313(6): Complete Details of the Proof of the Second Bianchi Identity from the Jacobi Identity.

The Jacobi identity is:

$$\left([D_\rho, [D_\mu, D_\nu]] + [D_\nu, [D_\rho, D_\mu]] + [D_\mu, [D_\nu, D_\rho]] \right) V^\kappa := 0 \quad -(1)$$

and is an exact identity of group theory.

Consider the first term: $\quad -(2)$

$$[D_\rho, [D_\mu, D_\nu]] V^\kappa = D_\rho \left([D_\mu, D_\nu] V^\kappa \right) - [D_\mu, D_\nu] D_\rho V^\kappa$$

and use:

$$[D_\mu, D_\nu] V^\kappa = R^\kappa{}_{\lambda\mu\nu} V^\lambda - T^\lambda{}_{\mu\nu} D_\lambda V^\kappa \quad -(3)$$

The conventional second Bianchi identity omits torsion, so

$$[D_\mu, D_\nu] V^\kappa =? R^\kappa{}_{\lambda\mu\nu} V^\lambda \quad -(4)$$

This is erroneous, but for the sake of illustration eqs. (2) and (4) give:

$$[D_\rho, [D_\mu, D_\nu]] V^\kappa = D_\rho R^\kappa{}_{\lambda\mu\nu} V^\lambda - [D_\mu, D_\nu] D_\rho V^\kappa \quad -(5)$$

The second term on the right hand side is

2)

evaluated using

$$[D_\rho, D_\sigma] X^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = -T^\lambda_{\rho\sigma} D_\lambda X^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}$$
$$+ R^{\mu_1}_{\lambda\rho\sigma} X^{\lambda\mu_2 \dots \mu_k}_{\nu_1 \dots \nu_l} + R^{\mu_2}_{\lambda\rho\sigma} X^{\mu_1 \lambda \dots \mu_k}_{\nu_1 \dots \nu_l} + \dots$$
$$- R^\lambda_{\nu_1\rho\sigma} X^{\mu_1 \dots \mu_k}_{\lambda\nu_2 \dots \nu_l} - R^\lambda_{\nu_2\rho\sigma} X^{\mu_1 \dots \mu_k}_{\nu_1 \lambda \dots \nu_l} - \quad -(6)$$

Eq. (6) defines the effect of the commutator on any tensor (Carroll chapter three). It can be seen that the commutator always produces torsion, but this is neglected in the conventional derivation. This is neglected in the conventional derivation.

From eq. (6), neglecting torsion:

$$[D_\rho, D_\sigma] X^{\mu_1}_{\nu_1} = R^{\mu_1}_{\lambda\rho\sigma} X^\lambda_{\nu_1} - R^\lambda_{\nu_1\rho\sigma} X^{\mu_1}_\lambda \quad -(7)$$

so:

$$[D_\mu, D_\nu] D_\rho V^\kappa = R^\kappa_{\lambda\mu\nu} D_\rho V^\lambda - R^\lambda_{\rho\mu\nu} D_\lambda V^\kappa \quad -(8)$$

This is sometimes known as the Ricci identity.

Therefore it is found that a cyclic sum of terms such as in eqs. (5) and (8) gives:

3)

$$\left([D_\rho,[D_\mu,D_\nu]] + [D_\nu,[D_\rho,D_\mu]] + [D_\mu,[D_\nu,D_\rho]]\right)V^\kappa$$

$$= \left(D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} + D_\mu R^\kappa{}_{\lambda\nu\rho}\right)V^\lambda$$

$$+ \left(R^\kappa{}_{\lambda\mu\nu} + R^\kappa{}_{\nu\mu\lambda} + R^\kappa{}_{\lambda\nu\mu}\right)D_\rho V^\lambda$$

$$- \left(R^\lambda{}_{\rho\mu\nu} + R^\lambda{}_{\nu\rho\mu} + R^\lambda{}_{\mu\nu\rho}\right)D_\lambda V^\kappa \quad -(9)$$

$$:= 0$$

By the first Bianchi identity the second two terms are both zero, so:

$$\left([D_\rho,[D_\mu,D_\nu]] + [D_\nu,[D_\rho,D_\mu]] + [D_\mu,[D_\nu,D_\rho]]\right)V^\kappa$$

$$= \left(D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} + D_\mu R^\kappa{}_{\lambda\nu\rho}\right)V^\lambda \quad -(10)$$

$$:= 0$$

Q.E.D

Finally it is assumed in the usual literature that: $-(11)$

$$D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} + D_\mu R^\kappa{}_{\lambda\nu\rho} := 0$$

