73(2): Analytical Orbit from the ECE2 Lagrangian

The use of the relativistic Lagrangian:

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U \quad -(1)$$

$$= -mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2} - U$$

means that the hamiltonian becomes:

$$H = \gamma mc^2 + U \quad -(2)$$

$$= mc^2\left(\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right) + U \quad -(3)$$

The non-relativistic kinetic energy is increased from

$$T_0 = \frac{1}{2}mv^2 \quad -(4)$$

to

$$T = (\gamma-1)mc^2 \quad -(5)$$

Using the binomial expansion:

$$(1-x)^{-1/2} = 1 + \frac{x}{2} + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots \quad -(6)$$

the relativistic kinetic energy (5) can be expressed as:

$$T = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \frac{5}{16}\left(\frac{v^2}{c^2}\right)^3 - 1\right)$$

$$= \frac{1}{2}mv^2\left(1 + \frac{3}{4}\frac{v^2}{c^2} + \frac{5}{8}\left(\frac{v^2}{c^2}\right)^2 + \cdots\right) \quad -(7)$$

$$\xrightarrow[v \ll c]{} \frac{1}{2}mv^2$$

Q.E.D.

) Therefore the use of a relativistic lagrangian increases the velocity as follows:

$$v_1^2 = v^2\left(1 + \frac{3}{4}\frac{v^2}{c^2} + \frac{5}{8}\left(\frac{v^2}{c^2}\right)^2 + \cdots\right) \quad -(8)$$

The Newtonian velocity v corresponds to:

$$H = \frac{1}{2}mv^2 + U \quad -(9)$$

$$\mathcal{L} = \frac{1}{2}mv^2 - U \quad -(10)$$

and

$$r = \frac{\alpha}{1+\epsilon\cos\phi}, \quad -(11)$$

with

$$v^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right). \quad -(12)$$

Here α is the half right latitude, ϵ is the eccentricity and a is the semi major axis of the ellipse (11):

$$a = \frac{\alpha}{1-\epsilon^2} \quad -(13)$$

Therefore the orbital velocity in the relativistic theory increases to:

$$v_1^2 = MG\left(\frac{2}{r} - \frac{1}{a}\right)\left(1 + \frac{3}{4}\frac{MG}{c^2}\left(\frac{2}{r} - \frac{1}{a}\right) + \cdots\right) \quad -(14)$$

Assume that:

$$v_1^2 = \frac{2}{r_1} - \frac{1}{a} = \left(\frac{2}{r} - \frac{1}{a}\right)\left(1 + \frac{3}{4}\frac{MG}{c^2}\left(\frac{2}{r} - \frac{1}{a}\right) + \cdots\right) \quad -(15)$$

It follows that:

$$\frac{2}{r_1} - \frac{1}{a} = \frac{2}{r} - \frac{1}{a} + \frac{3}{4}\frac{MG}{c^2}\left(\frac{2}{r} - \frac{1}{a}\right)^2 \quad -(16)$$

so the new orbit is:

$$\boxed{\frac{1}{r_1} = \frac{1}{r} + \frac{3}{8}\frac{MG}{c^2}\left(\frac{2}{r} - \frac{1}{a}\right)^2} \quad -(17)$$

It is known from numerical analysis that this is a precessing orbit. Therefore:

$$\frac{1}{r_1} = \frac{1}{\alpha}\left(1+\epsilon\cos\phi\right) + \frac{3}{8}\frac{MG}{c^2}\left(\frac{2}{\alpha}\left(1+\epsilon\cos\phi\right) - \frac{1}{a}\right)^2 \quad -(18)$$

Experimentally, a revolution of 2π radians is accompanied by an increase in ϕ of:

$$\Delta\phi = \frac{3MG}{c^2\alpha} \quad -(19)$$

By computer algebra, the effective potential is defined by:

$$F = -\frac{\partial U}{\partial r} = -\frac{MGm}{r^2}\left(1-\frac{v^2}{c^2}\right)^{1/2}\left(1-\left(\frac{\dot{r}}{c}\right)^2\right) \quad -(20)$$

and for $v \ll c$ reduces to:

$$F = -\frac{mMG}{r^2} \quad -(21)$$