425(2): Self Consistency Check with the Hamilton Canonical Equations.

As shown in UFT424, self consistency between the Hamiltonian and lagrangian m theories is obtained with

$$\mathcal{L} = p\dot{q} - H \quad -(1)$$

either in frame (r, ϕ). Applying the Hamilton Principle of Least Action to eq. (1):

$$\delta \int_{t_1}^{t_2} (p\dot{q} - H)\, dt = 0 \quad -(2)$$

i.e.

$$\int_{t_1}^{t_2} \left(\left(\dot{q} - \frac{\partial H}{\partial p} \right) \delta p - \left(\dot{p} + \frac{\partial H}{\partial q} \right) \delta q \right) dt = 0 \quad -(3)$$

Eq. (3) results in the Hamilton canonical equations of motion:

$$\dot{q} = \frac{\partial H}{\partial p} \quad -(4)$$

$$\dot{p} = -\frac{\partial H}{\partial q} \quad -(5)$$

Classical Level

In this case:

$$H = \frac{p^2}{2m} - \frac{mMG}{r} \quad -(6)$$

$$\mathcal{L} = \frac{p^2}{2m} + \frac{mMG}{r} \quad -(7)$$

) where

$$q := r \quad -(8)$$

Eq. (5) gives:

$$\underline{F} = \dot{p} = -\frac{mMG}{r^2} \quad -(9)$$

and Eq. (4) gives:

$$\dot{r} = \frac{\partial H}{\partial p} = v \quad -(10)$$

Eq. (9) is the equation of motion and Eq. (10) is the classical definition of ~~linear~~ velocity in an inertial frame. The Euler Lagrange equation gives:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \quad -(11)$$

where

$$\mathcal{L} = \frac{1}{2} m \dot{r}^2 + \frac{mMG}{r} \quad -(12)$$

So

$$\dot{p} = m\ddot{r} = -\frac{mMG}{r^2} \quad -(13)$$

Eqs. (9) and (13) are the same, so the two methods are self consistent.

<u>Special Relativity</u>

In this case:

$$H = \gamma mc^2 - \frac{mMG}{r} \quad -(14)$$

and

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{mMG}{r} \quad -(15)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(16)$$

) The Euler Lagrange equation in the inertial frame is:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \quad -(17)$$

here

$$v = \dot{r} \quad -(18)$$

Eq. (17) gives: $F = \frac{d}{dt}(\gamma m v) = -\frac{mMG}{r^2} \quad -(19)$

The Hamiltonian is:

$$H = mc^2\left(1 - \frac{p_N^2}{m^2c^2}\right)^{-1/2} - \frac{mMG}{r} \quad -(20)$$

so eq. (5) gives:

$$\dot{p} = -\frac{\partial H}{\partial r} = \frac{d}{dt}(\gamma m v) = -\frac{mMG}{r^2} \quad -(21)$$

So the two methods give the same result with:

$$\dot{p} = \frac{d}{dt}(\gamma m v) \quad -(22)$$

Q.E.D.

Eq. (4) gives:

$$\dot{r} = \frac{\partial H}{\partial p} = \frac{\partial H}{\partial v_N}\frac{\partial v_N}{\partial p} \quad -(23)$$

The relativistic momentum is:

$$p = \gamma m v_N \quad -(24)$$

so

$$p = m\left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} v_N \quad -(25)$$

4)

It follows that:

$$\frac{\partial p}{\partial v_N} = m\left(\gamma + v_N \frac{d}{dv_N}\left(1-\frac{v_N^2}{c^2}\right)^{-1/2}\right)$$

$$= m\left(\gamma + \gamma^3 \frac{v_N^2}{c^2}\right)$$

$$= m\gamma^3\left(\frac{1}{\gamma^2} + \frac{v_N^2}{c^2}\right)$$

$$= m\gamma^3 \qquad -(26)$$

Therefore:

$$\frac{\partial H}{\partial p} = \frac{1}{m\gamma^3}\frac{\partial H}{\partial v_N} \qquad -(27)$$

where:

$$\frac{\partial H}{\partial v_N} = m\gamma^3 v_N \qquad -(28)$$

So

$$\frac{\partial H}{\partial p} = v_N = \dot{r} \qquad -(29)$$

Q.E.D.

Therefore the Lagrangian and Hamilton canonical equations are self consistent in special relativity.