However there is a summation over repeated λ indices in the correct expression (10). Eq. (11) is the second Bianchi identity used by Einstein, but it is incorrect due to neglect of torsion.

4) Calculation with Torsion

This must proceed as follows:

$$[D_\mu, D_\nu] V^\kappa = R^\kappa{}_{\lambda\mu\nu} V^\lambda - T^\lambda{}_{\mu\nu} D_\lambda V^\kappa \quad -(12)$$

and:

$$[D_\mu, D_\nu] D_\rho V^\kappa = R^\kappa{}_{\lambda\mu\nu} D_\rho V^\lambda - R^\lambda{}_{\rho\mu\nu} D_\lambda V^\kappa - T^\lambda{}_{\mu\nu} D_\lambda D_\rho V^\kappa \quad -(13)$$

Eq. (13) is the correct expression for the Ricci identity.

Therefore:

$$\begin{aligned}
[D_\rho, [D_\mu, D_\nu]] V^\kappa &= D_\rho([D_\mu, D_\nu] V^\kappa) + [D_\mu, D_\nu](D_\rho V^\kappa) \\
&= D_\rho(R^\kappa{}_{\lambda\mu\nu} V^\lambda - T^\lambda{}_{\mu\nu} D_\lambda V^\kappa) \\
&\quad + R^\kappa{}_{\lambda\mu\nu} D_\rho V^\lambda - R^\lambda{}_{\rho\mu\nu} D_\lambda V^\kappa - T^\lambda{}_{\mu\nu} D_\lambda D_\rho V^\kappa
\end{aligned} \quad -(14)$$

in which:

5)
$$D_\rho \left(T^{\lambda}_{\mu\nu} D_\lambda V^\kappa \right) = D_\rho T^{\lambda}_{\mu\nu} \left(D_\lambda V^\kappa \right) + T^{\lambda}_{\mu\nu} \left(D_\rho D_\lambda V^\kappa \right) \quad -(15)$$

Therefore:

$$[D_\rho, [D_\mu, D_\nu]] V^\kappa = D_\rho R^\kappa{}_{\lambda\mu\nu} V^\lambda + R^\kappa{}_{\lambda\mu\nu} D_\rho V^\lambda - R^\lambda{}_{\rho\mu\nu} D_\lambda V^\kappa - \left(D_\rho T^{\lambda}_{\mu\nu} \right) \left(D_\lambda V^\kappa \right) - T^{\lambda}_{\mu\nu} [D_\rho, D_\lambda] V^\kappa \quad -(16)$$

therefore the Jacobi Cartan Evans identity

is:

$$\left(D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} + D_\mu R^\kappa{}_{\lambda\nu\rho} \right) V^\lambda := \left(D_\rho T^{\lambda}_{\mu\nu} + D_\nu T^{\lambda}_{\rho\mu} + D_\mu T^{\lambda}_{\nu\rho} \right) D_\lambda V^\kappa + \left(T^{\lambda}_{\mu\nu} [D_\rho, D_\lambda] + T^{\lambda}_{\rho\mu} [D_\nu, D_\lambda] + T^{\lambda}_{\nu\rho} [D_\mu, D_\lambda] \right) V^\kappa \quad -(17)$$

This is the same result as eq. (10) of the previous note with the addition of a new term on the right hand side.

