386(4): General Methodology for Magnetostatics.

In general:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(1)$$

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \quad -(2)$$

$$\underline{B} = \underline{\nabla} \times \underline{A} - \underline{\omega} \times \underline{A} \quad -(3)$$

with the antisymmetry laws:

$$\frac{\partial A_Z}{\partial Y} + \frac{\partial A_Y}{\partial Z} = \omega_Y A_Z + \omega_Z A_Y \quad -(4)$$

$$\frac{\partial A_X}{\partial Z} + \frac{\partial A_Z}{\partial X} = \omega_Z A_X + \omega_X A_Z \quad -(5)$$

$$\frac{\partial A_Y}{\partial X} + \frac{\partial A_X}{\partial Y} = \omega_X A_Y + \omega_Y A_X. \quad -(6)$$

Define:

$$\underline{\nabla} \times \underline{\alpha} = \underline{\omega} \times \underline{A} = \underline{B}_2 \quad -(7)$$

so

$$\underline{B} = B_1 - \underline{B}_2 \quad -(8)$$

where

$$\underline{B_1} = \underline{\nabla} \times \underline{A} \quad -(9)$$

$$\underline{B_2} = \underline{\nabla} \times \underline{\alpha} \quad -(10)$$

It follows that:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(11)$$

Q.E.D.

Define:

$$\underline{\nabla} \times \underline{B}_1 = \mu_0 \underline{J_1} \quad -(12)$$

$$\underline{\nabla} \times \underline{B_2} = \mu_0 \underline{J}_2 \quad -(13)$$

) Here $\underline{J}_1$ is the standard model current density and $\underline{J}_2$ is the current density due to the spin connection. This is the current density due to the matter or spacetime.

It follows as in Jackson chapter 5 that:

$$\underline{B}_1(\underline{x}) = \frac{\mu_0}{4\pi} \underline{\nabla} \times \int \frac{\underline{J}_1(\underline{x}')}{|\underline{x} - \underline{x}'|} d^3x' \quad -(14)$$

$$\underline{B}_2(\underline{x}) = \frac{\mu_0}{4\pi} \underline{\nabla} \times \int \frac{\underline{J}_2(\underline{x}')}{|\underline{x} - \underline{x}'|} d^3x' \quad -(15)$$

so

$$\underline{A} = \frac{\mu_0}{4\pi} \int \frac{\underline{J}_1(\underline{x}')}{|\underline{x} - \underline{x}'|} d^3x' \quad -(16)$$

$$\underline{\alpha} = \frac{\mu_0}{4\pi} \int \frac{\underline{J}_2(\underline{x}')}{|\underline{x} - \underline{x}'|} d^3x' \quad -(17)$$

Using eq. (7), it follows that:

$$\underline{\nabla} \cdot \underline{\nabla} \times \underline{\alpha} = \underline{\nabla} \cdot \underline{\omega} \times \underline{A} = \underline{0} \quad -(18)$$

Q.E.D.

Therefore the methodology is to choose a vector potential $\underline{A}$, use the antisymmetry relations to calculate $\underline{\omega}$, and calculate the extra magnetic field due to the spin connection, matter or vacuum:

$$\underline{B}_2 = \underline{\nabla} \times \underline{\alpha} = \underline{\omega} \times \underline{A} \quad -(19)$$

Compute $\underline{\alpha}$ from eq. (19) and find the current density $\underline{J}_2$ from:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{\alpha}) = \mu_0 \underline{J}_2 \quad -(20)$$

This procedure automatically conserves antisymmetry, eqs. (4) to (6), and automatically obeys eq. (1) and eq. (2). It is not necessary to compute $\underline{\alpha}$ because the vacuum, aether, or spacetime current density can be computed from:

$$\boxed{\underline{\nabla} \times (\underline{\omega} \times \underline{A}) = \mu_0 \underline{J}_2} \quad -(21)$$

The only constraint on this method is that eqs. (4) to (8) must be soluble, and must be exactly determined, i.e. three equations in three unknowns, ω_x, ω_y, ω_z. The components A_x, A_y and A_Z are known, because they are taken from any standard potential:

$$\underline{B}_1 = \underline{\nabla} \times \underline{A} \quad -(22)$$

For example, for a static magnetic field:

$$\underline{A} = \frac{B^{(0)}}{2} \left(-Y \underline{i} + X \underline{j} \right) \quad -(23)$$

+) and as shown in previous work:

$$\underline{\omega} = -\left(\frac{1}{X}\underline{i} + \frac{1}{Y}\underline{j}\right) \quad -(24)$$

It follows that:

$$\underline{B}_2 = -\frac{B^{(0)}}{2}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1/X & 1/Y & 0 \\ -Y & X & 0 \end{vmatrix} = -B^{(0)}\underline{k} \quad -(25)$$

and

$$\underline{\nabla} \times \underline{\alpha} = -B^{(0)}\underline{k} \quad -(26)$$

so

$$\underline{\alpha} = -\frac{B^{(0)}}{2}\left(-Y\underline{i} + X\underline{j}\right). \quad -(27)$$

It follows that:

$$\underline{B} = \underline{\nabla} \times \underline{A} - \underline{\nabla} \times \underline{\alpha} = 2B^{(0)}\underline{k} \quad -(28)$$

and is a static magnetic field, Q.E.D. It also follows that:

$$\underline{\nabla} \cdot \underline{B} = \underline{\nabla} \cdot (\underline{\nabla} \times \underline{A} - \underline{\nabla} \times \underline{\alpha}) = 0 \quad -(29)$$

Q.E.D. Finally eqs. (23) and (24) conserve the antisymmetry equations (4) to (6), Q.E.D.

In the case of the static magnetic field:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{\alpha}) = \mu_0 \underline{J}_2 = \underline{0} \quad -(30)$$

and

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \mu_0 \underline{J}_1 = \underline{0} \quad -(31)$$

5) so for the static magnetic field:

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(32)$$

$$\underline{\nabla} \times \underline{B} = \underline{0} \quad -(33)$$

but the spin connection is not zero.

Finally consider the potential of the static dipole field:

$$\underline{A} = \frac{\mu_0 I a^2}{4} \left(\frac{-Y \underline{i}}{(X^2+Y^2+Z^2)^{3/2}} + \frac{X \underline{j}}{(X^2+Y^2+Z^2)^{3/2}} \right) \quad -(34)$$

on constant surfaces:

$$R_0^2 = X^2 + Y^2 + Z^2 = \text{constant.} \quad -(35)$$

so:

$$\underline{A} = \frac{\mu_0 I a^2}{4 R_0^3} \left(-Y \underline{i} + X \underline{j} \right). \quad -(36)$$

This is the vector potential on a spherical surface at a constant distance R_0 far from a circular loop.

Note that eqs. (23) and (36) are the same provided that

$$B^{(0)} = \frac{2 \mu_0 I a^2}{4 R_0^3} \quad -(37)$$

so the above analysis of eq. (23) also applies to eq. (36). It would be interesting to use eq. (23) in eqs. (4) to (6) in order to find the general $\underline{\omega}$