313(7): Proof of the First Evans Identity

This is Note 313(6), Eq. (21):

$$T^{\lambda}_{\nu\rho}T^{\alpha}_{\mu\lambda} + T^{\lambda}_{\rho\mu}T^{\alpha}_{\nu\lambda} + T^{\lambda}_{\mu\nu}T^{\alpha}_{\rho\lambda} := 0 \quad -(1)$$

It was first discovered in UFT109 and in Note 313(6) it was proved to be part of the Jacobi Cartan Evans (JCE) identity.

Proof

Consider the covariant derivative acting on any tensor (Carroll Chapter Three):

$$D_\sigma T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} = \partial_\sigma T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} \quad -(2)$$

$$+ \Gamma^{\mu_1}_{\sigma\lambda}T^{\lambda\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda}T^{\mu_1\lambda\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} + \ldots$$

$$- \Gamma^{\lambda}_{\sigma\nu_1}T^{\mu_1\mu_2\ldots\mu_k}_{\lambda\nu_2\ldots\nu_l} - \Gamma^{\mu_2}_{\sigma\lambda}T^{\mu_1\lambda\ldots\mu_k}_{\nu_1\lambda\ldots\nu_l} - \ldots$$

It follows that:

$$D_\sigma T^{\kappa}_{\mu\nu} = \partial_\sigma T^{\kappa}_{\mu\nu} + \Gamma^{\kappa}_{\sigma\lambda}T^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\sigma\mu}T^{\kappa}_{\lambda\nu} - \Gamma^{\lambda}_{\sigma\nu}T^{\kappa}_{\mu\lambda}$$

$$D_\nu T^{\kappa}_{\sigma\mu} = \partial_\nu T^{\kappa}_{\sigma\mu} + \Gamma^{\kappa}_{\nu\lambda}T^{\lambda}_{\sigma\mu} - \Gamma^{\lambda}_{\nu\sigma}T^{\kappa}_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu}T^{\kappa}_{\sigma\lambda}$$

$$D_\mu T^{\kappa}_{\nu\sigma} = \partial_\mu T^{\kappa}_{\nu\sigma} + \Gamma^{\kappa}_{\mu\lambda}T^{\lambda}_{\nu\sigma} - \Gamma^{\lambda}_{\mu\nu}T^{\kappa}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma}T^{\kappa}_{\nu\lambda}$$

$$-(3)$$

It follows that:

$$D_\sigma T^\kappa_{\mu\nu} + D_\nu T^\kappa_{\sigma\mu} + D_\mu T^\kappa_{\nu\sigma}$$

$$= \left(\partial_\sigma T^\kappa_{\mu\nu} + \partial_\nu T^\kappa_{\sigma\mu} + \partial_\mu T^\kappa_{\nu\sigma} + \Gamma^\kappa_{\sigma\lambda} T^\lambda_{\mu\nu} + \Gamma^\kappa_{\nu\lambda} T^\lambda_{\sigma\mu} + \Gamma^\kappa_{\mu\lambda} T^\lambda_{\nu\sigma}\right)$$

$$- \left(\Gamma^\lambda_{\sigma\mu} T^\kappa_{\lambda\nu} + \Gamma^\lambda_{\sigma\nu} T^\kappa_{\mu\lambda} + \Gamma^\lambda_{\nu\sigma} T^\kappa_{\lambda\mu} + \Gamma^\lambda_{\nu\mu} T^\kappa_{\sigma\lambda} + \Gamma^\lambda_{\mu\nu} T^\kappa_{\lambda\sigma} + \Gamma^\lambda_{\mu\sigma} T^\kappa_{\nu\lambda}\right) \quad -(4)$$

$$: \; A - B$$

Using the definition of the torsion tensor:

$$A = \partial_\sigma\left(\Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu}\right) + \partial_\nu\left(\Gamma^\kappa_{\sigma\mu} - \Gamma^\kappa_{\mu\sigma}\right) + \partial_\mu\left(\Gamma^\kappa_{\nu\sigma} - \Gamma^\kappa_{\sigma\nu}\right)$$

$$+ \Gamma^\kappa_{\sigma\lambda}\left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right) + \Gamma^\kappa_{\nu\lambda}\left(\Gamma^\lambda_{\sigma\mu} - \Gamma^\lambda_{\mu\sigma}\right) + \Gamma^\kappa_{\mu\lambda}\left(\Gamma^\lambda_{\nu\sigma} - \Gamma^\lambda_{\sigma\nu}\right)$$

