424(1): Derivation of the Lagrangian of a Theory from the Hamiltonian

Reference: Marion and Thornton chapter 14, 3rd Edition.

The Hamiltonian and Lagrangian are related by:

$$H = \sum_i v_{Ni} p_i - \mathcal{L} \quad -(1)$$

For a single particle:

$$H = \gamma m v_N^2 - \mathcal{L} \quad -(2)$$

so the Lagrangian can be calculated from the Hamiltonian as follows:

$$\mathcal{L} = \gamma m v_N^2 - H \quad -(3)$$

In special relativity:

$$H = \gamma m c^2 + U \quad -(4)$$

where U is the potential energy, and the relativistic momentum is:

$$p = \gamma m v_N \quad -(5)$$

So the Lagrangian is:

$$\mathcal{L} = \frac{p^2 c^2}{\gamma m c^2} - \gamma m c^2 - U. \quad -(6)$$

Now use the Einstein energy equation:

$$E^2 = p^2 c^2 + m^2 c^4 \quad -(7)$$

to find that:

$$p^2 c^2 = (\gamma^2 - 1) m^2 c^4 \quad -(8)$$

so

$$\mathcal{L} = \frac{(\gamma^2 - 1)}{\gamma} m c^2 - \gamma m c^2 - U \quad -(9)$$

a)

i.e
$$\mathcal{L} = -\frac{mc^2}{\gamma} - U \quad -(10)$$

Q.E.D. So if the hamiltonian is eq. (4), the lagrangian is eq. (10).

In m theory, the hamiltonian is:
$$H = m(r)\gamma mc^2 + U \quad -(11)$$
where
$$\gamma = \left(m(r) - \frac{v_N^2}{m(r)c^2}\right)^{-1/2} \quad -(12)$$

From eqs. (3) and (11), the lagrangian is:
$$\mathcal{L} = \gamma m v_N^2 - m(r)\gamma mc^2 - U \quad -(13)$$
$$= \frac{p^2c^2}{\gamma mc^2} - \gamma m(r)mc^2 - U$$
using
$$p = \gamma m v_N. \quad -(14)$$

From eq. (14):
$$p^2c^2 = \gamma^2 m^2 v_N^2 c^2 = \gamma^2 m^2 c^4 \left(\frac{v_N^2}{c^2}\right). \quad -(15)$$

Now use:
$$\frac{1}{\gamma^2} = m(r) - \frac{v_N^2}{m(r)c^2} \quad -(16)$$

so
$$\frac{v_N^2}{c^2} = m(r)\left(m(r) - \frac{1}{\gamma^2}\right) \quad -(17)$$

From eqs. (15) and (17):

3)

$$p^2c^2 = \gamma^2 m^2 c^4 m(r)\left(m(r) - \frac{1}{\gamma^2}\right) \quad -(18)$$

$$= \gamma^2 m^2 c^4 m(r)^2 - m(r) m^2 c^4$$

$$= E^2 - m(r) m^2 c^4$$

So the single particle Einstein energy equation in n space is:

$$E^2 = p^2c^2 + m(r) m^2 c^4 \quad -(19)$$

From eqs. (13) and (19) the lagrangian is:

$$\mathcal{L} = \frac{E^2}{\gamma mc^2} - \frac{m(r) m^2 c^4}{\gamma mc^2} - \gamma m(r) mc^2 - U$$

$$= \frac{m(r)^2 \gamma^2 m^2 c^4}{\gamma mc^2} - \frac{m(r) mc^2}{\gamma} - \gamma m(r) mc^2 - U$$

$$\boxed{\mathcal{L} = m(r)\gamma mc^2 (m(r) - 1) - \frac{m(r) mc^2}{\gamma} - U}$$

The lagrangian (19) gives the hamiltonian (11)

Note that if $\quad m(r) \rightarrow 1 \quad -(20)$

$$\mathcal{L} \xrightarrow[m(r)\rightarrow 1]{} -\frac{mc^2}{\gamma} - U \quad -(21)$$

The Euler Lagrange equations for eq (19) are:

4)

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \quad -(22)$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi} \quad -(23)$$

The angular momentum in m theory is:

$$\underline{L} = \underline{p} \times \underline{r} \quad -(24)$$

where

$$\underline{r} = \frac{r}{m(r)^{1/2}} \underline{e}_r \quad -(25)$$

and

$$\underline{p} = \frac{\gamma m}{m(r)^{1/2}} \left(\dot{r} \underline{e}_r + r \dot{\phi} \underline{e}_\phi \right) \quad -(26)$$

so:

$$\underline{L} = \frac{\gamma m r^2 \dot{\phi}}{m(r)} \underline{e}_r \times \underline{e}_\phi$$

and

$$\boxed{L = \frac{\gamma m r^2 \dot{\phi}}{m(r)}} \quad -(27)$$

The Evans Eckardt equations are:

$$\frac{dH}{dt} = 0 \quad -(28)$$

$$\frac{dL}{dt} = 0 \quad -(29)$$

The Lagrangian:

$$\mathcal{L} = m(r)(m(r)-1)\gamma m c^2 - \frac{m(r) m c^2}{\gamma} - U \quad -(30)$$

5) must give the hamiltonian:

$$H = \gamma n(r) mc^2 + U \quad -(31)$$

and it follows that the lagrangian (30) must give eqs. (28) and (29).

By definition:

$$p = \frac{\partial \mathcal{L}}{\partial v} \quad -(32)$$

and from eqs. (6.162) and (6.163) of Marion and Thornton:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} = 0 \quad -(33)$$

Eqs. (30) and (32) give an equation for $\partial n(r)/\partial v$:

$$\frac{\partial}{\partial v}\left(n(r)(n(r)-1)\gamma mc^2 - n(r)\frac{mc^2}{\gamma} - U \right) = \gamma m v \quad -(34)$$

Eq. (1) is obtained from the Lagrange equations in generalized coordinates:

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} \quad -(35)$$

where

$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial q}\frac{dq}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q} \quad -(36)$$

$$= \dot{q}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q}$$

i.e.

$$\frac{d\mathcal{L}}{dt} - \frac{d}{dt}\left(\dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0 \quad -(37)$$

This means that:

$$\frac{d}{dt}\left(\mathcal{L} - \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = 0 \quad -(38)$$

The Hamiltonian is defined by:

$$\mathcal{L} - \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} = -H \quad -(39)$$

i.e.

$$H = \dot{q}p - \mathcal{L} \quad -(40)$$

This is eq. (3) if $p = \gamma m v_N, \ \dot{q} = v_N \quad -(41)$

Q.E.D.

In the theory the most useful procedure is to solve:

$$\frac{dH}{dt} = 0 \quad -(42)$$

and

$$\frac{dL}{dt} = 0 \quad -(43)$$

as in UFT420. The Lagrangian method leads to eq. (42), and eq. (43) is derived from first principles, in eqs (24) to (27). The Euler Lagrange equations (22) and (23) must be solved with the Lagrangian (19) together with eq. (34), which shows that:

$$\frac{\partial n(r)}{\partial r} \neq 0 \quad -(44)$$

as used in UFT420.