^4 T^4}{\hbar^4} \int_0^\infty \frac{y^3}{e^y - 1} dy \quad -(9)$$

and

$$\int_0^\infty \frac{\omega}{e^y - 1} d\omega = \frac{k^2 T^2}{\hbar^2} \int_0^\infty \frac{y}{e^y - 1} dy \quad -(10)$$

Fm Eq. (5):

$$\int_0^\infty \frac{y^3}{e^y - 1} dy = 2\pi^2 B_2 = \frac{\pi^2}{15} \quad -(11)$$

and

$$\int_0^\infty \frac{y}{e^y - 1} dy = \pi^2 B_1 = \frac{\pi^2}{6} \quad -(12)$$

So:

$$\underline{I} = \frac{\hbar}{c^3 \pi^2} \int_0^\infty \frac{\omega^3}{e^y - 1} d\omega$$

$$= \frac{\hbar}{c^3 \pi^2} \left(\frac{kT}{\hbar}\right)^4 \cdot \frac{\pi^2}{15} \quad -(13)$$

$$= \left(\frac{\pi^2 k^4}{15 c^3 \hbar^3}\right) T^4$$

and

3)

$$I_1 = \frac{\hbar\omega_0^2}{2c^3\pi^2}\int_0^\infty \frac{\omega}{e^y-1}\,d\omega$$

$$= \frac{\hbar\omega_0^2}{2c^3\pi^2}\left(\frac{kT}{\hbar}\right)^2\int_0^\infty \frac{y}{e^y-1}\,dy$$

$$= \frac{\hbar\omega_0^2}{12c^3}\left(\frac{kT}{\hbar}\right)^2$$

$$= \left(\frac{k^2\omega_0^2}{12c^3\hbar}\right)T^2 \quad -(14)$$

So the Stefan Boltzmann law is corrected by photon mass to:

$$\boxed{\frac{E}{V} = \left(\frac{\pi^2k^4}{15c^3\hbar^3}\right)T^4 - \left(\frac{k^2\omega_0^2}{12c^3\hbar}\right)T^2} \quad -(15)$$

Finally use:

$$\omega_0 = \frac{mc^2}{\hbar} \quad -(16)$$

to find:

$$\boxed{\frac{E}{V} = \left(\frac{\pi^2k^4}{15c^3\hbar^3}\right)T^4 - \left(\frac{k^2c}{12\hbar^3}\right)m^2T^2} \quad -(17)$$

Q.E.D.

4) The effect of photon mass m in the approximation (2) is to sustract a term proportional to $m^2 T^2$. Therefore the Stefan Boltzmann law is no longer proportional to T^4, but a sum of terms as in eq. (17).