326(3): Rotational Quantization

Starting from the relativistic lagrangian:

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U \quad -(1)$$

the Euler Lagrange equations give:

$$\frac{d}{dt}(\gamma m \dot{r}) - \gamma m r \dot{\theta}^2 = F(r) = -\frac{\partial U}{\partial r} \quad -(2)$$

and

$$L = \gamma m r^2 \dot{\theta} = \gamma L_0 \quad -(3)$$

in previous notation. The Lorentz factor is defined by

$$E^2 = \gamma^2 m^2 c^4 = p^2 c^2 + m^2 c^4$$
$$= (H - U)^2 \quad -(4)$$

and the potential energy is defined by:

$$U = H - \gamma mc^2 = H - \left(\frac{L}{L_0}\right) mc^2 \quad -(5)$$

Note that L is a constant of motion:

$$\frac{dL}{dt} = 0 \quad -(6)$$

but L_0 is not a constant of motion.

From eq. (4):

$$H - mc^2 - U = \frac{c^2 p^2}{H + mc^2 - U} \quad -(7)$$

2)

In the non-relativistic approximation:

$$v << c \quad - (8)$$

and

$$\gamma \rightarrow 1. \quad - (9)$$

Furthermore, assume that:

$$H \sim E \quad - (10)$$

so

$$U << E. \quad - (11)$$

In these approximations:

$$H_1 = H - mc^2 \sim \frac{c^2 p^2}{2mc^2 - U} + U \quad - (12)$$

$$= \frac{p^2}{2m} \left(1 - \frac{U}{2mc^2} \right)^{-1} + U$$

Now assume that:

$$U << 2mc^2 \quad - (13)$$

so:

$$H_1 = H - mc^2 \sim \frac{p^2}{2m} \left(1 + \frac{U}{2mc^2} \right) + U \quad - (14)$$

These approximations mean that:

$$E \sim mc^2 \text{ and } U \sim mc^2 \left(1 - \left(\frac{L}{L_0} \right)^2 \right) \quad - (15)$$

with

$$L \rightarrow L_0, \quad -(16)$$

So, self consistently:

$$\bar{U} \rightarrow 0. \quad -(17)$$

Eq. (14) quantizes to: $\quad -(18)$

$$\left(-\frac{\nabla^2}{2m} + \bar{U} \right) \phi = \left(H_1 + \frac{\bar{U}}{4m^2c^2} \nabla^2 \right) \phi$$

where

$$\bar{U} = mc^2 \left(1 - \left(\frac{L}{L_0} \right)^2 \right) \quad -(19)$$

with

$$\left(\frac{L}{L_0} \right)^2 = \gamma^2 = \left(\frac{L}{mr^2\dot{\theta}} \right)^2 \quad -(20).$$

However, it is known that:

$$\bar{U} = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(21)$$

So

$$1 - \left(\frac{L}{L_0} \right)^2 = -\frac{e^2}{4\pi\epsilon_0 mc^2 r} \quad -(22)$$

and

$$\left(\frac{L}{L_0} \right)^2 = \left(\frac{L}{mr^2\dot{\theta}} \right)^2 = 1 + \frac{e^2}{4\pi\epsilon_0 mc^2 r} \quad -(23)$$

$$\rightarrow 1$$

in the non-relativistic limit.

4) Eq. (23) gives:

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{L}{mr^2}\left(1 + \frac{e^2}{4\pi\epsilon_0 mc^2 r}\right)^{-1/2}$$

$$= \frac{L}{\gamma mr^2} \quad -(24)$$

Therefore

$$\dot{\theta} = \frac{d\theta}{dt} \sim \frac{L}{mr^2}\left(1 - \frac{e^2}{8\pi\epsilon_0 mc^2 r}\right) \quad -(25)$$

$$= \frac{L}{mr^2}\left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

$$\sim \frac{L}{mr^2}\left(1 - \frac{v^2}{2c^2}\right)$$

if

$$v << c. \quad -(26)$$

Therefore
$$v^2 = \frac{e^2}{4\pi\epsilon_0 mr} \quad -(27)$$

This leads to an orbital equation using:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(28)$$

and:

$$\frac{d\theta}{dt} = \frac{d\theta}{dr}\frac{dr}{dt} \quad -(29)$$

Fm eq. (28):

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \frac{e^2}{4\pi\epsilon_0 m r} \quad -(30)$$

So:

$$\left(\frac{dr}{dt}\right)^2 = \frac{e^2}{4\pi\epsilon_0 m r} - r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(31)$$

$$-(32)$$

and

$$\frac{d\theta}{dr} = \frac{dt}{dr}\frac{d\theta}{dt} = \frac{d\theta}{dt}\left(\frac{e^2}{4\pi\epsilon_0 m r} - r^2\left(\frac{d\theta}{dt}\right)\right)^{-1/2}$$

i.e.

$$\frac{d\theta}{dr} = \frac{d\theta/dt}{\left(\frac{e^2}{4\pi\epsilon_0 m r} - r^2\left(\frac{d\theta}{dt}\right)\right)^{1/2}} \quad -(33)$$

where

$$\frac{d\theta}{dt} = \frac{L}{mr^2}\left(1 - \frac{e^2}{8\pi\epsilon_0 m c^2 r}\right) \quad -(34)$$