580(3): Solutions for $\underline{\omega}$ and $\underline{Q}$

Consider the antisymmetry conditions:

$$(\partial_2+\omega_2)Q_3 = -(\partial_3+\omega_3)Q_2$$
$$(\partial_3+\omega_3)Q_1 = -(\partial_1+\omega_1)Q_3 \qquad -(1)$$
$$(\partial_1+\omega_1)Q_2 = -(\partial_2+\omega_2)Q_1$$

In Cartesian notation these become:

$$\left(\frac{\partial}{\partial Y}-\omega_Y\right)Q_Z = -\left(\frac{\partial}{\partial Z}-\omega_Z\right)Q_Y$$
$$\left(\frac{\partial}{\partial Z}-\omega_Z\right)Q_X = -\left(\frac{\partial}{\partial X}-\omega_X\right)Q_Z \qquad -(2)$$
$$\left(\frac{\partial}{\partial X}-\omega_X\right)Q_Y = -\left(\frac{\partial}{\partial Y}-\omega_Y\right)Q_X .$$

Using:

$$\underline{\Omega} = \underline{\nabla}\times\underline{Q} - \underline{\omega}\times\underline{Q} \qquad -(3)$$

and

$$\underline{\nabla}\cdot\underline{\Omega} = 0 \qquad -(4)$$

it is found that

$$\underline{\nabla}\cdot(\underline{\omega}\times\underline{Q}) = 0 \qquad -(5)$$

i.e.

$$\underline{Q}\cdot\underline{\nabla}\times\underline{\omega} = \underline{\omega}\cdot\underline{\nabla}\times\underline{Q} \qquad -(6)$$

i.e. $$Q_X\left(\frac{\partial\omega_Z}{\partial Y}-\frac{\partial\omega_Y}{\partial Z}\right)+Q_Y\left(\frac{\partial\omega_X}{\partial Z}-\frac{\partial\omega_Z}{\partial X}\right)+Q_Z\left(\frac{\partial\omega_Y}{\partial X}-\frac{\partial\omega_X}{\partial Y}\right)$$

$$= \omega_X\left(\frac{\partial Q_Z}{\partial Y}-\frac{\partial Q_Y}{\partial Z}\right)+\omega_Y\left(\frac{\partial Q_X}{\partial Z}-\frac{\partial Q_Z}{\partial X}\right)+\omega_Z\left(\frac{\partial Q_Y}{\partial X}-\frac{\partial Q_X}{\partial Y}\right)$$

$-(7)$

2) Eqs. (2) and (7) give five equations in six unknowns: $\dot{\omega}_x, \omega_y, \omega_z, Q_x, Q_y, Q_z$. One more equation at least is needed for a complete solution.

Using:

$$\underline{g} = -\bar{\nabla}\,\underline{\Phi} + \underline{\omega}\,\underline{\Phi} = -\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \quad -(8)$$

and

$$\underline{\nabla} \cdot \underline{g} = 4\pi G \rho_m \quad -(9)$$

it is found that:

$$\bar{\nabla} \cdot \left(\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \right) = -4\pi G \rho_m \quad -(10)$$

Now assume that:

$$\omega^\mu = (0, \underline{\omega}) \quad -(11)$$

So

$$\bar{\nabla} \cdot \frac{\partial \underline{Q}}{\partial t} = -4\pi G \rho_m \quad -(12)$$

i.e

$$\frac{\partial}{\partial t}\left(\bar{\nabla} \cdot \underline{Q} \right) = -4\pi G \rho_m \quad -(13)$$

or

$$\frac{\partial}{\partial t}\left(\frac{\partial Q_x}{\partial X} + \frac{\partial Q_y}{\partial Y} + \frac{\partial Q_z}{\partial Z} \right) = -4\pi G \rho_m \quad -(14)$$

<u>So there are six equations in six unknowns,</u> <u>$\omega_x, \omega_y, \omega_z$</u>, <u>$Q_x, Q_y, Q_z$</u>. These are eqs. (2),

3) (7) and (14).

Eq. (11) is similar to the concept of the so called radiatic gauge in electrodynamics:

$$A^\mu = (0, \underline{A}) \quad -(15)$$

In order to avoid the use of the radiatic gauge, use may be made of the lagrangian:

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{(X^2+Y^2+Z^2)^{1/2}} \quad -(16)$$

$$= -\frac{mc^2}{\gamma} + \frac{mMG}{|\underline{r}|}$$

in three dimensions. Here:

$$\gamma = \left(1 - \frac{v_0^2}{c^2}\right)^{-1/2} \quad -(17)$$

where

$$v_0^2 = \dot{\underline{r}} \cdot \dot{\underline{r}} = \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 \quad -(18)$$

There are two possible sets of Euler Lagrange equations.

1) $$\frac{\partial \mathcal{L}}{\partial \underline{r}} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\underline{r}}} \quad -(19)$$

which produces:

$$\underline{F} = \gamma^3 m \ddot{\underline{r}} = -mMG\frac{\underline{r}}{r^3} \quad -(20)$$

~~and~~ as in UFT377 and

2) $$\underline{g} = \ddot{\underline{r}} = \frac{MG}{\gamma r^3}\left(\frac{\dot{\underline{r}}(\dot{\underline{r}} \cdot \underline{r})}{c^2} - \underline{r}\right) \quad -(21)$$

4) Eq. (21) is obtained from:

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} \quad -(22)$$

$$\frac{\partial \mathcal{L}}{\partial Y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}} \quad -(23)$$

$$\frac{\partial \mathcal{L}}{\partial Z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Z}} \quad -(24)$$

and

$$\mathcal{L} = -mc^2\left(1 - \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}\right)^{1/2} + \frac{mMG}{(x^2+y^2+z^2)^{1/2}} \quad -(25)$$

Eq. (20) gives retrograde precession and eq. (21) gives forward precession.

Evaluating the Spin Connections

For retrograde precession:

$$\underline{g} = \gamma^3 \ddot{\underline{r}} = -MG\frac{\underline{r}}{r^3} = -\underline{\nabla}\,\underline{\Phi} + \underline{\omega}\,\underline{\Phi}$$

$$= -\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \quad -(26)$$

For forward precession:

$$\underline{g} = \ddot{\underline{r}} = \frac{MG}{\gamma r^3}\left(\frac{\dot{\underline{r}}(\dot{\underline{r}}\cdot\underline{r})}{c^2} - \underline{r}\right)$$

$$= -\underline{\nabla}\,\underline{\Phi} + \underline{\omega}\,\underline{\Phi} = -\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \quad -(27)$$

5) If the potential is approximated by:

$$\Phi \doteq -\frac{mG}{(x^2+y^2+z^2)^{1/2}} = -\frac{mG}{r} \quad - (28)$$

Then the three components of $\underline{\omega}$ can be found.

Eqs. (2) then give the three components of $\underline{Q}$.

Finally eq. (8) gives ω_0.

Knowing $\underline{Q}$ and $\underline{\omega}$ the gravitomagnetic field $\underline{\Omega}$ can be found from eq. (3').