The overall result is :

$$\dot{p} = \frac{d}{dt}(\gamma m v) = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} = -\frac{\partial H}{\partial r} \quad -(26)$$

and

$$\frac{\partial H}{\partial p} = \dot{r} \quad -(27).$$

In m theory, the frame (r_1, ϕ) must be used,

so

$$\dot{p}_1 = \frac{d}{dt}(\gamma m v_1) = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}_1} = \frac{\partial \mathcal{L}}{\partial r_1} = -\frac{\partial H}{\partial r_1} \quad -(28)$$

and

$$\dot{r}_1 = \frac{\partial H}{\partial p_1} \quad -(29)$$

The Hamiltonian and Lagrangian are related by:

$$H = \frac{p_1^2 c^2}{\gamma m c^2} - \mathcal{L} \quad -(30)$$

where

$$H = m(r_1)\gamma m c^2 - \frac{m M G}{r_1} \quad -(31)$$

$$\mathcal{L} = -\frac{m c^2}{\gamma} + \frac{m M G}{r_1} \quad -(32)$$

$$\gamma = \left(m(r_1) - \frac{v_1^2}{c^2}\right)^{-1/2} \quad -(33)$$

$$E = m(r_1)\gamma m c^2 \quad -(34)$$

and

$$E^2 = m(r_1)(p_1^2 c^2 + m^2 c^4) \quad -(35)$$

These equations describe the Euler / Lagrange /

6) Hamilton equations in the space.

From eq. (30):

$$\frac{\partial H}{\partial r_1} = \frac{\partial}{\partial r_1}\left(\frac{p_1^2}{\gamma m}\right) - \frac{\partial \mathcal{L}}{\partial r_1} \quad -(36)$$

From eq. (28):

$$\frac{\partial H}{\partial r_1} = -\frac{\partial \mathcal{L}}{\partial r_1} \quad -(37)$$

and it follows that:

$$\boxed{\frac{\partial}{\partial r_1}\left(\frac{p_1^2}{\gamma m}\right) = 0} \quad -(38)$$

This is a new equation of the theory.

On the classical level, eq. (36) is:

$$\frac{\partial H}{\partial r} = \frac{\partial}{\partial r}\left(\frac{p^2}{m}\right) - \frac{\partial \mathcal{L}}{\partial r} \quad -(39)$$

because:

$$H = \frac{p^2}{m} - \mathcal{L} = \frac{p^2}{m} - \frac{p^2}{2m} - \frac{mMG}{r}$$

$$= \frac{p^2}{2m} - \frac{mMG}{r} \quad -(30)$$

so eq. (38) on the classical level is:

$$\frac{\partial}{\partial r}\left(\frac{p^2}{m}\right) = 0 \quad -(31)$$

which is true because p has no dependence on r,

Q.E.D.

1) In special relativity:

$$H = \gamma m v_N^2 - \mathcal{L} \quad -(32)$$

So

$$\frac{\partial}{\partial r}\left(\gamma m v_N^2\right) = 0 \quad -(33)$$

This result follows because in general:

$$\mathcal{L} = p\dot{q} - H \quad -(34)$$

$$-(35)$$

with

$$q := r$$

In Lagrange dynamics $\dot{r}$ and r are independent variables:

$$\frac{\partial \dot{r}}{\partial r} = 0 \quad -(36)$$

and in the Hamilton dynamics p and r are independent variables,

$$\frac{\partial p}{\partial r} = 0 \quad -(37)$$

so

It follows that in special relativity:

$$\frac{\partial (p\dot{r})}{\partial r} = 0 \quad -(38)$$

Q.E.D., where

$$p = \gamma m v_N \quad -(39)$$

$$v_N = \dot{r} \quad -(40)$$

In special relativity eq. (33) is equivalent to:

$$\frac{\partial \gamma}{\partial r} = 0 \quad -(41)$$

where

$$\gamma = \left(1 - \frac{v_N^2}{c^2}\right)^{-1/2} \quad -(42)$$

$$= \left(1 - \frac{\dot{r}^2}{c^2}\right)^{-1/2} \quad -(43)$$

) In the theory:

$$H = p_1 \dot{r}_1 - \mathcal{L} \quad -(44)$$

and so

$$\frac{\partial}{\partial r_1}\left(p_1 \dot{r}_1\right) = 0 \quad -(45)$$

is true by definition because r_1 is independent of p_1 in Hamiltonian dynamics and independent of $\dot{r}_1$ in Lagrangian dynamics. From eq. (45):

$$\dot{r}_1 \frac{\partial p_1}{\partial r_1} + p_1 \frac{\partial \dot{r}_1}{\partial r_1} = 0 \quad -(46)$$

where:

$$p_1 = \gamma m \dot{r}_1 \quad -(47).$$

Therefore from these fundamentals:

$$\boxed{\frac{\partial p_1}{\partial r_1} = 0} \quad -(48)$$

Now use:

$$\frac{\partial p_1}{\partial r_1} = \frac{\partial p_1}{\partial r} \frac{\partial r}{\partial r_1} \quad -(49)$$

and

$$p_1 = \frac{p}{n(r)^{1/2}} \quad -(50)$$

to find that:

$$\frac{\partial}{\partial r}\left(\frac{\gamma m \dot{r}}{n(r)^{1/2}}\right) = 0 \quad -(51)$$

i.e.

