326(4): Relativistic Rotational Motion

Consider the hamiltonian and lagrangian of special relativity:

$$H = \gamma mc^2 + U \quad -(1)$$

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U \quad -(2)$$

where $E = \gamma mc^2 = (p^2c^2 + m^2c^4)^{1/2} \quad -(3)$

It follows that:

$$p^2c^2 + m^2c^4 = (H-U)^2 \quad -(4)$$

so $(H-U)^2 - m^2c^4 = p^2c^2, \quad -(5)$

so
$$H - U - mc^2 = \frac{p^2c^2}{H-U+mc^2} = \frac{p^2c^2}{E+mc^2} \quad -(6)$$

Therefore:

$$H_1 = H - mc^2 = \frac{p^2c^2}{E+mc^2} + U \quad -(7)$$

In the limit

$$E \longrightarrow mc^2 \quad -(8)$$

i.e.

$$\gamma \longrightarrow 1 \quad -(9)$$

eq. (7) becomes the Schroedinger equation,

2) based on the classical Hamiltonian:

$$H_1 = \frac{p^2}{2m} + U \quad -(10)$$

For rotational motion,

$$E = \frac{p^2}{2m} = \frac{1}{2} I\omega^2 = \frac{1}{2} m r^2 \omega^2 = \frac{1}{2} m v^2 \quad -(11)$$

so for circular motion:

$$v = r\omega \quad -(12)$$

The moment of inertia is defined as:

$$\underline{I} = m r^2 \quad -(13)$$

The classical kinetic energy is:

$$E = \frac{p^2}{2m} = \frac{L^2}{2I} \quad -(14)$$

where

$$\underline{L} = \underline{r} \times \underline{p} \quad -(15)$$

is the classical angular momentum.

The particle on a ring problem is defined by a particle moving on a circle of radius r in the XY plane. So:

$$X = r\cos\theta, \quad Y = r\sin\theta \quad -(16)$$

and the laplacian is:

3)

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad -(17)$$

In the particle on a ring problem derivatives with respect to r are not considered, so

$$\nabla^2 = \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad -(18)$$

and the potential energy U is considered to be zero, so

$$U = 0. \quad -(19)$$

The Schroedinger equation is:

$$-\frac{1}{2mr^2}\frac{\partial^2 \phi}{\partial \theta^2} = E\phi \quad -(20)$$

The solution is the quantized rotational energy:

$$E = m_l^2 \left(\frac{\hbar^2}{2I}\right), \quad m_l = 0, \pm 1, \pm 2, \pm 3, \ldots \quad -(21)$$

Comparing eqs. (14) and (21):

$$\boxed{L = \pm m_l \hbar} \quad -(22)$$

This is the Schroedinger quantization of rotational motion. It is similar to Sommerfeld quantization.

4) The relativistic Schroedinger equation with no potential is:

$$H_1 = H - mc^2 = \frac{p^2 c^2}{E + mc^2} \quad -(23)$$

$$= \frac{p^2 c^2}{mc^2 (1+\gamma)} = \frac{p^2}{m(1+\gamma)} \quad -(23a)$$

So

$$\boxed{\frac{p^2}{m} = (1+\gamma) H_1} \quad -(24)$$

As:

$$\gamma \to 1, \quad -(25)$$

eq. (25) becomes:

$$\frac{p^2}{2m} = H_1 \quad -(26)$$

which is the free particle Schroedinger equation. It is customary to denote:

$$H_1 := E \quad -(27)$$

Eq. (24) quantizes to:

$$\boxed{-\frac{\hbar^2}{m} \nabla^2 \psi = (1+\gamma) E \psi} \quad -(28)$$

The relativistic version of eq. (20) is:

$$-\frac{1}{mr^2}\frac{\partial^2\phi}{\partial\theta^2} = E_1\phi \quad -(29)$$

where

$$E_1 = (1+\gamma)E \quad -(30)$$

The Lorentz factor is:

$$\gamma = \left(1-\frac{v^2}{c^2}\right)^{-1/2} \quad -(31)$$

where

$$p = mv \quad -(32)$$

so

$$\frac{p^2}{m} = \left(1+\left(1-\left(\frac{p}{mc}\right)^2\right)^{-1/2}\right)H_1 \quad -(33)$$

If:

$$v \ll c \quad -(34)$$

then

$$\frac{p^2}{m} = \left(1+1+\frac{1}{2}\left(\frac{p}{mc}\right)^2\right)H_1 \quad -(35)$$

Using the notation of eq. (26):

$$\boxed{\frac{p^2}{m} = \left(2+\frac{1}{2}\left(\frac{p}{mc}\right)^2\right)E} \quad -(36)$$

which is the relativistic free particle Schroedinger

6)

equation ... the limit (34).

On quantization:

$$-\frac{\hbar^2}{m}\nabla^2\phi = \left(2 - \frac{\hbar^2}{2m^2c^2}\nabla^2\right)(E\phi) \quad -(37)$$

Since E is a constant of motion:

$$\nabla^2(E\phi) = E\nabla^2\phi \quad -(38)$$

so:

$$-\frac{\hbar^2}{m}\nabla^2\phi = \left(2 - \frac{\hbar^2}{2mc^2}\nabla^2\phi\right)E \quad -(39)$$

i.e.

$$\left(-\frac{\hbar^2}{m} + \frac{\hbar^2 E}{2mc^2}\right)\nabla^2\phi = 2E\phi \quad -(40)$$

or

$$\boxed{\left(-\frac{\hbar^2}{2m} + \frac{\hbar^2 E}{4mc^2}\right)\nabla^2\phi = E\phi} \quad -(41)$$

The Lorentz factor for the particle on a ring problem is calculated from:

$$v^2 = r^2\dot{\theta}^2 = \left(\frac{L_0}{mr}\right)^2 \quad -(42)$$

where L_0 is the non relativistic angular momentum defined by:

$$L_o = m r^2 \dot{\theta} \quad -(43)$$

So

$$\gamma = \left(1 - \left(\frac{L_o}{mrc} \right)^2 \right)^{-1/2} \quad -(44)$$

$$\sim \quad 2 + \frac{1}{2} \left(\frac{L_o}{mrc} \right)^2 \quad -(44a)$$

when

$$v << c \quad -(45)$$

So

$$E_1 = \left(2 + \frac{1}{2} \left(\frac{L_o}{mrc} \right)^2 \right) \quad -(46)$$

in eq. (29). The expectation value of E_1 is shifted to

$$\langle E_1 \rangle = \int \psi^* E_1 \psi \, d\tau \quad -(47)$$