333(6): Correction to Fine Structure For any Velocity

Consider the Hamiltonian:

$$H = \underline{\sigma} \cdot \underline{p_0} \frac{\gamma^2}{1+\gamma} \underline{\sigma} \cdot \underline{p_0} + U \quad -(1)$$

By computer algebra:

$$\underline{\nabla}\left(\frac{\gamma^2}{1+\gamma}\right) = -\left[\frac{2+\left(1-\frac{p_0^2}{m^2c^2}\right)^{-1/2}}{\left(\left(1-\frac{p_0^2}{m^2c^2}\right)^{1/2}+\left(1-\frac{p_0^2}{m^2c^2}\right)\right)^{1/2}}\right]\frac{e^2}{4\pi\epsilon_0 m^2c^2}\frac{\underline{r}}{r^3}$$

$$-(2)$$

where we have used:

$$p_0^2 = 2(H_0 - U) \quad -(3)$$

and

$$U = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(4)$$

Define:

$$A := \frac{2+\left(1-\frac{p_0^2}{m^2c^2}\right)^{-1/2}}{\left(\left(1-\frac{p_0^2}{m^2c^2}\right)^{1/2}+\left(1-\frac{p_0^2}{m^2c^2}\right)\right)^{1/2}} \quad -(5)$$

Now quantize eq. (1):

$$H\phi = -i\hbar\, \underline{\sigma} \cdot \underline{\nabla} \left( \frac{\gamma^2}{1+\gamma} \underline{\sigma} \cdot \underline{p}_0\, \phi \right) + U\phi \quad -(6)$$

$$= -i\hbar\, \underline{\sigma} \cdot \underline{\nabla} \left( \frac{\gamma^2}{1+\gamma} \right) \underline{\sigma} \cdot \underline{p}_0\, \phi + \dots + U\phi$$

From eqns. (2) and (6):

$$H\phi = \frac{i e^2 \hbar}{4\pi\epsilon_0 m^2 c^2 r^3} A\, \underline{\sigma} \cdot \underline{r}\, \underline{\sigma} \cdot \underline{p}_0 \phi + U\phi \quad -(7)$$

By Pauli algebra:

$$\underline{\sigma} \cdot \underline{\nabla}\, \underline{\sigma} \cdot \underline{p}_0 = \underline{\nabla} \cdot \underline{p}_0 + i\, \underline{r} \times \underline{p}_0 \quad -(8)$$

where the orbital angular momentum is:

$$\underline{L} = \underline{r} \times \underline{p}_0 \quad -(9)$$

So the spin orbit hamiltonian is:

$$\mathrm{Re}\, H_{so}\phi = \frac{-e^2\hbar}{4\pi\epsilon_0 m^2 c^2 r^3} A\, \underline{\sigma} \cdot \underline{L}\, \phi \quad -(10)$$

giving:

$$E = \langle H_{so} \rangle = \frac{-e^2\hbar A}{4\pi\epsilon_0 m^2 c^2} \left\langle \frac{\underline{\sigma} \cdot \underline{L}}{r^3} \right\rangle \quad -(11)$$

It is possible to vary $p_0$ from zero to

) infinity and plot the shifts in the spin orbit fine structure, or use

$$\frac{p_0^2}{m^2c^2} = \left\langle \frac{p_0^2}{m^2c^2} \right\rangle = \frac{\lambda_c}{a_0}\frac{\alpha}{n^2} \qquad -(12)$$

$$= \frac{2.662567 \times 10^{-5}}{n^2}$$

and plot the resulting spectrum.

So this is the first time that the theory of spin orbit coupling has been worked out for relativistic velocities.