376(3): Self Consistent Equations of ECE2 Gravitation Regarded as Gravitostatics.

The lagrangian is:

$$\mathcal{L} = -\frac{mc^2}{\gamma} - U \qquad -(1)$$

$$= -mc^2\left(1-\frac{\dot{\underline{r}}\cdot\dot{\underline{r}}}{c^2}\right)^{1/2} + \frac{mMG}{|\underline{r}|}.$$

The relativistic momentum is:

$$\underline{p} = \gamma m \underline{v} = \frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} = m\frac{d\underline{v}}{d\tau} \qquad -(2)$$

where $\tau$ is the proper time.

The Minkowski force equation is:

$$\underline{F} = m\frac{d\underline{v}}{d\tau} \qquad -(3)$$

where

$$\underline{v} = \gamma \underline{v}_0 \qquad -(4)$$

and where

$$\underline{v}_0 = \frac{d\underline{r}}{dt} \qquad -(5)$$

is the non relativistic velocity. Therefore:

$$F = \gamma m \frac{dv}{dt} = \gamma m \frac{d}{dt}(\gamma v_0) \qquad -(6)$$

$$= \gamma m\left(v_0\frac{d\gamma}{dt} + \gamma\frac{dv_0}{dt}\right)$$

where

$$\frac{d\gamma}{dt} = \frac{d\gamma}{dv_0}\frac{dv_0}{dt} \qquad -(7)$$

So

$$F = \gamma m \frac{dv_0}{dt}\left(v_0\frac{d\gamma}{dv_0} + \gamma\right) \qquad -(8)$$

2) where:

$$\frac{d\gamma}{dv_0} = \frac{d}{dv_0}\left(1-\frac{v_0^2}{c^2}\right)^{-1/2} = \frac{v_0}{c^2}\left(1-\frac{v_0^2}{c^2}\right)^{-3/2} = \gamma^3\frac{v_0}{c^2} \quad -(9)$$

So

$$F = \gamma m\frac{dv_0}{dt}\left(\gamma^3\frac{v_0^2}{c^2}+\gamma\right) \quad -(10)$$

$$= \gamma^2 m\frac{dv_0}{dt}\left(1+\gamma^2\frac{v_0^2}{c^2}\right)$$

where

$$\frac{v_0^2}{c^2} = 1-\frac{1}{\gamma^2} \quad -(11)$$

So

$$\underline{F} = \gamma^4 m\frac{d\underline{v}_0}{dt} = \gamma^4 m\frac{d^2\underline{r}}{dt^2}. \quad -(12)$$

The Minkowski force equation (12) is obtained from the ECE2 lagrangian (1) using the Euler Lagrange equation:

$$\frac{\partial\mathcal{L}}{\partial\underline{r}} = \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\underline{r}}} \quad -(13)$$

in which the proper time $\tau$ appears on the right hand side. From eqs. (2) and (13):

$$\underline{F} = \frac{\partial\mathcal{L}}{\partial\underline{r}} = -mMG\frac{\underline{r}}{r^3} = m\frac{d\underline{v}}{d\tau} \quad -(14)$$

Therefore:

$$\underline{F} = \gamma^4 m\frac{d^2\underline{r}}{dt^2} = -mMG\frac{\underline{r}}{r^3} \quad -(15)$$

3) Eq. (15) is the correctly relativistic force equation of ECE2 gravitostatics.

In Cartesian coordinates:

$$\ddot{X} = -MG\left(1 - \frac{\dot{X}^2 + \dot{Y}^2}{c^2}\right)^2 \frac{X}{(X^2+Y^2)^{3/2}} \quad -(16)$$

$$\ddot{Y} = -MG\left(1 - \frac{\dot{X}^2 + \dot{Y}^2}{c^2}\right)^2 \frac{Y}{(X^2+Y^2)^{3/2}} \quad -(17)$$

Eqs. (16) and (17) when solved simultaneously give a precessing elliptical orbit.

