51(4): Complete Solution for Orbital Precessions

Start with:

$$\underline{g} = -\underline{\nabla}\,\Phi + \underline{\omega}\,\Phi = -\underline{\nabla}\,\Phi - \frac{\partial \underline{Q}}{\partial t}(\text{total}) = -\frac{\partial \underline{Q}}{\partial t} - \omega_0 \underline{Q} \quad -(1)$$

In Cartesian coordinates:

$$\underline{g} = -\frac{mG}{(X^2+Y^2)^{3/2}}(X\underline{i} + Y\underline{j}) - \frac{mG}{(X^2+Y^2)^{1/2}}\underline{\omega}$$
$$= \ddot{X}\underline{i} + \ddot{Y}\underline{j} \quad -(2)$$

where the gravitational potential:

$$\Phi = -\frac{mG}{(X^2+Y^2)^{1/2}} \quad -(3)$$

is screened.

Therefore:

$$\ddot{X} = -mG\left(\frac{X}{(X^2+Y^2)^{3/2}} + \frac{\omega_X}{(X^2+Y^2)^{1/2}}\right) \quad -(4)$$

$$\ddot{Y} = -mG\left(\frac{Y}{(X^2+Y^2)^{3/2}} + \frac{\omega_Y}{(X^2+Y^2)^{1/2}}\right) \quad -(5)$$

As in recent work, the lagrangian of a planar orbit is:

$$\mathcal{L} = -mc^2\left(1 - \frac{\dot{X}^2+\dot{Y}^2}{c^2}\right)^{1/2} + \frac{mMG}{(X^2+Y^2)^{1/2}} \quad -(6)$$

and the Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}}, \quad \frac{\partial \mathcal{L}}{\partial Y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}} \quad -(7)$$

give the forward precession:

$$\ddot{x} = \frac{mG}{\gamma(x^2+y^2)^{3/2}}\left(\frac{\dot{x}\dot{y}y + x\dot{x}^2}{c^2} - x\right) \quad -(8)$$

$$\ddot{y} = \frac{mG}{\gamma(x^2+y^2)^{3/2}}\left(\frac{\dot{y}\dot{x}x + y\dot{y}^2}{c^2} - y\right) \quad -(9)$$

with the Lorentz factor:

$$\gamma = \left(1 - \frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{-1/2} \quad -(10)$$

The same lagrangian gives the retrograde precession:

$$\ddot{x} = -\frac{mG}{\gamma^3(x^2+y^2)^{3/2}}\,x \quad -(11)$$

$$\ddot{y} = -\frac{mG}{\gamma^3(x^2+y^2)^{3/2}}\,y \quad -(12)$$

For forward precession, comparison of eqns. (4), (5), (8) and (9) gives:

$$\omega_x = \frac{x}{x^2+y^2}\left(\frac{1}{\gamma}-1\right) - \frac{\dot{x}\dot{y}y + x\dot{x}^2}{\gamma c^2} \quad -(13)$$

$$\omega_y = \frac{y}{x^2+y^2}\left(\frac{1}{\gamma}-1\right) - \frac{\dot{y}\dot{x}x + y\dot{y}^2}{\gamma c^2} \quad -(14)$$

For retrograde precession, comparison of eqs. (4), (5), (11) and (12) gives:

$$\omega_X = \left(\frac{X}{X^2+Y^2}\right)\left(\frac{1}{\gamma^3} - 1\right) \quad -(15)$$

$$\omega_Y = \left(\frac{Y}{X^2+Y^2}\right)\left(\frac{1}{\gamma^3} - 1\right) \quad -(16)$$

Now calculate the vector potential $\underline{A}$ from the antisymmetry equations:

$$\frac{\partial A_Z}{\partial Y} + \frac{\partial A_Y}{\partial Z} = \omega_Y A_Z + \omega_Z A_Y \quad -(17)$$

$$\frac{\partial A_X}{\partial Z} + \frac{\partial A_Z}{\partial X} = \omega_Z A_X + \omega_X A_Z \quad -(18)$$

$$\frac{\partial A_Y}{\partial X} + \frac{\partial A_X}{\partial Y} = \omega_X A_Y + \omega_Y A_X. \quad -(19)$$

In general:

$$\underline{A} = A_X \underline{i} + A_Y \underline{j} + A_Z \underline{k} \quad -(20)$$

$$\underline{\omega} = \omega_X \underline{i} + \omega_Y \underline{j} + \omega_Z \underline{k} \quad -(21)$$

For the planar orbit:

$$\omega_Z = 0 \quad -(22)$$

Therefore:

$$\frac{\partial}{\partial Z}(A_X + A_Y) + \frac{\partial A_Z}{\partial Y} + \frac{\partial A_Z}{\partial X} = (\omega_Y + \omega_X) A_Z \quad -(23)$$

by adding eqs. (17) and (18). If it is considered that $\underline{A}$ is defined in the plane of the orbit, then:

$$\underline{A} = A_X \underline{i} + A_Y \underline{j} \quad -(24)$$

and eq. (23) reduces to zero on both sides.