328(4): More Accurate Theory of Orbital Precession in Special Relativity

The basic method is to use:

$$\frac{dr}{d\theta} = \frac{dr}{d\tau}\frac{d\tau}{d\theta} \quad -(1)$$

where the angular momentum in special relativity is defined by

$$L = mr^2\frac{d\theta}{d\tau} \quad -(2)$$

and is a constant of motion. So:

$$\frac{dr}{d\theta} = \frac{dr}{d\tau}\frac{mr^2}{L} \quad -(3)$$

$$= \gamma\frac{dr}{dt}\frac{mr^2}{L}$$

where

$$\gamma = \frac{dt}{d\tau} = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} \quad -(4)$$

is the Lorentz factor. The observer frame v_0^2 is defined by

$$v_0^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \quad -(5)$$

Now make the approximation:

$$L \sim L_0 \quad -(6)$$

in eq. (3), so:

$$\frac{dr}{d\theta} \sim \gamma\frac{dr}{dt}\frac{mr^2}{L_0} \quad -(7)$$

) In the classical theory:

$$\left(\frac{dr}{d\theta}\right)_0 = \frac{dr}{dt}\frac{mr^2}{L_0} = \frac{dr}{dt}\frac{dt}{d\theta} \quad -(8)$$

So

$$\boxed{\frac{dr}{d\theta} = \gamma\left(\frac{dr}{d\theta}\right)_0} \quad -(9)$$

Using

$$\frac{dr}{dt} = \left(\frac{dr}{d\theta}\right)_0 \frac{d\theta}{dt} \quad -(10)$$

it is found that:

$$v_0^2 = \left(\frac{d\theta}{dt}\right)^2 \left(\left(\frac{dr}{d\theta}\right)_0^2 + r^2\right) \quad -(11)$$

$$= \frac{L_0^2}{m^2 r^4}\left(\left(\frac{dr}{d\theta}\right)_0^2 + r^2\right).$$

Therefore:

$$\boxed{\frac{dr}{d\theta} = \left(\frac{dr}{d\theta}\right)_0 \left(1 - \frac{L_0^2}{m^2c^2r^4}\left(\left(\frac{dr}{d\theta}\right)_0^2 + r^2\right)\right)^{-1/2}} \quad -(12)$$

In the Newtonian theory:

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(13)$$

and

$$\left(\frac{dr}{d\theta}\right)_0 = \frac{\epsilon r^2}{\alpha}\sin\theta \quad -(14)$$

Therefore:

$$\frac{dr}{d\theta} = \frac{\epsilon r^2}{\alpha} \sin\theta \left(1 - \left(\frac{L_0}{mcr}\right)^2\right)\left(1 + \left(\frac{\epsilon r}{\alpha}\sin\theta\right)^2\right)\Bigg)^{-1/2} \quad -(15)$$

where

$$L_0^2 = m^2 MG\alpha \quad -(16)$$

From eq. (15), the ratio p/L can be worked out using:

$$\left(\frac{p}{L}\right)^2 = \frac{1}{r^4}\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) \quad -(17)$$

and the result can be compared with $(p/L)^2$ worked out from the lagrangian as in UFT324 and UFT325:

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U \quad -(18)$$

where

$$U = -mMG/r \quad -(19)$$

In the x theory:

$$r = \frac{\alpha}{1 + \epsilon\cos(x\theta)} \quad -(20)$$

and

$$\frac{dr}{d\theta} = \frac{x\epsilon r^2}{\alpha}\sin(x\theta) \quad -(21)$$

4) So eqs. (15) and (21) can be compared in a polar graph. The ρ / L from the lagrangian (18) can be compared directly with its equivalents from eq. (15) and eq. (21).

Finally, for the general precessing conical section:

$$r = \frac{\alpha}{1 + \epsilon \cos(\theta_1(\theta))} \quad -(22)$$

where θ_1 is a function of θ, then:

$$\frac{dr}{d\theta} = \left(\frac{d\theta_1}{d\theta}\right)\left(\frac{\epsilon}{\alpha}\right) r^2 \sin(\theta_1(\theta)) \quad -(23)$$

where

$$\cos(\theta_1(\theta)) = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(24)$$

and

$$\sin^2(\theta_1(\theta)) = 1 - \cos^2(\theta_1(\theta)) \quad -(25)$$

Therefore $d\theta_1 / d\theta$ can be found by comparing eqs. (12) and (23). The function $d\theta_1 / d\theta$ may also be found by comparing $(\rho / L)^2$ from eqs. (17) and (23) with $(\rho / L)^2$ from the lagrangian (18). In order to calculate the precession of the perihelion assume that the orbit is:

$$r = \frac{\alpha}{1+\epsilon\cos(\gamma\theta)} \quad -(26)$$

Then

$$\frac{dr}{d\theta} = \frac{\gamma r^2 \epsilon}{\alpha}\sin(\gamma\theta). \quad -(27)$$

If

$$\alpha \sim \alpha_0 \text{ and } \epsilon \sim \epsilon_0 \quad -(28)$$

then:

$$\frac{dr}{d\theta} = \gamma \frac{r^2\epsilon}{\alpha}\sin\theta = \gamma\left(\frac{dr}{d\theta}\right)_0 \quad -(29)$$

which is eq. (9), QED.

Therefore in this approximation:

$$\boxed{x = \gamma} \quad -(30)$$

The precession is therefore:

$$\Delta\theta = (x-1)\theta \quad -(31)$$

or

$$-\Delta\theta = (1-x)\theta \quad -(32)$$

depending on whether the rotation is clockwise or anticlockwise. In this theory:

$$x = \gamma \geq 1. \quad -(33)$$

For a rotation of 2π:

6)

$$\boxed{\Delta\theta = 2\pi(\gamma - 1).} \quad -(34)$$

where

$$\gamma = \left(1 - \left(\frac{L_0}{mcr}\right)^2 \left(1 + \left(\frac{\epsilon r}{\alpha}\sin\theta\right)^2\right)\right)^{-1/2} \quad -(35)$$

in which

$$\sin^2\theta = 1 - \cos^2\theta \quad -(36)$$

and

$$\cos\theta = \frac{1}{\epsilon}\left(\frac{\alpha_0}{r} - 1\right) \quad -(37)$$

with

$$L_0^2 = m^2 MG\alpha_0. \quad -(38)$$

At the perihelion:

$$r_{min} = a(1-\epsilon) \quad -(39)$$

$$= \frac{\alpha_0}{1+\alpha}$$

where a is the semi major axis. Finally:

$$a = \frac{\alpha}{1-\epsilon^2} \quad -(40)$$

In the approximation:

$$v_0 \ll c \quad -(41)$$

7) then:

$$\gamma = 1 + \frac{1}{2}\left(\frac{L_0}{mcr}\right)^2\left(1 + \left(\frac{\epsilon r}{\alpha}\sin\theta\right)^2\right) \quad -(42)$$

So:

$$\Delta\theta = \pi\left(\frac{L_0}{mcr}\right)^2\left(1 + \left(\frac{\epsilon r}{\alpha}\sin\theta\right)^2\right) \quad -(43)$$

for a rotation of 2π. Using eq. (37):

$$\boxed{\Delta\theta = \pi\left(\frac{L_0}{mcr}\right)^2\left(1 + \left(\frac{\epsilon r}{\alpha}\right)^2\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha_0}{r} - 1\right)^2\right)\right)} \quad -(44)$$

where

$$r = r_{min} = a(1-\epsilon) \quad -(45)$$

at the perihelion.