374(2): Lagrangian and Hamiltonian Formulation of UFT363.

Define the Hamiltonian as:

$$H = \frac{1}{2m}\underline{p}\cdot\underline{p} - \frac{mMG}{r} \quad -(1)$$

and the Lagrangian as

$$\mathcal{L} = \frac{1}{2m}\underline{p}\cdot\underline{p} + \frac{mMG}{r} \quad -(2)$$

Lagrangian dynamics lead to:

$$\underline{p} = \frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} \quad -(3)$$

and

$$\dot{\underline{p}} = \frac{\partial \mathcal{L}}{\partial \underline{r}} \quad -(4)$$

The Hamiltonian is defined as:

$$H = \underline{p}\cdot\dot{\underline{r}} - \mathcal{L} \quad -(5)$$

and the Hamilton equations are:

$$\dot{\underline{r}} = \frac{\partial H}{\partial \underline{p}} \quad -(6)$$

and

$$\dot{\underline{p}} = -\frac{\partial H}{\partial \underline{r}} \quad -(7)$$

In classical dynamics in plane polar coordinates:

$$\dot{\underline{r}} = \dot{r}\,\underline{e}_r + r\,\dot{\underline{e}}_r \quad -(8)$$

$$= \dot{r}\,\underline{e}_r + r\dot{\phi}\,\underline{e}_\phi$$

$$= \underline{v} \quad -(9)$$

So $$H = m\underline{v}\cdot\underline{v} - \mathcal{L} = \frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}} - \frac{mMG}{r}$$

From eqs. (2) and (3):

$$\underline{p} = \frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} = m\dot{\underline{r}} . \quad -(10)$$

The Euler Lagrange equation is:

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}}\right) \quad -(11)$$

where

$$\mathcal{L} = \frac{1}{2m}\dot{\underline{r}}\cdot\dot{\underline{r}} + \frac{mMG}{|\underline{r}|} \quad -(12)$$

Therefore

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}}\right) = m\ddot{\underline{r}} \quad -(13)$$

Note that:

$$\frac{1}{r} = \frac{1}{|\underline{r}|} = \frac{1}{(\underline{r}\cdot\underline{r})^{1/2}} \quad -(14)$$

So:

$$\frac{\partial}{\partial \underline{r}}\left((\underline{r}\cdot\underline{r})^{-1/2}\right) = -\frac{1}{2}(\underline{r}\cdot\underline{r})^{-3/2}(2\underline{r})$$

$$= -\frac{1}{r^2}\underline{e}_r \quad -(15)$$

where we have used:

$$\underline{r} = r\,\underline{e}_r \quad -(16)$$

Therefore

$$\frac{\partial \mathcal{L}}{\partial \underline{r}} = -\frac{mMG}{r^2}\underline{e}_r \quad -(17)$$

Therefore eq. (11) gives:

3)

$$\underline{F} = m\ddot{\underline{r}} = -\frac{mMG}{r^2}\underline{e}_r \quad -(18)$$

In classical dynamics:

$$\ddot{\underline{r}} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta\right) - (19)$$

$$= \ddot{r}\underline{e}_r + \dot{r}\dot{\underline{e}}_r + \dot{r}\dot{\theta}\underline{e}_\theta + r\ddot{\theta}\underline{e}_\theta + r\dot{\theta}\dot{\underline{e}}_\theta$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\underline{e}_\theta$$

Fm eqs. (18) and (19):

$$\ddot{r} - r\dot{\theta}^2 = -\frac{mMG}{r^2} \quad -(20)$$

and

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \quad -(21)$$

Eq. (20) is the Leibnitz equation. It can be obtained from the Lagrangian (2), which is:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{mMG}{r} \quad -(22)$$

using the Lagrange variables r and θ, so:

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} \quad -(23)$$

and

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} \quad -(24)$$

Eq. (23) gives eq. (20), and eq. (24)

gives

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{L}{mr^2} \quad -(25)$$

where L is the constant angular momentum. However, eqs. (22) to (24) do not give eq. (21).

The complete analysis therefore needs eq. (11) and the lagrangian (12).

