436(4): Separation of Variables Solution of the Schroedinger Equation.

The Schroedinger equation is the diffusion equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial r^2} + U(r)\psi = i\hbar\frac{\partial\psi}{\partial t} \quad -(1)$$

where ψ is the complete wavefunction and $U(r)$ is the potential energy. The wavefunction is expressed as the product:

$$\psi = \psi_1(r)\psi_2(t) \quad -(2)$$

into an r dependent and t dependent part. It follows that:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}(\psi_1\psi_2) + U(r)\psi_1\psi_2 = i\hbar\frac{\partial}{\partial t}(\psi_1\psi_2) \quad -(3)$$

i.e.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_1}{\partial r^2}\psi_2 + U(r)\psi_1\psi_2 = i\hbar\psi_1\frac{\partial\psi_2}{\partial t} \quad -(4)$$

Divide by $\psi_1\psi_2$:

$$-\frac{\hbar^2}{2m}\frac{1}{\psi_1}\frac{\partial^2\psi_1}{\partial r^2} + U(r) = \frac{i\hbar}{\psi_2}\frac{\partial\psi_2}{\partial t} \quad -(5)$$

This divides into two equations:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_1}{\partial r^2} + U(r)\psi_1 = E\psi_1 \quad -(6)$$

and

$$i\hbar\frac{\partial\psi_2}{\partial t} = E\psi_2 \quad -(7)$$

which give the complete solution:

$$\psi(r,t) = \psi_1(r)\psi_2(t) \quad -(8)$$

$$= \exp\left(-i\frac{Et}{\hbar}\right)\psi_1(r)$$

2) In m space, the generally covariant solution is:

$$\phi\left(\frac{r}{n(r)^{1/2}}, m^{1/2} t\right) = \exp\left(-i\frac{E}{\hbar} m^{1/2}(r) t\right) \phi\left(\frac{r}{n(r)^{1/2}}\right) \quad -(9)$$

This is the general solution of the Schroedinger equation in the presence of the vacuum. The replacement rule in the wave function is:

$$t \rightarrow m^{1/2}(r)\, t \quad -(10)$$

$$r \rightarrow \frac{r}{n(r)^{1/2}} \quad -(11)$$

The complete solutions of the standard problems of quantum mechanics are as follows:

1) Free Particle

The energy levels are:

$$E = \hbar\omega \quad -(12)$$

and

$$\phi_2(t) = \exp(-i\omega t) \quad -(13)$$

2) Harmonic Oscillator

$$E = \left(n + \frac{1}{2}\right)\hbar\omega \quad -(14)$$

and

$$\phi_2(t) = \exp\left(-i\left(n + \frac{1}{2}\right)\omega t\right) \quad -(15)$$

3) Anharmonic Oscillator

$$E = \hbar\omega\left(n + \frac{1}{2}\right)\left(1 - \left(n + \frac{1}{2}\right)x\right) \quad -(16)$$

$$\phi_2(t) = \exp\left(-i\omega t\left(n + \frac{1}{2}\right)\left(1 - \left(n + \frac{1}{2}\right)x\right)\right) \quad -(17)$$

+) <u>Particle in a Box</u>

$$E = \frac{n^2 \hbar^2 \pi^2}{2mL^2} \quad -(18)$$

So

$$\psi_2(t) = \exp\left(- \frac{i n^2 \hbar \pi^2 t}{2mL^2} \right) \quad -(19)$$

<u>Hydrogen Atom</u>

$$E = - \frac{\mu e^4}{32\pi^2 \epsilon_0^2 \hbar^2 n^2} \quad -(20)$$

So

$$\psi_2(t) = \exp\left(- i \frac{\mu e^4 t}{32\pi^2 \epsilon_0^2 \hbar^3 n^2} \right) \quad -(21)$$

In each case the time dependent wave function in the presence of the vacuum is found from

$$t \longrightarrow n(r)^{1/2}\, t \quad -(22)$$

and this is the time dependent wave function in general relativity: the generally covariant wave function.

In the usual quantum mechanics, eq. (7) gives

$$\langle E \rangle = i\hbar \int \psi_2^* \frac{d\psi_2}{dt}\, d\tau. \quad -(23)$$

Using:

$$\psi_2(t) = \exp\left(- i \frac{Et}{\hbar} \right) \quad -(24)$$

then

$$\frac{d\psi_2}{dt} = - i \frac{E}{\hbar} \psi_2 \quad -(25)$$

5)

From eqs. (23) and (25):

$$\langle E \rangle = E \int \psi_2^* \psi_2 \, d\tau \quad -(26)$$

If the wavefunction is correctly normalized then:

$$\int \psi_2^* \psi_2 \, d\tau = 1 \quad -(27)$$

So

$$\langle E \rangle = E \, . \quad -(28)$$

However, in the face:

$$\psi_2(t) = \exp\left(-\frac{iE}{\hbar} m(r)^{1/2} t\right) \quad -(29)$$

So

$$\frac{d\psi_2(t)}{dt} = -\frac{iE}{\hbar} m(r)^{1/2} \psi_2(t) \quad -(30)$$

and

$$\boxed{\langle E \rangle = E \int \psi_2^* m(r)^{1/2} \psi_2 \, d\tau} \quad -(31)$$

It is clear from this equation that energy levels are shifted by the vacuum. <u>This is a general law of quantum mechanics.</u>