333(3): The Class Four Hamiltonian without Approximation

The class four hamiltonian is:

$$H = \left(\frac{\gamma^2}{1+\gamma}\right)\frac{p_0^2}{m} + U = \frac{1}{m}\left(\frac{\gamma^2}{1+\gamma}\right)\underline{\sigma}\cdot\underline{p_0}\,\underline{\sigma}\cdot\underline{p_0} + U \quad -(1)$$

in which:

$$\frac{\gamma^2}{1+\gamma} = \left(\frac{1}{\gamma^2}+\frac{1}{\gamma}\right)^{-1} \quad -(2)$$

and

$$\gamma = \left(1-\frac{p_0^2}{m^2c^2}\right)^{-1/2} \quad -(3)$$

So:

$$H = \left(\left(1-\frac{p_0^2}{m^2c^2}\right)^{1/2} + \left(1-\frac{p_0^2}{m^2c^2}\right)\right)^{-1}\frac{p_0^2}{m} + U \quad -(4)$$

where $\underline{p_0}$ is the non relativistic linear momentum:

$$\underline{p_0} = m\underline{v}_0, \quad -(5)$$

where m is the particle mass and where U is the potential energy.

On quantization, eq. (4) becomes:

$$H\phi = -\left(1-\frac{p_0^2}{m^2c^2} + \left(1-\frac{p_0^2}{m^2c^2}\right)^{1/2}\right)^{-1}\frac{\hbar^2\nabla^2}{m}\phi + U\phi \quad -(6)$$

2) The energy levels are:

$$E = \langle H \rangle = -\left(1 - \frac{p_0^2}{m^2c^2} + \left(1 - \frac{p_0^2}{m^2c^2}\right)^{1/2}\right)^{-1} \left\langle \frac{\hbar^2\nabla^2}{m} \right\rangle + \langle u \rangle \quad -(7)$$

In the H atom:

$$\left\langle \frac{p_0^2}{m} \right\rangle = \left\langle -\frac{\hbar^2\nabla^2}{m} \right\rangle = \frac{mc^2}{\pi}\frac{\lambda_c}{a_0}\frac{\alpha}{n^2} \quad -(8)$$

$$\langle U \rangle = -\frac{mc^2}{\pi}\frac{\lambda_c}{a_0}\frac{\alpha}{n^2} \quad -(9)$$

where λ_c is the Compton wavelength of the electron, a_0 is the Bohr radius and α the fine structure constant. Here n is the principal quantum number.

So in this class four theory the energy levels of the H atom are:

$$E = \left(\left(1 - \frac{p_0^2}{m^2c^2} + \left(1 - \frac{p_0^2}{m^2c^2}\right)^{1/2}\right)^{-1} - 1\right) \frac{mc^2}{\pi}\frac{\lambda_c}{a_0}\frac{\alpha}{n^2} \quad -(10)$$

$$\xrightarrow[\gamma \to 1]{} -\frac{mc^2}{2\pi}\frac{\lambda_c}{a_0}\frac{\alpha}{n^2} = \frac{-me^4}{32\pi^2\hbar^2\epsilon_0^2 n^2}$$

which is the correct non relativistic result <u>Q.E.D</u>.

3) Now use:

$$\frac{p_0^2}{n^2 c^2} = \frac{1}{mc^2}\left\langle \frac{p_0^2}{m} \right\rangle = \frac{\lambda_c}{\pi a_0}\frac{\alpha}{n^2} \quad -(11)$$

to find that:

$$E = \left(\frac{1}{1 - \frac{1}{2}\frac{\lambda_c}{a_0}\frac{\alpha}{n^2} + \left(1 - \frac{1}{2}\frac{\lambda_c \alpha}{a_0 n^2}\right)^{1/2}} - 1 \right) \frac{mc^2}{\pi}\frac{\lambda_c}{a_0}\frac{\alpha}{n^2} \quad -(12)$$

in which:

$$\frac{1}{2}\left(\frac{\lambda_c}{\pi a_0}\right)\alpha = \frac{2.662567 \times 10^{-5}}{\pi} \quad -(13)$$

so

$$E = \left(\frac{1}{1 - \frac{2.662567 \times 10^{-5}}{\pi n^2} + \left(1 - \frac{2.662567 \times 10^{-5}}{\pi n^2}\right)^{1/2}} - 1 \right)$$

$$\times \frac{mc^2}{\pi}\frac{\lambda_c}{a_0}\frac{\alpha}{n^2} \quad -(14)$$

i.e.

$$E = \left(\frac{1}{1 - \frac{2.662567 \times 10^{-5}}{\pi n^2} + \left(1 - \frac{2.662567 \times 10^{-5}}{\pi n^2}\right)^{1/2}} - 1 \right) \frac{me^4}{16\pi^2 \hbar^2 \epsilon_0^2 n^2} \quad -(15)$$

This does not give the observed fine structure of the H atom. It produces energy level shifts that depend on the principal quantum number n.

This result is already enough to show that the Dirac method is not rigorous or fundamental, because no approximations are used in deriving eq. (15).

In the next note this method will be applied to the classical hamiltonian:

$$H = \frac{1}{m} \underline{\sigma} \cdot \underline{p}_0 \left(\frac{\gamma^2}{1+\gamma} \right) \underline{\sigma} \cdot \underline{p}_0 + U \quad - (16)$$

again without approximation.