This realization is important in fluid dynamics. In UFT363 it was inferred that:

$$\underline{v} = \dot{\underline{r}} = x\dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(26)$$

where

$$x = 1 + \frac{\partial R_r}{\partial r}. \quad -(27)$$

Therefore:

$$\ddot{\underline{r}} = x\ddot{r}\,\underline{e}_r + x\dot{r}\,\dot{\underline{e}}_r + \dot{r}\dot{\theta}\,\underline{e}_\theta + r\ddot{\theta}\,\underline{e}_\theta + r\dot{\theta}\,\dot{\underline{e}}_\theta$$

$$= \left(x\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left((x+1)\dot{r}\dot{\theta} + r\ddot{\theta}\right)\underline{e}_\theta \quad -(28)$$

where it has been assumed that x is independent of time. If x depends on time then additional terms appear in Eq. (28). It follows that:

$$\underline{F} = m\ddot{\underline{r}} = m\left(\left(x\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left((x+1)\dot{r}\dot{\theta} + r\ddot{\theta}\right)\underline{e}_\theta\right)$$

$$= -\frac{mMG}{r^2}\,\underline{e}_r \quad -(29)$$

Therefore:

5)

$$x\ddot{r} - r\dot{\theta}^2 = -\frac{MG}{r^2} \quad -(30)$$

and

$$(x+1)\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \quad -(31)$$

These equation reduce to classical dynamics if

$$x \rightarrow 1 \quad -(32)$$

Therefore:

$$\boxed{\begin{aligned} \left(1 + \frac{\partial R_r}{\partial r}\right)\ddot{r} &= r\dot{\theta}^2 - \frac{MG}{r^2} \quad -(33) \\ \left(2 + \frac{\partial R_r}{\partial r}\right)\dot{r}\dot{\theta} + r\ddot{\theta} &= 0 \quad -(34) \end{aligned}}$$

and

Note carefully that eqs. (33) and (34) are obtained from the Lagrangian (2) and the Euler Lagrange equation (11). This is the correct procedure because it correctly gives the momentum from the Lagrangian, using eq. (10).

If the Lagrangian is defined as:

$$\mathcal{L} = \frac{1}{2} m \left(\left(1 + \frac{\partial R_r}{\partial r}\right)^2 \dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{mMG}{r} \quad -(35)$$

it must be re-expressed as:

$$\mathcal{L} = \frac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}} + \frac{mMG}{r} \quad -(36)$$

6) in order to obtain the correct momentum:

$$\underline{p} = \frac{\partial \mathcal{L}}{\partial \underline{\dot{r}}} = m \underline{\dot{r}} \quad -(37)$$

The use of this correct momentum leads to eqs. (33) and (34). It is not possible to chose [illegible] as because the equation of Lagrange variables in eq. (35),

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \quad -(38)$$

leads to:

$$\left(1 + \frac{\partial R_r}{\partial r}\right)^2 \ddot{r} = r\dot{\theta}^2 - \frac{MG}{r^2} \quad -(39)$$

whereas the correct equation is Eq. (33). the θ variable can still be used, so

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \quad -(40)$$

is still valid. This gives:

$$\dot{\theta} = \frac{L}{mr^2} \quad -(41)$$

The reason for this is that the angular part of the lagrangian:

$$\mathcal{L}_\theta = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{mMG}{r} \quad -(42)$$

is the same in classical dynamics and fluid dynamics. Therefore there are three equations in three unknowns:

$$\left(1 + \frac{\partial R_r}{\partial r}\right)\ddot{r} = r\dot{\theta}^2 - \frac{MG}{r^2} \quad -(43)$$

$$\left(2 + \frac{\partial R_r}{\partial r}\right)\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \quad -(44)$$

and

$$\dot{\theta} = \frac{L}{mr^2} \quad -(45)$$

here L is a constant of motion.

The above three equations can be solved for $\dot{r}$, $\dot{\theta}$ and $\partial R_r / \partial r$. The differential orbital function is:

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} \quad -(46)$$

and the orbit is

$$r = \int \frac{dr}{d\theta}\, d\theta. \quad -(47)$$

Various equations of the fluid spacetime may now be added to eqs. (43) to (45), namely the continuity equation (conservation of matter), the Navier-Stokes equation, the conservation of entropy per unit mass, and the conservation of angular momentum.

Conclusions

The complete range of Lagrangian and Hamiltonian dynamics, eqs. (1) to (12), is needed for the problem.

The Lagrange variables are $\underline{r}$ and $\underline{r}$ g in eq. (11), i.e. the proper Lagrange variable must be $\underline{r}$.

The angle θ may also be a proper Lagrange variable.