(%i1) `kill(all);`

(%o0) done

(%i1) `assume(c>0, m>0);`

(%o1) [c>0, m>0]

Non-relativistic theory

1 Equations of central motion in coordinates (r,phi)

(%i2) `depends([r,phi],t);`

(%o2) $[r(t), \varphi(t)]$

(%i3) `diff(r,t,2) = diff(phi,t)^2*r - M*G/r^2;`

(%o3) $$\frac{d^2}{d\,t^2}\,r = \left(\frac{d}{d\,t}\,\varphi\right)^2 r - \frac{G\,M}{r^2}$$

(%i4) `diff(phi,t,2) = -2*diff(phi,t)*diff(r,t)/r;`

(%o4) $$\frac{d^2}{d\,t^2}\,\varphi = -\frac{2\left(\frac{d}{d\,t}\,\varphi\right)\left(\frac{d}{d\,t}\,r\right)}{r}$$

2 Non-relativistic Hamilton equations I inertial frame v^2 = p_r^2/

(%i5) `H: (p_r^2)/(2*m)-m*M*G/q_r;`

(H) $$\frac{{p_r}^2}{2\,m} - \frac{G\,M\,m}{q_r}$$

2.1 First Hamilton equations

(%i6) `H1: q_rd = diff(H,p_r);`

(H1) $$q_{rd} = \frac{p_r}{m}$$

(%i7) `H2: q_phid = diff(H,p_phi);`

(H2) $q_{phid} = 0$

2.2 Second Hamilton equations

(%i8) `H3: p_rd = -diff(H,q_r);`

(H3) $$p_{rd} = -\frac{G\,M\,m}{{q_r}^2}$$

(%i9) `H4: p_phid = -diff(H,q_phi);`

(H4) $p_{phid} = 0$

3 Non-relativistic Hamilton equations II general frame p^2 = p_r^2 + p_phi^2 / q_r^2

(%i10) `H: (p_r^2+1/q_r^2*p_phi^2)/(2*m)-m*M*G/q_r;`

(H) $$\frac{\frac{{p_{phi}}^2}{{q_r}^2} + {p_r}^2}{2\,m} - \frac{G\,M\,m}{q_r}$$

3.1 First Hamilton equations

(%i11) `H1: q_rd = diff(H,p_r);`

(H1) $$q_{rd} = \frac{p_r}{m}$$

```
(%i12) H2: q_phid = diff(H,p_phi);
```

(H2) $$q_{phid}=\frac{p_{phi}}{m\,q_r^2}$$

3.2 Second Hamilton equations

```
(%i13) H3: p_rd = -diff(H,q_r);
```

(H3) $$p_{rd}=\frac{p_{phi}^2}{m\,q_r^3}-\frac{G\,M\,m}{q_r^2}$$

```
(%i14) H4: p_phid = -diff(H,q_phi);
```

(H4) $$p_{phid}=0$$

Relativistic theory

1 Relativistic Hamilton equations I inertial frame gamma defined by velocity q_rd^2

```
(%i15) gamma: (1-(q_rd^2)/c^2)^(-1/2);
```

(gamma) $$\frac{1}{\sqrt{1-\frac{q_{rd}^2}{c^2}}}$$

```
(%i16) H: 1/(2*gamma)*((p_r^2)*c^2/(m*c^2)+m*c^2)-m*M*G/q_r;
```

(H) $$\frac{\left(\frac{p_r^2}{m}+c^2\,m\right)\sqrt{1-\frac{q_{rd}^2}{c^2}}}{2}-\frac{G\,M\,m}{q_r}$$

1.1 First Hamilton equations

```
(%i17) H1: q_rd = diff(H,p_r);
```

(H1) $$q_{rd}=\frac{p_r\sqrt{1-\frac{q_{rd}^2}{c^2}}}{m}$$

```
(%i18) H2: q_phid = diff(H,p_phi);
```

(H2) $$q_{phid}=0$$

1.2 Second Hamilton equations

```
(%i19) H3: p_rd = (-diff(H,q_r));
```

(H3) $$p_{rd}=-\frac{G\,M\,m}{q_r^2}$$

```
(%i20) H4: p_phid = -diff(H,q_phi);
```

(H4) $$p_{phid}=0$$

1.3 Re-insert gamma

```
(%i24) ratsubst(%gamma, gamma, H1);
       ratsubst(%gamma, gamma, H2);
       expand(ratsubst(%gamma, gamma, H3));
       ratsubst(%gamma, gamma, H4);
```

