373(6): The Most Fundamental Expression for the Relativistic Orbit

The nonrelativistic hamiltonian is:

$$H = \frac{1}{2} m v^2 + U \quad -(1)$$

and the relativistic hamiltonian is:

$$H_0 = H - mc^2 = (\gamma - 1) mc^2 + U \quad -(2)$$

Therefore:

$$\boxed{(\gamma - 1) mc^2 - \frac{1}{2} m v^2 = H_0 - H} \quad -(3)$$

where $H_0 - H$ is a constant of motion. Therefore:

$$\left(\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right) mc^2 - \frac{1}{2} m v^2 = H_0 - H \quad -(4)$$

In Eq. (4), v is the Newtonian orbital velocity:

$$v^2 = \left(\frac{L}{mr} \right)^2 \left(1 + \frac{1}{r^2} \left(\frac{dr}{d\phi} \right)^2 \right) \quad -(5)$$

where L is a constant of motion:

$$L = mr^2 \frac{d\phi}{dt} \quad -(6)$$

Using eqs. (4) and (5) gives the relativistic $\left(\frac{dr}{d\phi} \right)^2$ in terms of the constants $H_0 - H$ and L.

This solves the problem of finding an

) analytical solution for the relativistic $(dr/d\phi)^2$,

Q.E.D.

Computer algebra can be used to find an analytical expression for $(dr/d\phi)^2$ from eqns. (4) and (5). In the limit:

$$v << c \quad -(7)$$

it must reduce to:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\epsilon^2 r^4}{\alpha^2}\sin^2\phi = \frac{\epsilon^2 r^4}{\alpha^2}\left(1-\frac{1}{\epsilon^2}\left(\frac{\alpha}{r}-1\right)^2\right) \quad -(8)$$

The function $(dr/d\phi)^2$ from eqs (4) and (5) is known from numerical analysis to be due to a precessing ellipse.

In the limit (7):

$$H_0 \longrightarrow H \quad -(9)$$