In addition to the above equations, the gravitostatic field equations of ECE2 covariant relativity are automatically relativistic and are:

$$\underline{\nabla} \times \underline{g} = \underline{0} \quad -(18)$$

$$\underline{\nabla} \cdot \underline{g} = \underline{\kappa} \cdot \underline{g} = 4\pi G \rho_m = \frac{4\pi MG}{r^3} \quad -(19)$$

$$\frac{\partial \underline{g}}{\partial t} = \underline{0} \quad -(20)$$

and from the continuity equation in the absence of $\underline{\Omega}$:

$$\frac{\partial}{\partial t}\left(\underline{\kappa} \cdot \underline{g}\right) = 0 \quad -(21)$$

Eq. (18) gives:

$$\frac{\partial g_Y}{\partial X} - \frac{\partial g_X}{\partial Y} = 0 \quad -(22)$$

4) Eq. (19) gives:

$$\frac{\partial g_x}{\partial X} + \frac{\partial g_Y}{\partial Y} = \kappa_x g_x + \kappa_Y g_Y = \frac{4\pi M G}{(X^2+Y^2)^{3/2}} \quad -(23)$$

Eq. (20) gives:

$$\frac{\partial g_x}{\partial t}\underline{i} + \frac{\partial g_Y}{\partial t}\underline{j} = \underline{0} \quad -(24)$$

i.e.

$$\left(\left(\frac{\partial g_x}{\partial t}\right)^2 + \left(\frac{\partial g_Y}{\partial t}\right)^2\right) = 0 \quad -(25)$$

Eq. (21) gives:

$$\frac{\partial}{\partial t}\left(\kappa_x g_x + \kappa_Y g_Y\right) = 0 \quad -(26)$$

In these equations:

$$\underline{F} = m\underline{g} = \gamma^4 m \frac{d^2\underline{r}}{dt^2} \quad -(27)$$

So

$$g_x = \left(1 + \frac{\dot{X}^2+\dot{Y}^2}{c^2}\right)^{-2}\ddot{X} \quad -(28)$$

$$g_Y = \left(1 + \frac{\dot{X}^2+\dot{Y}^2}{c^2}\right)^{-2}\ddot{Y} \quad -(29)$$

Finally use:

$$\frac{\partial g_Y}{\partial X} = \frac{\partial g_Y}{\partial t}\frac{\partial t}{\partial X} = \frac{1}{\dot{X}}\frac{\partial g_Y}{\partial t} \quad -(30)$$

$$\frac{\partial g_x}{\partial Y} = \frac{\partial g_x}{\partial t}\frac{\partial t}{\partial Y} = \frac{1}{\dot{Y}}\frac{\partial g_x}{\partial t} \quad -(31)$$

to find that eq. (22) becomes:

$\Rightarrow$)

$$\frac{1}{\dot{x}}\frac{dg_y}{dt} - \frac{1}{\dot{y}}\frac{dg_x}{dt} = 0 \quad -(32)$$

i.e. $$\dot{x}\frac{dg_x}{dt} = \dot{y}\frac{dg_y}{dt} \quad -(33)$$

Similarly, Eq. (23) becomes:

$$\frac{1}{\dot{x}}\frac{dg_x}{dt} + \frac{1}{\dot{y}}\frac{dg_y}{dt} = \kappa_x g_x + \kappa_y g_y = \frac{4\pi m G}{(x^2+y^2)_v^{3/2}} \quad -(34)$$

Complete Set of Equations

$$g_x = \left(1 + \frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{-2}\ddot{x} \quad -(35)$$

$$g_y = \left(1 + \frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{-2}\ddot{y} \quad -(36)$$

$$\ddot{x} = -mG\left(1 - \frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^2 \frac{x}{(x^2+y^2)^{3/2}} \quad -(37)$$

$$\ddot{y} = -mG\left(1 - \frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^2 \frac{y}{(x^2+y^2)^{3/2}} \quad -(38)$$

$$\dot{x}\frac{dg_x}{dt} = \dot{y}\frac{dg_y}{dt} \quad -(39)$$

$$\left(\frac{dg_x}{dt}\right)^2 + \left(\frac{dg_y}{dt}\right)^2 = 0 \quad -(40)$$

$$\frac{1}{\dot{x}}\frac{dg_x}{dt} + \frac{1}{\dot{y}}\frac{dg_y}{dt} = \kappa_x g_x + \kappa_y g_y = \frac{4\pi m G}{(x^2+y^2)_v^{3/2}} \quad -(41)$$

$$\frac{d}{dt}\left(\kappa_x g_x + \kappa_y g_y\right) = 0 \quad -(42)$$

5) There are eight equations in all. These can be solved in various ways by computer.