(%o21) $$q_{rd}=\frac{p_r}{\gamma\,m}$$

(%o22) $$q_{phid}=0$$

(%o23) $$p_{rd}=-\frac{G\,M\,m}{q_r^2}$$

(%o24) $$p_{phid}=0$$

2 Relativistic Hamilton equations II
inertial frame
gamma defined by velocities p_r^2/m^2

(%i25) `gamma: (1-(p_rd^2)/(m^2*c^2))^(-1/2);`

(gamma) $$\frac{1}{\sqrt{1-\frac{{p_{rd}}^{2}}{c^{2}\,m^{2}}}}$$

(%i26) `H: 1/(2*gamma)*((p_r^2)*c^2/(m*c^2)+m*c^2)-m*M*G/q_r;`

(H) $$\frac{\left(\frac{{p_r}^{2}}{m}+c^{2}\,m\right)\sqrt{1-\frac{{p_{rd}}^{2}}{c^{2}\,m^{2}}}}{2}-\frac{G\,M\,m}{q_r}$$

2.1 First Hamilton equations

(%i27) `H1: q_rd = diff(H,p_r);`

(H1) $$q_{rd}=\frac{p_r\sqrt{1-\frac{{p_{rd}}^{2}}{c^{2}\,m^{2}}}}{m}$$

(%i28) `H2: q_phid = diff(H,p_phi);`

(H2) $$q_{phid}=0$$

2.2 Second Hamilton equations

(%i29) `H3: p_rd = (-diff(H,q_r));`

(H3) $$p_{rd}=-\frac{G\,M\,m}{{q_r}^{2}}$$

(%i30) `H4: p_phid = -diff(H,q_phi);`

(H4) $$p_{phid}=0$$

2.3 Re-insert gamma

(%i34)
```
ratsubst(%gamma, gamma, H1);
ratsubst(%gamma, gamma, H2);
expand(ratsubst(%gamma, gamma, H3));
ratsubst(%gamma, gamma, H4);
```

(%o31) $$q_{rd}=\frac{p_r}{\gamma\,m}$$

(%o32) $$q_{phid}=0$$

(%o33) $$p_{rd}=-\frac{G\,M\,m}{{q_r}^{2}}$$

(%o34) $$p_{phid}=0$$

3 Relativistic Hamilton equations III
general frame
gamma defined by velocities p_r^2 + p_phi^2/q_r^2

(%i35) `gamma: (1-(p_rd^2+1/q_r^2*p_phid^2)/(m^2*c^2))^(-1/2);`

(gamma) $$\frac{1}{\sqrt{1-\frac{\frac{{p_{phid}}^{2}}{{q_r}^{2}}+{p_{rd}}^{2}}{c^{2}\,m^{2}}}}$$

```
(%i36) H: 1/(2*gamma)*((p_r^2+p_phi^2/q_r^2)*c^2/(m*c^2)+m*c^2)-m*M*G/q_r;
```

(H) $$\frac{\left(\frac{\frac{p_{phi}^2}{q_r^2}+p_r^2}{m}+c^2\,m\right)\sqrt{1-\frac{\frac{p_{phid}^2}{q_r^2}+p_{rd}^2}{c^2\,m^2}}}{2}-\frac{G\,M\,m}{q_r}$$

3.1 First Hamilton equations

```
(%i37) H1: q_rd = diff(H,p_r);
```

(H1) $$q_{rd}=\frac{p_r\sqrt{1-\frac{\frac{p_{phid}^2}{q_r^2}+p_{rd}^2}{c^2\,m^2}}}{m}$$

```
(%i38) H2: q_phid = diff(H,p_phi);
```

(H2) $$q_{phid}=\frac{p_{phi}\sqrt{1-\frac{\frac{p_{phid}^2}{q_r^2}+p_{rd}^2}{c^2\,m^2}}}{m\,q_r^2}$$

3.2 Second Hamilton equations

```
(%i39) H3: p_rd = (-diff(H,q_r));
```

(H3) $$p_{rd}=-\frac{G\,M\,m}{q_r^2}+\frac{p_{phi}^2\sqrt{1-\frac{\frac{p_{phid}^2}{q_r^2}+p_{rd}^2}{c^2\,m^2}}}{m\,q_r^3}-\frac{p_{phid}^2\left(\frac{\frac{p_{phi}^2}{q_r^2}+p_r^2}{m}+c^2\,m\right)}{2\,c^2\,m^2\sqrt{1-\frac{\frac{p_{phid}^2}{q_r^2}+p_{rd}^2}{c^2\,m^2}}\,q_r^3}$$

```
(%i40) H4: p_phid = -diff(H,q_phi);
```

(H4) $$p_{phid}=0$$

3.3 Re-insert gamma

```
(%i44) Hr1: ratsubst(%gamma, gamma, H1);
       Hr2: ratsubst(%gamma, gamma, H2);
       Hr3: expand(ratsubst(%gamma, gamma, H3));
       Hr4: ratsubst(%gamma, gamma, H4);
```