$$= \partial_\sigma \Gamma^\kappa_{\mu\nu} - \partial_\mu \Gamma^\kappa_{\sigma\nu} + \Gamma^\kappa_{\sigma\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}$$

$$+ \partial_\nu \Gamma^\kappa_{\sigma\mu} - \partial_\sigma \Gamma^\kappa_{\nu\mu} + \Gamma^\kappa_{\nu\lambda}\Gamma^\lambda_{\sigma\mu} - \Gamma^\kappa_{\sigma\lambda}\Gamma^\lambda_{\nu\mu}$$

$$+ \partial_\mu \Gamma^\kappa_{\nu\sigma} - \partial_\nu \Gamma^\kappa_{\mu\sigma} + \Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\kappa_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$= R^\kappa{}_{\mu\nu\sigma} + R^\kappa{}_{\sigma\mu\nu} + R^\kappa{}_{\nu\sigma\mu} \quad -(5)$$

where the three Riemann tensors are defined by:

3)

$$R^{\kappa}{}_{\mu\nu\sigma} = \partial_\mu \Gamma^{\kappa}_{\nu\sigma} - \partial_\nu \Gamma^{\kappa}_{\mu\sigma} + \Gamma^{\kappa}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\kappa}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

$$R^{\kappa}{}_{\sigma\mu\nu} = \partial_\sigma \Gamma^{\kappa}_{\mu\nu} - \partial_\mu \Gamma^{\kappa}_{\sigma\nu} + \Gamma^{\kappa}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\kappa}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\nu}$$

$$R^{\kappa}{}_{\nu\sigma\mu} = \partial_\nu \Gamma^{\kappa}_{\sigma\mu} - \partial_\sigma \Gamma^{\kappa}_{\nu\mu} + \Gamma^{\kappa}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu} - \Gamma^{\kappa}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu} \quad -(6)$$

The Cartan identity is:

$$D_\sigma T^{\kappa}{}_{\mu\nu} + D_\nu T^{\kappa}{}_{\sigma\mu} + D_\mu T^{\kappa}{}_{\nu\sigma} \quad -(7)$$

$$:= R^{\kappa}{}_{\mu\nu\sigma} + R^{\kappa}{}_{\sigma\mu\nu} + R^{\kappa}{}_{\nu\sigma\mu}.$$

It follows that $B := 0 \quad -(8)$

This means that:

$$\Gamma^{\lambda}_{\sigma\mu} T^{\kappa}{}_{\lambda\nu} + \Gamma^{\lambda}_{\sigma\nu} T^{\kappa}{}_{\mu\lambda} + \Gamma^{\lambda}_{\nu\sigma} T^{\kappa}{}_{\lambda\mu}$$

$$+ \Gamma^{\lambda}_{\nu\mu} T^{\kappa}{}_{\sigma\lambda} + \Gamma^{\lambda}_{\mu\nu} T^{\kappa}{}_{\lambda\sigma} + \Gamma^{\lambda}_{\mu\sigma} T^{\kappa}{}_{\nu\lambda} := 0 \quad -(9)$$

i.e.

$$\left(\Gamma^{\lambda}_{\sigma\nu} T^{\kappa}{}_{\mu\lambda} + \Gamma^{\lambda}_{\nu\sigma} T^{\kappa}{}_{\lambda\mu}\right)$$

$$+ \left(\Gamma^{\lambda}_{\sigma\mu} T^{\kappa}{}_{\lambda\nu} + \Gamma^{\lambda}_{\mu\sigma} T^{\kappa}{}_{\nu\lambda}\right) := 0$$

$$+ \left(\Gamma^{\lambda}_{\nu\mu} T^{\kappa}{}_{\sigma\lambda} + \Gamma^{\lambda}_{\mu\nu} T^{\kappa}{}_{\lambda\sigma}\right) \quad -(10)$$

4) From torsion antisymmetry:

$$\left(\Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\lambda}_{\nu\sigma}\right)T^{\kappa}_{\mu\lambda} + \left(\Gamma^{\lambda}_{\sigma\mu} - \Gamma^{\lambda}_{\mu\sigma}\right)T^{\kappa}_{\lambda\nu} + \left(\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\lambda}_{\mu\nu}\right)T^{\kappa}_{\sigma\lambda} := 0 \quad -(11)$$

i.e

$$T^{\lambda}_{\sigma\nu}T^{\kappa}_{\mu\lambda} + T^{\lambda}_{\sigma\mu}T^{\kappa}_{\lambda\nu} + T^{\lambda}_{\nu\mu}T^{\kappa}_{\sigma\lambda} := 0 \quad -(12)$$

