

PROOF OF THE LORENTZ COVARIANCE OF THE B CYCLIC THEOREM.

As in S.P. Carroll, “Spacetime and Geometry: Introduction to General Relativity” (Addison Wesley, New York, 2004, define the complete vector field by:

$$V = V^\mu e_{\mu'} = V^{\nu'} e_{\nu'} \quad (1)$$

Where V^μ denotes its components and where e_μ denotes its basis elements. Eq. (1) shows that the complete vector field is invariant under the general transformation of coordinates. For example, considering a Lorentz boost in the X axis , set up the column two vector:

$$V^\mu = \begin{pmatrix} c t \\ X \end{pmatrix} \quad (2)$$

where c is the speed of light and t is the time. The X and Y axes remain the same so we need only consider Eq. (2). The vector field is:

$$\underline{V} = c t \underline{e}_0 + X \underline{i} \quad (3)$$

in vector notation. The components are ct and X, and the basis elements are the unit vectors \underline{e}_0 and \underline{i} . The latter is part of the Cartesian system \underline{i} , \underline{j} and \underline{k} , and the former is the timelike unit vector in four dimensional spacetime. So there are four unit vectors in four dimensions. The invariance of Eq. (1) (tensor notation) translates into the following vector notation:

$$\underline{V} = \underline{V}' = (c t)' \underline{e}_0' + X' \underline{i}' \quad (4)$$

The Lorentz transform is described in a textbook such as J.D.Jackson, “Classical Electrodynamics”(Wiley, 3rd Edition, 1999). Consider the frame K' moving at v in the positive Z direction with respect to a frame K. The speed of light in a vacuum, c , is postulated to be the same in both frames. In frame K:

$$X^2 + Y^2 + Z^2 = c^2 t^2 \quad (5)$$

and in frame K' :

$$X'^2 + Y'^2 + Z'^2 = c^2 t'^2 \quad (6)$$

Lorentz postulated that:

$$c^2 t'^2 - (X'^2 + Y'^2 + Z'^2) = \lambda^2 (c^2 t^2 - (X^2 + Y^2 + Z^2)) \quad (7)$$

And it may be shown that for all v:

$$\lambda = 1 \quad (8)$$

Use the notation:

$$X_0 = c t \quad , \quad X_1 = X \quad , \quad X_2 = Y \quad , \quad X_3 = Z \quad (9)$$

It follows from Eq. (7) that,

$$\left. \begin{aligned} X_0' &= \gamma (X_0 - \beta X_1) \\ X_1' &= \gamma (X_1 - \beta X_0) \\ X_2' &= X_2 \\ X_3' &= X_3 \end{aligned} \right\} \quad (10)$$

where

$$\beta = \frac{v}{c} \quad , \quad \gamma = (1 - v^2/c^2)^{-1/2} \quad (11)$$

Finally parameterize as follows:

$$\beta = \tanh \varphi \quad , \quad \gamma = \cosh \varphi \quad , \quad \gamma \beta = \sinh \varphi \quad (12)$$

The X axis Lorentz boost matrix is then:

$$\Lambda = \begin{pmatrix} \cosh \varphi & -\sinh \varphi \\ -\sinh \varphi & \cosh \varphi \end{pmatrix} \quad (13)$$

The inverse Lorentz boost matrix is denoted Λ^{-1} and is defined by:

$$\Lambda \Lambda^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (14)$$

So:

$$\Lambda^{-1} = \begin{pmatrix} \cosh \varphi & \sinh \varphi \\ \sinh \varphi & \cosh \varphi \end{pmatrix} \quad (15)$$

The components of V transform under the Lorentz transformation as:

$$\begin{pmatrix} c t' \\ X' \end{pmatrix} = \begin{pmatrix} \cosh \varphi & -\sinh \varphi \\ -\sinh \varphi & \cosh \varphi \end{pmatrix} \quad (16)$$

i.e:

$$c t' = c t \cosh \varphi - X \sinh \varphi \quad (17)$$

$$X' = -c t \sinh \varphi + X \cosh \varphi \quad (18)$$

The unit vectors transform under the inverse Lorentz transformation as:

$$\begin{pmatrix} \underline{e_0'} \\ \underline{i'} \end{pmatrix} = \begin{pmatrix} \cosh \varphi & \sinh \varphi \\ \sinh \varphi & \cosh \varphi \end{pmatrix} \quad (19)$$

i.e.

$$\underline{e}_0' = \underline{e}_0 \cosh \varphi + \underline{i}' \sinh \varphi \quad (20)$$

$$\underline{i}' = \underline{e}_0 \sinh \varphi + \underline{i}' \cosh \varphi \quad (21)$$

Both components and unite vectors transform covariantly according to Eq. (1) and (4).