(Hr1) $$q_{rd}=\frac{p_r}{\gamma\,m}$$

(Hr2) $$q_{phid}=\frac{p_{phi}}{\gamma\,m\,q_r^2}$$

(Hr3) $$p_{rd}=-\frac{G\,M\,m}{q_r^2}-\frac{\gamma\,p_{phid}^2\,p_r^2}{2\,c^2\,m^3\,q_r^3}-\frac{\gamma\,p_{phid}^2}{2\,m\,q_r^3}+\frac{p_{phi}^2}{\gamma\,m\,q_r^3}-\frac{\gamma\,p_{phi}^2\,p_{phid}^2}{2\,c^2\,m^3\,q_r^5}$$

(Hr4) $$p_{phid}=0$$

```
(%i45) expand(first(first((solve([Hr3,Hr4], [p_rd, p_phid])))));
```

(%o45) $$p_{rd}=\frac{p_{phi}^2}{\gamma\,m\,q_r^3}-\frac{G\,M\,m}{q_r^2}$$

4 *Relativistic Hamilton equations IV general frame gamma with q_rd^2 + q_phid^2*q_r^2*

(%i46) `gamma: (1-(q_rd^2+q_r^2*q_phid^2)/(c^2))^(-1/2);`

(gamma) $$\frac{1}{\sqrt{1-\frac{q_{rd}^2+q_{phid}^2\,q_r^2}{c^2}}}$$

(%i47) `H: 1/(2*gamma)*((p_r^2+p_phi^2/q_r^2)*c^2/(m*c^2)+m*c^2)-m*M*G/q_r;`

(H) $$\frac{\left(\frac{\frac{p_{phi}^2}{q_r^2}+p_r^2}{m}+c^2\,m\right)\sqrt{1-\frac{q_{rd}^2+q_{phid}^2\,q_r^2}{c^2}}}{2}-\frac{G\,M\,m}{q_r}$$

4.1 First Hamilton equations

(%i48) `H1: q_rd = diff(H,p_r);`

(H1) $$q_{rd}=\frac{p_r\sqrt{1-\frac{q_{rd}^2+q_{phid}^2\,q_r^2}{c^2}}}{m}$$

(%i49) `H2: q_phid = diff(H,p_phi);`

(H2) $$q_{phid}=\frac{p_{phi}\sqrt{1-\frac{q_{rd}^2+q_{phid}^2\,q_r^2}{c^2}}}{m\,q_r^2}$$

4.2 Second Hamilton equations

(%i50) `H3: p_rd = -diff(H,q_r);`

(H3) $$p_{rd}=\frac{p_{phi}^2\sqrt{1-\frac{q_{rd}^2+q_{phid}^2\,q_r^2}{c^2}}}{m\,q_r^3}+\frac{q_{phid}^2\left(\frac{\frac{p_{phi}^2}{q_r^2}+p_r^2}{m}+c^2\,m\right)q_r}{2\,c^2\sqrt{1-\frac{q_{rd}^2+q_{phid}^2\,q_r^2}{c^2}}}-\frac{G\,M\,m}{q_r^2}$$

(%i51) `H4: p_phid = -diff(H,q_phi);`

(H4) $$p_{phid}=0$$

4.3 Re-insert gamma

(%i55)
```
Hr1: ratsubst(%gamma, gamma, H1);
Hr2: ratsubst(%gamma, gamma, H2);
Hr3: expand(ratsubst(%gamma, gamma, H3));
Hr4: ratsubst(%gamma, gamma, H4);
```

(Hr1) $$q_{rd}=\frac{p_r}{\gamma\,m}$$

(Hr2) $$q_{phid}=\frac{p_{phi}}{\gamma\,m\,q_r^2}$$

(Hr3) $$p_{rd}=\frac{\gamma\,p_r^2\,q_{phid}^2\,q_r}{2\,c^2\,m}+\frac{\gamma\,m\,q_{phid}^2\,q_r}{2}+\frac{\gamma\,p_{phi}^2\,q_{phid}^2}{2\,c^2\,m\,q_r}-\frac{G\,M\,m}{q_r^2}+\frac{p_{phi}^2}{\gamma\,m\,q_r^3}$$

(Hr4) $$p_{phid}=0$$

(%i56)
```
Hr: expand((first((solve([Hr1,Hr2,Hr3,Hr4],
    [q_rd, q_phid, p_rd, p_phid])))))$
```

(%i60) Hr[1]; Hr[2]; Hr[3]; Hr[4];

(%o57) $q_{rd}=\frac{p_r}{\gamma\, m}$

(%o58) $q_{phid}=\frac{p_{phi}}{\gamma\, m\, {q_r}^2}$

(%o59) $p_{rd}=-\frac{G\, M\, m}{{q_r}^2}+\frac{{p_{phi}}^2\, {p_r}^2}{2\,\gamma\, c^2\, m^3\, {q_r}^3}+\frac{3\, {p_{phi}}^2}{2\,\gamma\, m\, {q_r}^3}+\frac{{p_{phi}}^4}{2\,\gamma\, c^2\, m^3\, {q_r}^5}$

(%o60) $p_{phid}=0$