$$m \dot{r} \frac{\partial}{\partial r}\left(\frac{\gamma}{n(r)^{1/2}}\right) = 0 \quad -(52)$$

Therefore:

$$\frac{\partial}{\partial r}\left(\frac{\gamma}{m(r)^{1/2}}\right) = 0 \quad -(53)$$

In inertial coordinates:

$$\gamma = \left(m(r) - \frac{\dot{r}^2}{m(r)c^2}\right)^{-1/2} \quad -(54)$$

Eq. (53) is a new equation for $dm(r)/dr$.

Finally, using the equation:

$$\frac{dH}{\partial r} = -\frac{\partial \mathcal{L}}{\partial r} \quad -(55)$$

it is found that:

$$\frac{\partial}{\partial r_1}\left(m(r_1)\gamma m c^2\right) = \frac{\partial}{\partial r_1}\left(\frac{mc^2}{\gamma}\right) \quad -(56)$$

i.e.

$$\boxed{\gamma \frac{dm(r_1)}{dr_1} + m(r_1)\frac{d\gamma}{dr_1} = \frac{d}{dr_1}\frac{1}{\gamma}} \quad -(57)$$

This is another new equation for $dm(r_1)/dr_1$.

In the (r_1, ϕ) system:

$$\gamma = \left(m(r_1) - \frac{\dot{r}_1^2 + r_1^2\dot{\phi}^2}{c^2}\right)^{-1/2} \quad -(58)$$

$$\frac{1}{\gamma} = \left(m(r_1) - \frac{\dot{r}_1^2 + r_1^2\dot{\phi}^2}{c^2}\right)^{1/2} \quad -(59)$$

It follows that:

$$\frac{d}{dr_1}\left(\frac{1}{\gamma}\right) = \frac{1}{2}\left(\frac{dm(r_1)}{dr_1} - \frac{2r_1\dot{\phi}^2}{c^2}\right)\gamma \quad -(60)$$

and:

$$\frac{d\gamma}{dr_1} = -\frac{1}{2}\left(\frac{dm(r_1)}{dr_1} - \frac{2r_1\dot{\phi}^2}{c^2}\right)\gamma^3 \quad -(61)$$

So:

$$\gamma\frac{dm(r_1)}{dr_1} - \frac{1}{2}m(r_1)\left(\frac{dm(r_1)}{dr_1} - \frac{2r_1\dot{\phi}^2}{c^2}\right)\gamma^3 \quad -(62)$$

$$= \frac{1}{2}\left(\frac{dm(r_1)}{dr_1} - \frac{2r_1\dot{\phi}^2}{c^2}\right)\gamma \quad -(63)$$

$$\gamma\left(\frac{1}{2}\frac{dm(r_1)}{dr_1} + \frac{r_1\dot{\phi}^2}{c^2}\right) = \frac{1}{2}m(r_1)\left(\frac{dm(r_1)}{dr_1} - \frac{2r_1\dot{\phi}^2}{c^2}\right)\gamma^3 \quad -(64)$$

i.e.

$$\frac{1}{2}\frac{dm(r_1)}{dr_1} = \frac{1}{2}m(r_1)\frac{dm(r_1)}{dr_1}\gamma^2 - \frac{r_1\dot{\phi}^2}{c^2}\left(1+\gamma^2 m(r_1)\right)$$

$$\frac{1}{2}\frac{dm(r_1)}{dr_1}\left(\gamma^2 m(r_1) - 1\right) = \frac{r_1\dot{\phi}^2}{c^2}\left(1+\gamma^2 m(r_1)\right) \quad -(65)$$

So

$$\frac{dm(r_1)}{dr_1} = \frac{r_1\dot{\phi}^2}{c^2}\left(\frac{1+m(r_1)\gamma^2}{\gamma^2 m(r_1)-1}\right) \quad -(66)$$

i.e

$$\frac{dm(r_1)}{dr_1} = -\frac{r_1\dot{\phi}^2}{c^2} \quad -(67)$$

The angular momentum is:

$$L = \gamma m r_1^2\dot{\phi} \quad -(68)$$

so

$$r_1\dot{\phi}^2 = \frac{L^2}{\gamma^2 m^2 r_1^3} \quad -(69)$$

and

$$\boxed{\frac{dm(r_1)}{dr_1} = -\frac{L^2}{c^2\gamma^2 m^2 r_1^3}} \quad -(70)$$