424(3): Combined Lagrangian and Hamiltonian Analysis

It was shown in Note 424(1) that the lagrangian equivalent to the hamiltonian:

$$H = m(r)\gamma mc^2 - m(r)^{1/2}\frac{mMG}{r} \quad -(1)$$

is:

$$\mathcal{L} = m(r)\left(m(r)-1\right)\gamma mc^2 - \frac{m(r)mc^2}{\gamma} + m(r)^{1/2}\frac{mMG}{r} \quad -(2)$$

The Evans Eckardt equations are:

$$\frac{dH}{dt} = 0 \quad -(3)$$

and

$$\frac{dL}{dt} = 0 \quad -(4)$$

where

$$L = \frac{\gamma m r^2 \dot{\phi}}{m(r)} \quad -(5)$$

Eqs. (3) to (5) must give the same results as the Euler Lagrange equations with lagrangian (2):

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \quad -(6)$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial \phi} \quad -(7)$$

This is because the lagrangian (2) gives the hamiltonian (1) through:

$$H = \frac{p^2 c^2}{\gamma mc^2} - \mathcal{L} \quad -(8)$$

where p is the relativistic momentum of m theory:

$$p = \frac{\gamma m v}{m(r)^{1/2}} \quad -(9)$$

In m theory:

$$\gamma = \left(m(r) - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{m(r)c^2} \right)^{-1/2} \quad -(10)$$

and the relativistic total energy is:

$$E = m(r)\gamma m c^2 \quad -(11)$$

It follows as in Note 424(1) that:

$$E^2 = p^2c^2 + m(r)m^2c^4 \quad -(11)$$

By expanding eqs. (3) and (4) using computer algebra, and comparing the results with the expansion of eqs. (6) and (7) by computer algebra, new equations of motion will be found. These are essentially constraint equations.

For example, from eqs. (2) and (7):

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0 \quad -(12)$$

and the angular momentum from the Lagrangian method is:

$$L_1 = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad -(13)$$

$$\frac{dL_1}{dt} = 0 \quad -(14)$$

In general:

$$L_1 \neq L \quad -(15)$$

and

$$\frac{dL_1}{dt} = 0 \text{ and } \frac{dL}{dt} = 0 \quad -(16)$$

The result (5) is obtained from first principles:

$$\underline{L} = \underline{r} \times \underline{p} \quad -(17)$$

in plane polar coordinates, so it is assumed that the lagrangian (2) must produce the angular momentum (5). Therefore

$$\boxed{\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\gamma m r^2}{n(r)} \dot{\phi}} \quad -(18)$$

This is the constraint equation. From eqs. (2) and (18):

$$\frac{\partial}{\partial \dot{\phi}} \left(n(r)(n(r)-1)\gamma m c^2 - n(r) \frac{mc^2}{\gamma} - U \right) = \frac{\gamma m r^2 \dot{\phi}}{n(r)} \quad -(19)$$

In the limit of flat spacetime:

$$n(r) = 1 \quad -(20)$$

so:

$$-\frac{\partial}{\partial \dot{\phi}} \left(\frac{mc^2}{\gamma} \right) = \gamma m r^2 \dot{\phi} \quad -(21)$$

$$-(22)$$

i.e

$$-mc^2 \frac{\partial}{\partial \dot{\phi}} \left(1 - \frac{\dot{r}^2 + r^2 \dot{\phi}^2}{c^2} \right)^{1/2} = \gamma m r^2 \dot{\phi}$$

This is the correct result. Eqn. (19) is the n space generalization of eq. (22).

In eq. (19), careful consideration has to be given to the differentiation of $n(r)$ with respect to $\dot{\phi}$. In flat spacetime:

)

$$U = -\frac{mMG}{r} \quad -(23)$$

and

$$\frac{\partial U}{\partial \dot{\phi}} = 0 \quad -(24)$$

But in space:

$$U = -m(r)^{1/2} \frac{mMG}{r} \quad -(25)$$

The m function is a function of r, but r is also a function of $\dot{\phi}$ and t. In flat spacetime for example:

$$\mathcal{L} = -mc^2 \left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)^{1/2} + \frac{mMG}{r} \quad -(26)$$

So

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \gamma m r^2 \dot{\phi} \quad -(27)$$

and

$$\frac{\partial \mathcal{L}}{\partial r} = \gamma m r \dot{\phi}^2 - \frac{mMG}{r^2} \quad -(28)$$

where

$$\gamma = \left(1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right)^{-1/2} \quad -(29)$$

So

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial r} \frac{\partial r}{\partial \dot{\phi}} \quad -(30)$$

and

$$\gamma m r^2 \dot{\phi} = \left(\gamma m r \dot{\phi}^2 - \frac{mMG}{r^2}\right) \frac{\partial r}{\partial \dot{\phi}} \quad -(31)$$

i.e.

$$\frac{\partial r}{\partial \dot{\phi}} = \frac{\gamma m r^2 \dot{\phi}}{\gamma m r \dot{\phi}^2 - \frac{mMG}{r^2}} \quad -(32)$$

So r is a <u>function of $\dot{\phi}$</u>.

