373(3): Analytical Expression for the Orbit of the ECE2 Lagrangian

The hamiltonian is:

$$H_1 = \frac{1}{2} m v^2 \left(1 + \frac{3}{4} \frac{v^2}{c^2} + \frac{5}{8} \left(\frac{v^2}{c^2} \right)^2 + \dots \right) + U \quad -(1)$$

where

$$U = - \frac{mMG}{r} \quad -(2)$$

is the gravitational potential. To first order in v^2/c^2:

$$H_1 = \frac{1}{2} m v^2 + U + \frac{3}{8} m \frac{v^4}{c^2} + \dots \quad -(2)$$

where

$$v^2 = MG \left(\frac{2}{r} - \frac{1}{a} \right) \quad -(3)$$

where a is the semi major axis of the ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos \phi}, \quad -(4)$$

i.e.

$$a = \frac{\alpha}{1 - \epsilon^2}. \quad -(5)$$

Here α is the half right latitude and ϵ is the eccentricity. The plane polar coordinates (r, ϕ) have been used. The mass m orbits a mass M, separated by a distance r. Here G is Newton's constant.

Now note that:

$$H := \frac{1}{2} m v^2 - \frac{mMG}{r} \quad -(6)$$

$$= \frac{1}{2} mMG \left(\frac{2}{r} - \frac{1}{a} \right) - \frac{mMG}{r}$$

$$= - \frac{mMG}{2a}$$

so:

$$H_1 = -\frac{mMG}{2a} + \frac{3}{8} m \left(\frac{MG}{c}\right)^2 \left(\frac{2}{r} - \frac{1}{a}\right)^2 \quad -(7)$$

where H_1 is a constant of motion by definition. The hamiltonian is therefore:

$$H_1 = -\frac{mMG}{2a} + \frac{3}{8} m \left(\frac{MG}{c}\right)^2 \left(\frac{2}{r} - \frac{1}{a}\right)^2$$

$$:= -\frac{mMG}{2a} + A \quad -(8)$$

where

$$A := \frac{3}{8} m \left(\frac{MG}{c}\right)^2 \left(\frac{2}{r} - \frac{1}{a}\right)^2 \quad -(9)$$

$$= \frac{3}{8} m \left(\frac{MG}{c}\right)^2 \left(\frac{2}{\alpha}\left(1+\epsilon\cos\phi\right) - \frac{1}{a}\right)^2$$

In eq. (8):

$$a = \frac{\alpha}{1-\epsilon^2} = \left(\frac{1+\epsilon\cos\phi}{1-\epsilon^2}\right)\alpha \quad -(10)$$

so in eq. (8):

$$H_1 = -\frac{mMG}{2r}\left(\frac{1-\epsilon^2}{1+\epsilon\cos\phi}\right) + A \quad -(11)$$

i.e.

$$\frac{mMG}{2r}\left(\frac{1-\epsilon^2}{1+\epsilon\cos\phi}\right) = A - H_1 \quad -(12)$$

and

$$\boxed{r = \frac{mMG(1-\epsilon^2)}{2(A-H_1)(1+\epsilon\cos\phi)}} \quad -(13)$$

where:

$$A = \frac{3}{8} m \left(\frac{MG}{c}\right)^2 \left(\frac{2}{\alpha}(1+\epsilon\cos\phi) - \frac{1}{a}\right)^2 \quad -(14)$$

Since A is of order $1/c^2$ it is a small correction. In the non relativistic limit:

$$A \rightarrow 0 \quad -(15)$$

so

$$r \rightarrow -\frac{mMG(1-\epsilon^2)}{2H(1+\epsilon\cos\phi)} \quad -(16)$$

because

$$H_1 \rightarrow H. \quad -(17)$$

Using

$$H = -\frac{mMG}{2a} \quad -(18)$$

and

$$\alpha = a(1-\epsilon^2) \quad -(19)$$

it follows that eq. (16) is:

$$r = \frac{\alpha}{1+\epsilon\cos\phi} \quad -(20)$$

Q.E.D.

4) From eqs. (13) and (14):

$$r = \frac{mMG(1-\epsilon^2)}{2\left(\frac{3}{8}m\left(\frac{mG}{c}\right)^2\left(\frac{2}{\alpha}(1+\epsilon\cos\phi)-\frac{1}{a}\right)^2-H_1\right)(1+\epsilon\cos\phi)} \quad (21)$$

in which H_1 is a constant that can be found by comparison with experimental data.

To an excellent approximation:

$$H_1 \sim H = -\frac{mMG}{2a}. \quad (22)$$

All the other quantities are known from astronomy for a given orbit: α, ϵ, a and MG.

Graphical Work

The orbit (21) can be graphed as a plot of r against ϕ. Experimentally the orbit is a precessing ellipse.