336(2): Wavefunction of the Relativistic Electron, and Interaction with the Vacuum.

The classical equation of the relativistic electron is:

$$E^2 = p^2c^2 + m^2c^4 \quad -(1)$$

and the relativistic quantization is:

$$E\phi = i\hbar \frac{\partial \phi}{\partial t}, \quad -(2)$$

$$\underline{p}\phi = -i\hbar \underline{\nabla}\phi. \quad -(3)$$

so $$\hbar^2\left(-\frac{\partial^2}{\partial t^2} + c^2\nabla^2\right)\phi = m^2c^4\phi \quad -(4)$$

i.e. $$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\phi = 0 \quad -(5).$$

which is a limit of the ECE wave equation obtained from the tetrad postulate. The d'Alembertian is:

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \quad -(6)$$

The Compton wavenumber is:

$$\kappa_c = \frac{mc}{\hbar}. \quad -(7)$$

The de Broglie Einstein relations follow:

$$E = \hbar\omega = \gamma mc^2, \quad \underline{p} = \hbar\underline{\kappa} = \gamma\underline{p}_0 \quad -(8)$$

A solution of eq. (5) is:

a)

$$\psi = \psi_0 \exp\left(-i(\omega t - \kappa z)\right) \quad -(9)$$

From eqs. (5) and (9):

$$\left(-\frac{\omega^2}{c^2} + \kappa^2\right)\psi = -\left(\frac{mc}{\hbar}\right)^2 \psi \quad -(10)$$

i.e

$$\frac{\omega^2}{c^2} = \kappa^2 + \frac{m^2 c^2}{\hbar^2} \quad -(11)$$

so

$$\hbar^2 \omega^2 = c^2 \hbar^2 \kappa^2 + m^2 c^4 \quad -(12)$$

Eqs. (1), (8) and (12) are self consistent, Q.E.D.

Eq. (1) can be rewritten as:

$$E - mc^2 = \frac{p^2 c^2}{(1+\gamma)mc^2} = \frac{p^2 c^2}{E + mc^2} \quad -(13)$$

so the relativistic kinetic energy is:

$$T = (\gamma - 1)mc^2 = \frac{p^2}{m(1+\gamma)} \quad -(14)$$

In the classical non-relativistic limit:

$$v \ll c \quad -(15)$$

so

$$T \rightarrow \frac{1}{2}mv^2 = \frac{p_0^2}{2m} \quad -(16)$$

Q.E.D.

In the SU(2) basis:

3)

$$E - mc^2 = \frac{\underline{\sigma}\cdot\underline{p}\ \underline{\sigma}\cdot\underline{p}}{m\,(1+\gamma)} = \frac{(\underline{\sigma}\cdot\underline{p}\ \underline{\sigma}\cdot\underline{p})c^2}{E+mc^2} \quad -(17)$$

The interaction of the electron beam with the ECE vacuum is described by the relativistic minimal prescription:

$$p^\mu \rightarrow p^\mu - eA^\mu_{vac} \quad -(18)$$

where:

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right),\ A^\mu_{vac} = \left(\frac{\phi_{vac}}{c}, \underline{A}_{vac}\right) \quad -(19)$$

Therefore:

$$E \rightarrow E - e\phi_{vac} \quad -(20)$$

$$\underline{p} \rightarrow \underline{p} - e\underline{A}_{vac} \quad -(21)$$

If, in addition, an external magnetic field is applied, defined conventionally as:

$$\underline{B} = \underline{\nabla}\times\underline{A} \quad -(22)$$

then

$$E \rightarrow E - e\phi_{vac}, \quad -(23)$$

$$\underline{p} \rightarrow \underline{p} - e(\underline{A} + \underline{A}_{vac}) \quad -(24)$$

From the de Broglie / Einstein equations:

$$\gamma = \frac{\hbar\omega}{mc^2} \quad -(25)$$

In the usual assumption the quantization of Eq. (17) takes place as follows:

$$(E - mc^2)\phi = \frac{1}{m}\left(\underline{\sigma}\cdot(-i\hbar\underline{\nabla})\left(\frac{1}{1+\gamma}\,\underline{\sigma}\cdot\underline{p}\,\phi\right)\right) \quad -(26)$$

Now note that:

$$\underline{\nabla}\left(\frac{\hbar\omega}{mc^2}\right) = \underline{0} \quad -(27)$$

so

$$\underline{\nabla}\left(\frac{1}{1+\gamma}\right) = \underline{0} \quad -(28)$$

It follows that:

$$(E - mc^2)\phi = -\frac{i\hbar}{m(1+\gamma)}\,\underline{\sigma}\cdot\underline{\nabla}\,(\underline{\sigma}\cdot\underline{p}\,\phi)$$

$$= -\frac{i\hbar}{m(1+\gamma)}\left((\underline{\sigma}\cdot\underline{\nabla}\,\underline{\sigma}\cdot\underline{p})\phi + (\underline{\sigma}\cdot\underline{\nabla}\phi)\,\underline{\sigma}\cdot\underline{p}\right) \quad -(29)$$

where

$$\phi = \phi_0 \exp(-i(\omega t - \kappa Z)) \quad -(30)$$

and

$$\underline{p} = \gamma\,\underline{p}_0 \quad -(31)$$

Therefore:

$$\underline{\nabla}\phi = \frac{\partial\phi}{\partial Z}\,\underline{k} \quad -(32)$$

$$= i\kappa\phi\,\underline{k}$$

The real part of eq. (29) is therefore:

$$\mathrm{Re}\,(E - mc^2)\,\phi = \frac{\hbar\kappa}{m(1+\gamma)}\left(\underline{\sigma}\cdot\underline{k}\;\underline{\sigma}\cdot\underline{p}\right)\phi \quad -(33)$$