) Therefore in the constraint equation (19):

$$\boxed{\frac{\partial n(r)}{\partial \dot{\phi}} \neq 0} \quad -(33)$$

Eq. (19) is therefore an equation for $n(r)$.

In m theory the (r_1, ϕ) coordinate system must be used for a self consistent analysis, as shown in the immediately preceding papers. In this case the hamiltonian is

$$H = n(r_1)\,\gamma m c^2 - \frac{mMG}{r_1} \quad -(34)$$

and the angular momentum is:

$$L = \gamma m r_1^2 \dot{\phi} \quad -(35)$$

The Evans Eckardt equations are eqs. (3) and (4). In (r_1, ϕ)

$$v_1 = \frac{v}{n(r)^{1/2}} \quad -(36)$$

so the lagrangian is:

$$\mathcal{L} = \gamma m v_1^2 - H \quad -(37)$$

where γ in the (r_1, ϕ) system is:

$$\gamma = \left(n(r_1) - \frac{\dot{r}_1^2 + r_1^2 \dot{\phi}^2}{c^2} \right)^{-1/2} \quad -(38)$$

The relativistic momentum in the (r_1, ϕ) coordinate system is

$$p_1 = \gamma m v_1 \quad -(39)$$

so

$$v_1^2 = \frac{p_1^2}{m^2 \gamma^2} \quad -(40)$$

) and
$$\gamma m v_1^2 = \frac{p_1^2}{m\gamma} = \frac{p_1^2 c^2}{\gamma m c^2} \quad -(41)$$

Using:
$$p_1 = \gamma m v_1 \quad -(42)$$

it follows that
$$p_1^2 c^2 = \gamma^2 m^2 v_1^2 c^2 = \gamma^2 m^2 c^4 \frac{v_1^2}{c^2} \quad -(42)$$

Now use:
$$\frac{1}{\gamma^2} = m(r_1) - \frac{v_1^2}{c^2} \quad -(43)$$

so
$$\frac{v_1^2}{c^2} = m(r_1) - \frac{1}{\gamma^2} \quad -(44)$$

From eqs. (42) and (44).
$$p_1^2 c^2 = \gamma^2 m^2 c^4 \left(m(r_1) - \frac{1}{\gamma^2} \right)$$
$$= \gamma^2 m^2 c^4 m(r_1) - m^2 c^4$$
$$= \frac{E^2}{m(r_1)} - m^2 c^4 \quad -(45)$$

where
$$E = m(r_1)\gamma m c^2 \quad -(46)$$

, the total relativistic energy. So:
$$E^2 = m(r_1)\left(p_1^2 c^2 + m^2 c^4 \right) \quad -(47)$$

The lagrangian in (r_1, ϕ) corresponding to the hamiltonian (34) is:
$$\mathcal{L} = \frac{p_1^2 c^2}{\gamma m c^2} - m(r_1)\gamma m c^2 + \frac{mMG}{r_1} \quad -(48)$$

From eqs. (45) and (48):

$$\mathcal{L} = \frac{E^2}{\gamma m(r_1) mc^2} - \frac{mc^2}{\gamma} - m(r_1)\gamma mc^2 + \frac{mMG}{r_1} \quad (49)$$

$$= m(r_1)\gamma mc^2 - \frac{mc^2}{\gamma} - m(r_1)\gamma mc^2 + \frac{mMG}{r_1}$$

$$= -\frac{mc^2}{\gamma} + \frac{mMG}{r_1}$$

Therefore the correct Lagrangian (49) is obtained from the hamiltonian (34) in the (r_1, ϕ) system.

Conclusions

1) The m theory is rigorously self consistent if and only if the (r_1, ϕ) coordinate system is used

2) The Evans Eckardt equations must be used in the (r_1, ϕ) system.

3) The hamiltonian is eq. (34) and the angular momentum is eq. (35).

4) The velocity is eq. (36).

5) The Lagrangian is:

$$\mathcal{L} = \frac{p_1^2 c^2}{\gamma mc^2} - H = -\frac{mc^2}{\gamma} + \frac{mMG}{r_1}$$

6) All previous work is rigorously correct.