80(4): Simultaneous Equations for $\underline{\omega}$ and $\underline{\Phi}$ and for $\underline{Q}$ and ω_0.

Retrograde Precession, $\underline{\omega}$ and $\underline{\Phi}$.

There are two equations in two variables, $\underline{\omega}$ and $\underline{\Phi}$:

$$\underline{g} = -\underline{\nabla}\,\underline{\Phi} + \underline{\omega}\,\underline{\Phi} = \gamma^3 \ddot{\underline{r}} = -m_1 G \frac{\underline{r}}{r^3} \quad -(1)$$

and

$$\underline{\nabla} \cdot \left(-\underline{\nabla}\,\underline{\Phi} + \underline{\omega}\,\underline{\Phi}\right) = 4\pi G \rho_m \quad -(2)$$

In these equations, γ, $\underline{r}$ and ρ_m are known, so $\underline{\Phi}$ and $\underline{\omega}$ can be found.

Forward Precession, $\underline{\omega}$ and $\underline{\Phi}$

In this case:

$$\underline{g} = \ddot{\underline{r}} = \frac{mG}{\gamma r^3}\left(\frac{\dot{\underline{r}}\,(\dot{\underline{r}} \cdot \underline{r})}{c^2} - \underline{r}\right) \quad -(3)$$

$$= -\underline{\nabla}\,\underline{\Phi} + \underline{\omega}\,\underline{\Phi}$$

and

$$\underline{\nabla} \cdot \left(-\underline{\nabla}\,\underline{\Phi} + \underline{\omega}\,\underline{\Phi}\right) = 4\pi G \rho_m \quad -(4)$$

Again there are two equations in two unknowns, $\underline{\omega}$ and $\underline{\Phi}$.

Retrograde Precession, $\underline{Q}$ and ω_0

There are two equations in two variables, Q and ω_0:

$$\underline{g} = -\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} = \gamma^3 \ddot{\underline{r}} = -mG\frac{\underline{r}}{r^3} \quad -(5)$$

and

$$\underline{\nabla} \cdot \left(-\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \right) = 4\pi G \rho_m \quad -(6)$$

<u>Forward Precession, $\underline{Q}$ and ω_0</u>

In this case:

$$\underline{g} = \ddot{\underline{r}} = \frac{mG}{\gamma r^3}\left(\frac{\dot{\underline{r}}(\dot{\underline{r}} \cdot \underline{r})}{c^2} - \underline{r} \right) \quad -(7)$$

$$= -\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q}$$

and

$$\underline{\nabla} \cdot \left(-\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \right) = 4\pi G \rho_m \quad -(8)$$

Additionally, these are the equations of Note 380(3). There is also the gravitational Faraday law of induction:

$$\underline{\nabla} \times \underline{g} + \frac{\partial \underline{\Omega}}{\partial t} = \underline{0} \quad -(9)$$

in which:

$$\underline{g} = -\underline{\nabla} \underline{\Phi} + \underline{\omega}\, \underline{\Phi} = -\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \quad -(10)$$

and

$$\underline{\Omega} = \underline{\nabla} \times \underline{Q} - \underline{\omega} \times \underline{Q} \quad -(11)$$

) Therefore:

$$\underline{\nabla} \times \left(-\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \right) + \frac{\partial}{\partial t} \left(\underline{\nabla} \times \underline{Q} - \underline{\omega} \times \underline{Q} \right) = \underline{0} \quad -(12)$$

i.e.

$$\boxed{\frac{\partial}{\partial t} \left(\underline{\omega} \times \underline{Q} \right) = \underline{\nabla} \times \left(\omega_0 \underline{Q} \right)} \quad -(13)$$

Eq. (13) provide another equation to solve for six unknowns, $\omega_X, \omega_Y, \omega_Z, Q_X, Q_Y, Q_Z$, plus ω_0. In Eq. (13):

$$\underline{\nabla} \times \left(\omega_0 \underline{Q} \right) = \omega_0 \underline{\nabla} \times \underline{Q} + \underline{\nabla} \omega_0 \times \underline{Q} \quad -(14)$$

So

$$\boxed{\frac{\partial}{\partial t} \left(\underline{\omega} \times \underline{Q} \right) = \omega_0 \underline{\nabla} \times \underline{Q} + \underline{\nabla} \omega_0 \times \underline{Q}} \quad -(15)$$

Comparing $\underline{i}$, $\underline{j}$ and $\underline{k}$ components of eq. (15) gives three equations.

$$\omega_Y Q_Z - \omega_Z Q_Y = \omega_0 \left(\frac{\partial Q_Z}{\partial Y} - \frac{\partial Q_Y}{\partial Z} \right) + Q_Z \frac{\partial \omega_0}{\partial Y} - Q_Y \frac{\partial \omega_0}{\partial Z} \quad -(16)$$

$$\omega_Z Q_X - \omega_X Q_Z = \omega_0 \left(\frac{\partial Q_X}{\partial Z} - \frac{\partial Q_Z}{\partial X} \right) + Q_X \frac{\partial \omega_0}{\partial Z} - Q_Z \frac{\partial \omega_0}{\partial X} \quad -(17)$$

$$\omega_X Q_Y - \omega_Y Q_X = \omega_0 \left(\frac{\partial Q_Y}{\partial X} - \frac{\partial Q_X}{\partial Y} \right) + Q_Y \frac{\partial \omega_0}{\partial X} - Q_X \frac{\partial \omega_0}{\partial Y}$$