6) This is made up of terms such as:

$$[D_\rho, D_\lambda]V^\kappa = R^\kappa{}_{\alpha\rho\lambda} V^\alpha - T^\alpha{}_{\rho\lambda} D_\alpha V^\kappa \quad -(18)$$

So:

$$\left(T^\lambda{}_{\mu\nu}[D_\rho, D_\lambda] + T^\lambda{}_{\rho\mu}[D_\nu, D_\lambda] + T^\lambda{}_{\nu\rho}[D_\mu, D_\lambda]\right)V^\kappa$$

$$= T^\lambda{}_{\mu\nu} R^\kappa{}_{\alpha\rho\lambda} V^\alpha - T^\lambda{}_{\mu\nu} T^\alpha{}_{\rho\lambda} D_\alpha V^\kappa \quad -(19)$$

$$+ \cdots$$

$$= T^\lambda{}_{\mu\nu}\left(R^\kappa{}_{\alpha\rho\lambda} V^\alpha - T^\alpha{}_{\rho\lambda} D_\alpha V^\kappa\right)$$

$$+ T^\lambda{}_{\rho\mu}\left(R^\kappa{}_{\alpha\nu\lambda} V^\alpha - T^\alpha{}_{\nu\lambda} D_\alpha V^\kappa\right)$$

$$+ T^\lambda{}_{\nu\rho}\left(R^\kappa{}_{\alpha\mu\lambda} V^\alpha - T^\alpha{}_{\mu\lambda} D_\alpha V^\kappa\right)$$

Therefore the JEC identity is:

$$\left(D_\rho R^\kappa{}_{\lambda\mu\nu} + D_\nu R^\kappa{}_{\lambda\rho\mu} + D_\mu R^\kappa{}_{\lambda\nu\rho}\right)V^\lambda$$

$$:= \left(D_\rho T^\lambda{}_{\mu\nu} + D_\nu T^\lambda{}_{\rho\mu} + D_\mu T^\lambda{}_{\nu\rho}\right) D_\lambda V^\kappa$$

$$+ \left(T^\lambda{}_{\mu\nu} R^\kappa{}_{\alpha\rho\lambda} + T^\lambda{}_{\rho\mu} R^\kappa{}_{\alpha\nu\lambda} + T^\lambda{}_{\nu\rho} R^\kappa{}_{\alpha\mu\lambda}\right) V^\alpha$$

$$- \left(T^\lambda{}_{\mu\nu} T^\alpha{}_{\rho\lambda} + T^\lambda{}_{\rho\mu} T^\alpha{}_{\nu\lambda} + T^\lambda{}_{\nu\rho} T^\alpha{}_{\mu\lambda}\right) D_\alpha V^\kappa$$

$$-(20)$$

7) However, as in UFT109 it can be shown that:

$$T^{\lambda}_{\mu\nu} T^{\alpha}_{\rho\lambda} + T^{\lambda}_{\rho\mu} T^{\alpha}_{\nu\lambda} + T^{\lambda}_{\nu\rho} T^{\alpha}_{\mu\lambda} := 0 \quad -(21)$$

and:

$$T^{\lambda}_{\mu\nu} R^{\kappa}{}_{\alpha\rho\lambda} + T^{\lambda}_{\rho\mu} R^{\kappa}{}_{\alpha\nu\lambda} + T^{\lambda}_{\nu\rho} R^{\kappa}{}_{\alpha\mu\lambda} := 0 \quad -(22)$$

so we obtain the same result as the previous note:

$$\left(D_\rho R^{\kappa}{}_{\lambda\mu\nu} + D_\nu R^{\kappa}{}_{\lambda\rho\mu} + D_\mu R^{\kappa}{}_{\lambda\nu\rho} \right) V^{\lambda} := \left(D_\rho T^{\lambda}_{\mu\nu} + D_\nu T^{\lambda}_{\rho\mu} + D_\mu T^{\lambda}_{\nu\rho} \right) D_\lambda V^{\kappa} \quad -(23)$$

It is by no means easy to derive the second Bianchi identity from the Jacobi identity, and in almost all the textbooks, torsion is incorrectly omitted.