316(4): Development of the Vector Theory for the Magnetic Monopole.

Consider the Cartan identity:

$$D_\mu T^a_{\nu\rho} + D_\rho T^a_{\mu\nu} + D_\nu T^a_{\rho\mu} := R^a_{\mu\nu\rho} + R^a_{\rho\mu\nu} + R^a_{\nu\rho\mu}. \quad -(1)$$

With the definitions:

$$F^a_{\nu\rho} = A^{(0)} T^a_{\nu\rho} \quad -(2)$$

and

$$F^a_{\mu\nu\rho} = W^{(0)} T^a_{\mu\nu\rho} \quad -(3)$$

eq. (1) becomes:

$$D_\mu F^a_{\nu\rho} + D_\rho F^a_{\mu\nu} + D_\nu F^a_{\rho\mu} := \frac{A^{(0)}}{W^{(0)}} \left(F^a_{\mu\nu\rho} + F^a_{\rho\mu\nu} + F^a_{\nu\rho\mu} \right) \quad -(4)$$

In vector notation eq. (1) is:

$$\underline{\nabla} \cdot \underline{T}^a(\text{spin}) + \underline{\omega}^a{}_b \cdot \underline{T}^b(\text{spin}) = \underline{q}^b \cdot \underline{R}^a{}_b(\text{spin}) \quad -(5)$$

Now use:

$$\underline{B}^a = A^{(0)} \underline{T}^a(\text{spin}) \quad -(6)$$

and

$$\underline{B}^a{}_b = W^{(0)} \underline{R}^a{}_b(\text{spin}) \quad -(7)$$

to find that:

$$\underline{\nabla} \cdot \underline{B}^a + \underline{\omega}^a{}_b \cdot \underline{B}^a = \left(\frac{A^{(0)}}{W^{(0)}} \right) \underline{q}^b \cdot \underline{B}^a{}_b \quad -(8)$$

2) Define:

$$\underline{A}^{b} = A^{(0)} \underline{V}^{b} \quad -(9)$$

so

$$\underline{\nabla} \cdot \underline{B}^{a} = \frac{1}{W^{(0)}} \underline{A}^{b} \cdot \underline{B}^{a}{}_{b} - \underline{\omega}^{a}{}_{b} \cdot \underline{B}^{b} \quad -(10)$$

This is the magnetic monopole equation. The magnetic monopole can be defined as:

$$\rho_{B} = \frac{1}{W^{(0)}} \underline{A}^{b} \cdot \underline{B}^{a}{}_{b} - \underline{\omega}^{a}{}_{b} \cdot \underline{B}^{b} \quad -(11)$$

The tangent index a can be removed from $\underline{B}^{a}$ using:

$$\boxed{\underline{B} = - e_{a} \underline{B}^{a}} \quad -(12)$$

where e_a is the unit vector in the tangent space.

In the complex circular basis:

$$e_{a} = \left(1, -\frac{1}{\sqrt{2}}(1+i), -\frac{1}{\sqrt{2}}(1-i), -1\right) \quad -(13)$$

In the Cartesian basis:

$$e_{a} = (1, -1, -1, -1) \quad -(14)$$

Therefore:

$$\underline{B} = e_{(0)} \underline{B}^{(0)} - e_{(1)} \underline{B}^{(1)} - e_{(2)} \underline{B}^{(2)} - e_{(3)} \underline{B}^{(3)} \quad -(15)$$

By definition:

$$\underline{B}^{(0)} = \underline{0} \quad -(16)$$

i.e. the spacelike vector $\underline{B}$ has no timelike component.

