Note 336(1): Rigorous Theory of ESR in an Electron Beam, Use of the Einstein / de Broglie Equations.

Consider the rigorous energy equation of a free electron:

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(1)$$

where E is the total energy:

$$E = \gamma m c^2 = \hbar \omega \quad -(2)$$

and $\underline{p}$ the relativistic momentum:

$$\underline{p} = \gamma \underline{p_0} = \hbar \underline{\kappa} \quad -(3)$$

In this case there is no potential energy, so the relativistic hamiltonian is:

$$H = E. \quad -(4)$$

Eq. (1) is:

$$E^2 - m^2 c^4 = (E - mc^2)(E + mc^2) = c^2 p^2 \quad -(5)$$

so

$$E = \frac{c^2 p^2}{E + mc^2} + mc^2 \quad -(6)$$

$$= \frac{\gamma^2}{1+\gamma} \frac{p_0^2}{m} + mc^2$$

From eq. (2)

$$\boxed{\gamma = \frac{\hbar \omega}{mc^2}} \quad -(7)$$

where ω is the angular frequency of the electron

2) matter wave, and m is the electron mass.

Therefore:

$$x := \frac{\gamma^2}{1+\gamma} = \left(\frac{\hbar\omega}{mc^2}\right)^2 \left(1 + \frac{\hbar\omega}{mc^2}\right)^{-1} \quad -(8)$$

Eq. (8) is true for any particle of mass m, including the photon.

Now apply a magnetic flux density $\underline{B}$ to the electron beam. The resulting hamiltonian is:

$$E = H = \frac{x}{m}\left(\underline{p_0} - e\underline{A}\right)\cdot\left(\underline{p_0} - e\underline{A}\right) + mc^2 \quad -(9)$$

This is quantized as follows:

$$H\phi = \frac{x}{m}\left(-i\hbar\underline{\nabla} - e\underline{A}\right)\cdot\left(\underline{p_0} - e\underline{A}\right) + mc^2\,\phi \quad -(10)$$

In the SU(2) basis:

$$H\phi = \frac{x}{m}\,\underline{\sigma}\cdot\left(-i\hbar\underline{\nabla} - e\underline{A}\right)\underline{\sigma}\cdot\left(\underline{p_0} - e\underline{A}\right)\phi + mc^2\phi \quad -(11)$$

giving the interaction hamiltonian:

$$H_{int}\,\phi = -\frac{xe\hbar}{m}\,\underline{\sigma}\cdot\underline{\nabla}\times\underline{A}\,\phi \quad -(12)$$

3) The spin angular momentum of the electron is:

$$\underline{S} = \frac{\hbar}{2} \underline{\sigma} \quad -(13)$$

so:

$$H_{int} \phi = -2x \frac{e}{m} \underline{S} \cdot \underline{B} \quad -(14)$$

where

$$x = \left(\frac{\hbar\omega}{mc^2}\right)^2 \left(1 + \frac{\hbar\omega}{mc^2}\right)^{-1} \xrightarrow[\hbar\omega \to mc^2]{} \frac{1}{2} \quad -(15)$$

The usual theory assumes:

$$\hbar\omega = mc^2 \quad -(16)$$

In eq. (14):

$$S_Z \phi = m_s \hbar \phi \quad -(17)$$

For a magnetic field aligned in the Z axis:

$$E = -2x \frac{\hbar e}{m} m_s B_Z \quad -(18)$$

where

$$m_s = \frac{1}{2} \text{ and } -\frac{1}{2} \quad -(19)$$

The ESR resonance frequency is

$$\omega_{ESR} = 2x \frac{e}{m} B_Z \quad -(20)$$

i.e.

$$\boxed{\omega_{ESR} = 2\left(\frac{\hbar\omega}{mc^2}\right)^2\left(1+\frac{\hbar\omega}{mc^2}\right)^{-1}\frac{e}{m}B_Z} \quad -(21)$$

This result is directly testable in a relativistic electron beam.

In the presence of a vacuum potential $\underline{A}_{vac}$, there is an additional magnetic flux density:

$$\underline{B}_{vac} = \underline{\nabla} \times \underline{A}_{vac} \quad -(22)$$

and the next note will apply the theory of 4FT 318 to this situation.
