313(5): Second Method of Reducing the Jacobi Identity to the Second Bianchi Identity.

Consider the Jacobi identity to act on a vector V^κ:

$$\left([D_\rho, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] \right) V^\kappa = 0 \quad -(1)$$

The first term is:

$$[D_\rho, [D_\mu, D_\nu]] V^\kappa = D_\rho \left([D_\mu, D_\nu] V^\kappa \right) - [D_\mu, D_\nu] \left(D_\rho V^\kappa \right)$$

$$= D_\rho [D_\mu, D_\nu] V^\kappa + [D_\mu, D_\nu] D_\rho V^\kappa - [D_\mu, D_\nu] D_\rho V^\kappa \quad -(2)$$

$$= D_\rho [D_\mu, D_\nu] V^\kappa$$

In general:

$$[D_\mu, D_\nu] V^\kappa = R^\kappa{}_{\lambda\mu\nu} V^\lambda - T^\lambda{}_{\mu\nu} D_\lambda V^\kappa \quad -(3)$$

where the torsion tensor is:

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu} \quad -(4)$$

It is obvious that if:

$$\mu = \nu \quad -(5)$$

a) The commutator vanishes and the torsion vanishes, and therefore the curvature vanishes.

This is the basic error of the Einstein era.

However if we accept the Einsteinian argument for the sake of illustration then:

$$[D_\mu, D_\nu] V^\kappa = R^\kappa{}_{\lambda\mu\nu} V^\lambda \quad -(6)$$

From eqs (2) and (6):

$$[D_\rho, [D_\mu, D_\nu]] V^\kappa = D_\rho R^\kappa{}_{\lambda\mu\nu} V^\lambda \quad -(7)$$

So Eq. (1) becomes:

$$\left(D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\mu R^\kappa{}_{\lambda\nu\rho} + D_\nu R^\kappa{}_{\lambda\rho\mu}\right) V^\lambda = 0 \quad -(8)$$

The usual second Bianchi identity is the solution:

$$D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\mu R^\kappa{}_{\lambda\nu\rho} + D_\nu R^\kappa{}_{\lambda\rho\mu} = 0 \quad -(9)$$

However the correct result from eqs. (2) and (3) is the Jacobi Cartan Evans (JCE) identity:

$$\left(D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\mu R^\kappa{}_{\lambda\nu\rho} + D_\nu R^\kappa{}_{\lambda\rho\mu}\right) V^\lambda := \left(D_\rho T^\lambda{}_{\mu\nu} + D_\nu T^\lambda{}_{\rho\mu} + D_\mu T^\lambda{}_{\nu\rho}\right) D_\lambda V^\kappa \quad -(10)$$

This is the same result as in Note 313(3).