In general:

$$\underline{B}^{(1)} = \frac{1}{\sqrt{2}}\left(B_x \underline{i} - i B_y \underline{j}\right) \quad -(17)$$

and

$$\underline{B}^{(2)} = \frac{1}{\sqrt{2}}\left(B_x \underline{i} + i B_y \underline{j}\right) - (18)$$

So in eq. (15):

$$\underline{B} = \frac{1}{\sqrt{2}}(1+i) \cdot \frac{1}{\sqrt{2}}\left(B_x \underline{i} - i B_y \underline{j}\right) + \frac{1}{\sqrt{2}}(1-i) \cdot \frac{1}{\sqrt{2}}\left(B_x \underline{i} + i B_y \underline{j}\right) + B_Z \underline{k} \quad -(19)$$

$$= B_x \underline{i} + B_y \underline{j} + B_Z \underline{k} \quad -(20)$$

Q.E.D., where we have used:

$$\underline{B}^{(3)} = B_Z \underline{k} \quad -(21)$$

Now multiply both sides of eq. (10) by $-e_a$ to obtain:

$$\underline{\nabla} \cdot \underline{B} = \frac{1}{W^{(0)}} \underline{A}^b \cdot \underline{B}_b - \underline{\omega}_b \cdot \underline{B}^b \quad -(22)$$

Finally, we:

$$\underline{A} = -e_b \underline{A}^b \quad -(23)$$

4) and $\underline{B} = -e^b \underline{B}_b$. — (24)

Similarly: $\underline{\omega} = -e^b \omega_b$ — (25)

and $\underline{B} = -e_b \underline{B}^b$ — (26)

— (27)

to find that:

$$\underline{\nabla} \cdot \underline{B} = \frac{e_b e^b}{W^{(0)}} \left(\underline{A} \cdot \underline{B} - \underline{\omega} \cdot \underline{B} \right)$$

where we have used

$$e_b e^b = e^b e_b \quad -(28)$$

Eq. (28) must be an invariant, and must not depend on the coordinate system. In the Cartesian basis

$$e_b e^b = e^b e_b = -2 \quad -(29)$$

In order to obtain the same result in the complex circular system we must use:

$$e_b e^{b*} = e^{b*} e_b = -2 \quad -(30)$$

where $*$ denotes complex conjugate.

For self consistency the property (30)

5) implies that $\underline{A}$ in eq. (23) must be defined by:

$$\underline{A}^* = -\left(e_b \underline{A}^b\right)^* \quad -(24)$$

Similarly $\underline{\omega}$ in eq. (25) must be defined by:

$$\underline{\omega}^* = -\left(e^b \omega_b\right)^* \quad -(25)$$

Therefore eq. (22) becomes: $\quad -(26)$

$$\underline{\nabla} \cdot \underline{B} = -2\left(\frac{\underline{A}^*}{W^{(0)}} \cdot \underline{B} - \underline{\omega}^* \cdot \underline{B}\right)$$

where we have used:

$$e^{b*} e_b = e_b^* e^b = -2 \quad -(27)$$

So the monopole equation reduces to:

$$\boxed{\underline{\nabla} \cdot \underline{B} = 2\, \underline{B} \cdot \left(\underline{\omega}^* - \frac{\underline{A}^*}{W^{(0)}}\right)} \quad -(28)$$

This can be written as:

$$\underline{\nabla} \cdot \underline{B} = 2\, \underline{B} \cdot \left(\underline{\omega}^* - \frac{A^{(0)}}{W^{(0)}} \underline{V}^*\right) \quad -(29)$$

This equation has the correct units because $A^{(0)} / W^{(0)}$ has the units of inverse metres.

6) We may rewrite:

$$\frac{A^{(0)}}{W^{(0)}} = \frac{1}{r^{(0)}} \quad -(30)$$

So

$$\boxed{\underline{\nabla} \cdot \underline{B} = 2\underline{B} \cdot \left(\underline{\omega}^* - \frac{1}{r^{(0)}} \underline{V}^* \right)} \quad -(31)$$

The magnetic monopole is therefore:

$$\boxed{\rho_B = 2\underline{B} \cdot \left(\underline{\omega}^* - \frac{1}{r^{(0)}} \underline{V}^* \right)} \quad -(32)$$

and vanishes if and only if:

$$\underline{\omega}^* = \frac{1}{r^{(0)}} \underline{V}^* \quad -(33)$$

This result can be compared with that of the engineering model:

$$\rho_B = \underline{\omega}^a{}_b \cdot \underline{B}^b - \underline{A}^b \cdot \underline{R}^a{}_b(\text{spin}) \quad -(34)$$

on page 26.

The result (32) is very much clearer and simpler.