Finally use $\kappa \rightarrow \alpha,\ \sigma \rightarrow \rho$ $\quad -(13)$
and torsion antisymmetry to find:

$$\boxed{T^{\lambda}_{\nu\rho}T^{\alpha}_{\mu\lambda} + T^{\lambda}_{\rho\mu}T^{\alpha}_{\nu\lambda} + T^{\lambda}_{\mu\nu}T^{\alpha}_{\rho\lambda} := 0} \quad -(14)$$

which is eq. (1), <u>QED</u>.

This is named <u>the first Evans identity</u> to distinguish it from the Evans Cartan identity in four dimensions:

$$D \wedge \tilde{T} := \tilde{R} \wedge q \quad -(15)$$

5) Note that eq. (14) has cyclical symmetry in μ, ν and ρ. In electrodynamics and gravitation it becomes a cyclical identity in field tensor.

Eq. (14) is the wedge product of a tensor valued one form $T_\rho{}^a{}_b$ and a vector valued two form $T^b{}_{\mu\nu}$, so:

$$T^b{}_{\mu\nu} \wedge T_\rho{}^a{}_b := 0 \quad -(16)$$

i.e. in the notation of differential geometry:

$$\boxed{T^b \wedge T^a{}_b := 0} \quad -(17)$$

as in UFT 109.

The Second Evans Identity

This is also part of the Jacobi Cartan Evans identity and is eq. (22) of note 313(6):

$$T^\lambda{}_{\mu\nu} R^\kappa{}_{\rho\lambda} + T^\lambda{}_{\rho\mu} R^\kappa{}_{\nu\lambda} + T^\lambda{}_{\nu\rho} R^\kappa{}_{\mu\lambda} := 0 \quad -(18)$$

It also has cyclical symmetry in μ, ν and ρ and is the wedge product of a vector valued two form $T^b{}_{\mu\nu}$ and a tensor valued one form $R^a{}_{b\rho c}$:

$$\boxed{T^b \wedge R^a{}_{bc} := 0} \quad -(19)$$

6) Eq. (18) is true if there is a special relation between the curvature and torsion:

$$R^{\kappa}{}_{\alpha\rho\lambda} = \left(T^{\kappa}{}_{\rho\lambda}\right)_{\alpha} \quad -(20)$$

so for each α the curvature is the torsion.

In general:

$$T^{\lambda}{}_{\mu\nu} R^{\kappa}{}_{\alpha\rho\lambda} + T^{\lambda}{}_{\rho\mu} R^{\alpha}{}_{\alpha\nu\lambda} + T^{\lambda}{}_{\nu\rho} R^{\kappa}{}_{\alpha\mu\lambda} \neq 0 \quad -(21)$$

so the complete JCE identity is:

$$\boxed{\begin{aligned}&\left(D_\rho R^{\kappa}{}_{\lambda\mu\nu} + D_\nu R^{\kappa}{}_{\lambda\rho\mu} + D_\mu R^{\kappa}{}_{\lambda\nu\rho}\right) V^{\lambda} \\ &:= \left(D_\rho T^{\lambda}{}_{\mu\nu} + D_\nu T^{\lambda}{}_{\rho\mu} + D_\mu T^{\lambda}{}_{\nu\rho}\right) D_\lambda V^{\kappa} \\ &+ \left(T^{\lambda}{}_{\mu\nu} R^{\kappa}{}_{\alpha\rho\lambda} + T^{\lambda}{}_{\rho\mu} R^{\kappa}{}_{\alpha\nu\lambda} + T^{\lambda}{}_{\nu\rho} R^{\kappa}{}_{\alpha\mu\lambda}\right) V^{\alpha}\end{aligned}} \quad -(22)$$

with:

$$T^{\lambda}{}_{\mu\nu} T^{\alpha}{}_{\rho\lambda} + T^{\lambda}{}_{\rho\mu} T^{\alpha}{}_{\nu\lambda} + T^{\lambda}{}_{\nu\rho} T^{\alpha}{}_{\mu\lambda} := 0 \quad -(23)$$