The expectation value of this equation is:

$$E - mc^2 = \frac{\hbar\kappa\,\underline{p}\cdot\underline{k}}{m(1+\gamma)} = \frac{\hbar\kappa\,p_z}{m(1+\gamma)} \quad -(34)$$

Using:

$$p_z = \hbar\kappa \quad -(35)$$

and

$$m(1+\gamma) = (E + mc^2)/c^2 \quad -(36)$$

eq. (34) becomes, self consistently:

$$E - mc^2 = \frac{c^2 p_z^2}{E + mc^2} \quad -(37)$$

Q.E.D.

The imaginary part of Eq. (29) is:

$$\mathrm{Im}\,(E - mc^2)\,\phi = -\frac{i\hbar}{m(1+\gamma)}\left(\underline{\sigma}\cdot\underline{\nabla}\;\underline{\sigma}\cdot\underline{p}\right)\phi \quad -(38)$$

where

$$\underline{\sigma}\cdot\underline{\nabla}\;\underline{\sigma}\cdot\underline{p} = \underline{\nabla}\cdot\underline{p} + i\,\underline{\sigma}\cdot\underline{\nabla}\times\underline{p} \quad -(39)$$

This gives a real part:

$$(E - mc^2)\,\phi = \frac{\hbar}{m(1+\gamma)}\,\underline{\sigma}\cdot\underline{\nabla}\times\underline{p}\,\phi \quad -(40)$$

where the spin angular momentum of the electron is:

$$\underline{S} = \frac{\hbar}{2} \underline{\sigma} \quad -(41)$$

For an electron matter wave moving in Z:

$$\underline{p} = p_Z \underline{k} \quad -(42)$$

so

$$\underline{\nabla} \times \underline{p} = \underline{0}. \quad -(43)$$

However, in the presence of an external magnetic field $\underline{B}$ applied to the relativistic electron beam:

$$\underline{p} \rightarrow \underline{p} - e\underline{A} \quad -(44)$$

and

$$(E - mc^2)\phi = -\frac{e\hbar}{m(1+\gamma)} \underline{\sigma} \cdot \underline{\nabla} \times \underline{A} \, \phi$$

$$= -\frac{e\hbar}{m(1+\gamma)} \underline{\sigma} \cdot \underline{B} \, \phi \quad -(45)$$

This term gives rise to electron spin resonance in the relativistic electron beam as follows:

$$\boxed{(E - mc^2)\phi = -\frac{2e}{m(1+\gamma)} \underline{S} \cdot \underline{B} \, \phi} \quad -(46)$$

where

$$\gamma = \frac{\hbar\omega}{mc^2} \quad -(47)$$

and ω is the angular frequency of the electron wave.

) For an external magnetic field aligned in the same direction, Z, as the electron beam:

$$\left(E - mc^2\right)\phi = -\frac{2eB_z}{m\left(1+\frac{\hbar\omega}{mc^2}\right)} S_z\phi \quad -(48)$$

$$= -\frac{2e\hbar B_z}{m\left(1+\frac{\hbar\omega}{mc^2}\right)} m_s\phi$$

where $m_s = \pm 1/2$.

The ESR resonance frequency is:

$$\boxed{\omega_{ESR} = \frac{2eB_z}{m\left(1+\frac{\hbar\omega}{mc^2}\right)}} \quad -(49).$$

In the non-relativistic limit:

$$\hbar\omega \rightarrow mc^2 \quad -(50)$$

so

$$\omega_{ESR} \rightarrow \frac{eB_z}{m} \quad -(51)$$

which is the usual result, QED.

Eq. (49) is a rigorous test of relativistic quantum mechanics and the de Broglie / Einstein equations. It is also a rigorous test of the quantization condition:

$$p^\mu = i\hbar \partial^\mu \quad -(52)$$

It is implicitly assumed but rarely made clear that the quantization condition refers to the relativistic four momentum p^μ. This is an axiom, or assumption, of quantum mechanics.

<u>The Effect of the Vacuum Potential</u>

This is to change eq. (45) to:

$$(E - mc^2)\phi = -\frac{e\hbar}{m(1+\gamma)} \underline{\sigma} \cdot \underline{\nabla} \times (\underline{A} + \underline{A}_{vac}) \phi \quad -(53)$$

so:

$$(E - mc^2)\phi = -\frac{2e}{m(1+\gamma)} \underline{S} \cdot (\underline{B} + \underline{\nabla} \times \underline{A}_{vac}) \quad -(54)$$

and the ESR resonance frequency is:

$$\boxed{\omega_{ESR} = \frac{2e}{m\left(1 + \frac{\hbar\omega}{mc^2}\right)} \left(B_Z + (\underline{\nabla} \times \underline{A}_{vac})_Z\right) \quad -(55)}$$

So the vacuum potential has an effect on the ESR resonance frequency. This demonstrates the existence of energy from spacetime.