) As in Note 380(3), there are three more antisymmetry equations:

$$\left(\frac{\partial}{\partial Y} - \omega_Y\right) Q_Z = -\left(\frac{\partial}{\partial Z} - \omega_Z\right) Q_Y \quad -(19)$$

$$\left(\frac{\partial}{\partial Z} - \omega_Z\right) Q_X = -\left(\frac{\partial}{\partial X} - \omega_X\right) Q_Z \quad -(20)$$

$$\left(\frac{\partial}{\partial X} - \omega_X\right) Q_Y = -\left(\frac{\partial}{\partial Y} - \omega_Y\right) Q_X \quad -(21)$$

and from the laws: $$\underline{\nabla} \cdot \underline{\Omega} = 0 \quad -(22)$$

there is one more equation:

$$Q_X\left(\frac{\partial \omega_Z}{\partial Y} - \frac{\partial \omega_Y}{\partial Z}\right) + Q_Y\left(\frac{\partial \omega_X}{\partial Z} - \frac{\partial \omega_Z}{\partial X}\right) + Q_Z\left(\frac{\partial \omega_Y}{\partial X} - \frac{\partial \omega_X}{\partial Y}\right)$$

$$= \omega_X\left(\frac{\partial Q_Z}{\partial Y} - \frac{\partial Q_Y}{\partial Z}\right) + \omega_Y\left(\frac{\partial Q_X}{\partial Z} - \frac{\partial Q_Z}{\partial X}\right) + \omega_Z\left(\frac{\partial Q_Y}{\partial X} - \frac{\partial Q_X}{\partial Y}\right) \quad -(23)$$

Eqs. (16) − (18), (19 − 21), and (23) are seven equations in seven unknowns, $\omega_X, \omega_Y, \omega_Z, Q_X, Q_Y, Q_Z$ and ω_0. The system is <u>exactly</u> determined without any other information.

Having found ω_0, $\underline{\omega}$ and $\underline{Q}$, the scalar

5) potential Φ can be found from:

$$\underline{g} = -\underline{\nabla}\,\underline{\Phi} + \underline{\omega}\,\underline{\Phi} = -\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \quad -(24)$$

so $\underline{g}$ can be found unequivocally.

The general orbit may be found from eq. (24), using the methods developed in UFT378.

<u>Two Dimensional Orbits and the Newtonian limit</u>

In two dimensions:

$$Q_Z = \omega_Z = 0 \quad -(25)$$

and

$$\frac{\partial Q_Y}{\partial Z} = \frac{\partial Q_X}{\partial Z} = \frac{\partial \omega_Y}{\partial Z} = \frac{\partial \omega_X}{\partial Z} = 0 \quad -(26).$$

so the problem reduces to:

$$\left(\frac{\partial}{\partial X} - \omega_X\right) Q_Y = -\left(\frac{\partial}{\partial Y} - \omega_Y\right) Q_X \quad -(27)$$

and

$$\omega_X Q_Y - \omega_Y Q_X = \omega_0 \left(\frac{\partial Q_Y}{\partial X} - \frac{\partial Q_X}{\partial Y}\right) + Q_Y \frac{\partial \omega_0}{\partial X} - Q_X \frac{\partial \omega_0}{\partial Y} \quad -(28)$$

In the Newtonian limit:

$$\underline{g} = -\frac{mG}{r^3}\underline{r} = -\underline{\nabla}\,\underline{\Phi} + \underline{\omega}\,\underline{\Phi}$$

$$= -\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \quad -(29)$$

so:

$$g_x = -\frac{mGX}{(X^2+Y^2)^{3/2}} = -\frac{\partial Q_x}{\partial t} - \omega_0 Q_x \quad -(30)$$

$$g_Y = \frac{-mGY}{(X^2+Y^2)^{3/2}} = -\frac{\partial Q_Y}{\partial t} - \omega_0 Q_Y \quad -(31)$$

Eqs. (27), (28), (30) and (31) are four equations in five unknowns, ω_X, ω_Y, Q_X, Q_Y and ω_0.

In the Newtonian limit:

$$\underline{g} = -\underline{\nabla}\,\underline{\Phi}_N \quad -(32)$$

where

$$\underline{\Phi}_N = -\frac{mG}{r} = -\frac{mG}{(X^2+Y^2)^{1/2}} \quad -(33)$$

In order to find a fifth equation use:

$$\underline{\nabla}\cdot\underline{g} = \underline{\nabla}\cdot\left(-\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q}\right) = 4\pi G\rho_m \quad -(34)$$

so

$$-\frac{\partial}{\partial t}(\underline{\nabla}\cdot\underline{Q}) - \omega_0\,\underline{\nabla}\cdot\underline{Q} = 4\pi G\rho_m \quad -(35)$$

which means that:

$$-\frac{\partial}{\partial t}\left(\frac{\partial Q_X}{\partial X} + \frac{\partial Q_Y}{\partial Y}\right) - \omega_0\left(\frac{\partial Q_X}{\partial X} + \frac{\partial Q_Y}{\partial Y}\right) = 4\pi G\rho_m \quad -(36)$$

So there are five equations in five unknowns, eqs. (27), (28), (30), (31) and (36), in Q_X, Q_Y, ω_X, ω_Y and ω_0. In the next note, the Ampère Maxwell law will be