1) The force equations are eqs. (37) and (38).

2) The field equations are eqs. (39) to (42).

a) Eqs. (37) and (38) can be solved simultaneously to produce a precessing elliptical orbit.

b) Eqs. (39) and (40) can be solved simultaneously to find $dg_x/dt$ and $dg_y/dt$, in terms of $\dot{x}$ and $\dot{y}$.

c) Eq. (41) must be interpreted as

$$\rho_m = \frac{m}{V} = \frac{m}{(x^2+y^2)^{3/2}_V} = \text{constant}. \quad -(43)$$

This is consistent with the fact that from eqs. (41) and (42):

$$\frac{d}{dt}\left(4\pi G\rho_m\right) = 0 \quad -(44)$$

i.e. the source mass density $\rho_m$ does not change with time in gravitostatics.

In order to obtain gravitational radiation a time independent source is needed. A time independent source does not radiate.

the gravitomagnetic field

d) The field equations simplify when eqs. (41) and (42) are solved to give:

$$\frac{d}{dt}\left(\frac{1}{\dot{x}}\frac{dg_x}{dt} + \frac{1}{\dot{y}}\frac{dg_y}{dt}\right) = 0 \quad -(45)$$

7) and when Eqs. (39) and (40) are solved to give:

$$\dot{X}^2 \left(\frac{\partial g_x}{\partial t}\right)^2 = \dot{Y}^2 \left(\frac{\partial g_Y}{\partial t}\right)^2 \quad -(43)$$

i.e.

$$\left(\dot{X}^2 - \dot{Y}^2\right)\left(\frac{\partial g_x}{\partial t}\right)^2 = 0 \quad -(44)$$

Therefore the field equations reduce to:

$$\frac{\partial}{\partial t}\left(\frac{1}{\dot{X}}\frac{\partial g_x}{\partial t} + \frac{1}{\dot{Y}}\frac{\partial g_Y}{\partial t}\right) = 0 \quad -(45)$$

$$\left(\dot{X}^2 - \dot{Y}^2\right)\left(\frac{\partial g_x}{\partial t}\right)^2 = 0 \quad -(46)$$

and

$$\left(\frac{\partial g_x}{\partial t}\right)^2 = -\left(\frac{\partial g_Y}{\partial t}\right)^2 \quad -(47)$$

Therefore:

$$\left(\frac{\partial g_x}{\partial t}\right)^2 = -\left(\frac{\partial g_Y}{\partial t}\right)^2 = 0 \quad -(48)$$

or

$$\dot{X}^2 = \dot{Y}^2 \quad -(49)$$

Neither eq. (48) nor eq. (49) is true in general.
Therefore the correct interpretation is:

$$\left(\frac{\partial g_Y}{\partial t}\right)^2 = \frac{\dot{X}^2}{\dot{Y}^2}\left(\frac{\partial g_x}{\partial t}\right)^2 \quad -(50)$$

so

$$\frac{\partial g_Y}{\partial t} = \pm \frac{\dot{X}}{\dot{Y}}\left(\frac{\partial g_x}{\partial t}\right) \quad -(51)$$

From eqs. (45) and (51):

$$\frac{\partial}{\partial t}\left(\frac{1}{\dot{X}}\frac{\partial g_x}{\partial t} \pm \frac{\dot{X}}{\dot{Y}^2}\left(\frac{\partial g_x}{\partial t}\right)\right) = 0 \quad -(52)$$

i.e.

$$\boxed{\frac{\partial}{\partial t}\left(\left(\frac{1}{\dot{X}} \pm \frac{\dot{X}}{\dot{Y}^2}\right)\left(\frac{\partial g_x}{\partial t}\right)\right) = 0} \quad -(53)$$

So the field equations reduce to eq. (53).

e) Eqs. (35) and (53) can be solved for $\partial g_x / \partial t$ in terms of $\dot{X}$ and $\dot{Y}$

f) Eqs. (36) and (53) can be solved for $\partial g_y / \partial t$ in terms of $\dot{X}$ and $\dot{Y}$.

g) Eqs (41) and (42) can be solved finally to give the spin connections $\kappa_x$ and $\kappa_y$.