The complete vector field remains constant. We can demonstrate this constancy as follows. We have:

$$\underline{V} = c t \underline{e}_0 + X \underline{i} \quad (22)$$

$$\underline{V}' = (c t)' \underline{e}_0' + X' \underline{i}' \quad (23)$$

Eq.(23) is:

$$\begin{aligned} \underline{V}' &= (c t \cosh \varphi - X \sinh \varphi) (\underline{e}_0 \cosh \varphi + \underline{i}' \sinh \varphi) \\ &\quad + (X \cosh \varphi - c t \sinh \varphi) (\underline{e}_0 \sinh \varphi + \underline{i}' \cosh \varphi) \\ &= c \underline{t} \underline{e}_0 (\cosh^2 \varphi - \sinh^2 \varphi) + X \underline{i} (\cosh^2 \varphi - \sinh^2 \varphi) \\ &= c \underline{t} \underline{e}_0 + X \underline{i} = \underline{V} \end{aligned} \quad (24)$$

Q.E.D. (quod erat demonstrandum, that which we wanted to prove).

Application to the B Cyclic Theorem:

The complex circular basis is defined as follows:

$$\left. \begin{aligned} \underline{e}^{(1)} &= \frac{1}{\sqrt{2}} (\underline{i} - i \underline{j}) \\ \underline{e}^{(2)} &= \frac{1}{\sqrt{2}} (\underline{i} + i \underline{j}) \\ \underline{e}^{(3)} &= \underline{k} \\ \underline{e}^{(0)} &= \underline{e}_0 \end{aligned} \right\} \quad (25)$$

It is a fundamental basis set, as fundamental as the Cartesian basis set. The spacelike unit vectors are related cyclically:

$$\left. \begin{aligned}
 \underline{e}^{(1)} \times \underline{e}^{(2)} &= i \underline{e}^{(3)*} \\
 \underline{e}^{(3)} \times \underline{e}^{(1)} &= i \underline{e}^{(2)*} \\
 \underline{e}^{(2)} \times \underline{e}^{(3)} &= i \underline{e}^{(1)*}
 \end{aligned} \right\} \quad (26)$$

where * denotes “complex conjugate”. Similarly the Cartesian unit vectors are related cyclically:

$$\left. \begin{aligned}
 \underline{i} \times \underline{j} &= \underline{k} \\
 \underline{k} \times \underline{i} &= \underline{j} \\
 \underline{j} \times \underline{k} &= \underline{i}
 \end{aligned} \right\} \quad (27)$$

The B Cyclic Theorem is given in the Omnia Opera of this site and is:

$$\left. \begin{aligned}
 \underline{B}^{(1)} \times \underline{B}^{(2)} &= i \underline{B}^{(0)} \underline{B}^{(3)*} \\
 \underline{B}^{(3)} \times \underline{B}^{(1)} &= i \underline{B}^{(0)} \underline{B}^{(2)*} \\
 \underline{B}^{(2)} \times \underline{B}^{(3)} &= i \underline{B}^{(0)} \underline{B}^{(1)*}
 \end{aligned} \right\} \quad (28)$$

where the magnetic flux densities are defined by:

$$\underline{B}^{(1)} = \underline{B}^{(0)} \underline{e}^{(1)} e^{i\varphi}, \quad \underline{B}^{(2)} = \underline{B}^{(0)} \underline{e}^{(2)} e^{-i\varphi}, \quad \underline{B}^{(3)} = \underline{B}^{(0)} \underline{e}^{(3)} \quad (29)$$

Here φ is the electromagnetic phase. So the B Cyclic Theorem is the relation (26) between basis elements of the complex circular basis. These basis elements are Lorentz covariant by definition, so the B Cyclic Theorem is Lorentz covariant, Q.E.D.

Similarly, the B Cyclic Theorem is generally covariant.