DEFINITIVE PROOF 3: THE TETRAD POSTULATE

The tetrad postulate is the name given to the equation that results from the very fundamental requirement that the complete vector field be independent of its components and basis elements. Without this mathematical property, physics would be inconceivable, because, for example, a vector field in three dimensional space in Cartesian coordinates would not be the same vector field if expressed in spherical polar coordinates. Every proof of Cartan geometry relies on the tetrad postulate, which was introduced in about 1925 or earlier, and has been taught ever since. ECE uses this standard tetrad postulate.

Proof.

Consider the covariant derivative of Riemann geometry:

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{1}$$

and express this covariant derivative in a spacetime labelled by Latin indices. This was introduced by Cartan as the Minkowski spacetime tangential to the base manifold at point P, but has been generalized in ECE theory to describe spin. The tetrad is defined by:

$$V^a = q^a_\mu V^\mu \tag{2}$$

and the covariant derivative becomes:

$$D_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} V^b \tag{3}$$

where $\omega^a_{\mu b}$ is the spin connection. Any basis elements can be used with the index a, so this introduces an advantage over Riemann geometry, whose basis elements are ∂_μ. The complete vector field is the same, so:

$$D\,V = D_\mu V^\nu \, \mathrm{d}x^\mu \otimes \partial_\nu = D_\mu V^a \, \mathrm{d}x^\mu \otimes \hat{q}_a \tag{4}$$

where the component and basis elements of the vector field are given. The basis elements and components are related by equations similar to (2):

$$\hat{q}_a = q^\sigma_a \, \partial_\sigma \tag{5}$$

and

$$V^a = q^a_\nu V^\nu \tag{6}$$

Therefore, eq. (4) may be expanded as:

$$D\,V = (\partial_\mu \, (q^a_\nu V^\nu) + \omega^a_{\mu b} q^b_\lambda V^\lambda) \, \mathrm{d}x^\mu \otimes (q^\sigma_a \partial_\sigma) \tag{7}$$

Now use the commutative property of matrices to rewrite eq. (7):

$$D\,V = q_a^\sigma\ (\partial_\mu (q_\nu^a V^\nu) + \omega_{\mu b}^a q_\lambda^b V^\lambda)\ \mathrm{d}x^\mu \otimes \partial_\sigma \tag{8}$$

The dummy indices σ are now re-labelled as ν to give:

$$D\,V = q_a^\nu\ (q_\nu^a\ \partial_\mu V^\nu + V^\nu \partial_\mu q_\nu^a + \omega_{\mu b}^a q_\lambda^b V^\lambda\ )\ \mathrm{d}x^\mu \otimes \partial_\nu \tag{9}$$

The Cartan convention is that q_a^σ is defined as the inverse of q_σ^a , so:

$$q_a^\sigma\ .\ q_\sigma^a\ = 1 \tag{10}$$

Therefore, eq. (9) becomes:

$$D\,V = (\ \partial_\mu V^\nu + q_a^\nu \partial_\mu q_\nu^a V^\nu + q_a^\nu \omega_{\mu b}^a q_\lambda^b V^\lambda)\ \mathrm{d}x^\mu \otimes \partial_\nu \tag{11}$$

The dummy indices ν in the second term on the right hand side are re-labelled as λ to give:

$$D\,V = (\ \partial_\mu V^\nu + q_a^\nu (\partial_\mu q_\lambda^a + \omega_{\mu b}^a q_\lambda^b) V^\lambda)\ \mathrm{d}x^\mu \otimes \partial_\nu \tag{12}$$

Comparing this with eq. (1) gives:

$$\Gamma_{\mu\lambda}^\nu = q_a^\nu (\partial_\mu q_\lambda^a + \omega_{\mu b}^a q_\lambda^b) \tag{13}$$

and using eq.(10) we obtain the tetrad postulate, quod erat demonstratum:

$$D_\mu\ q_\lambda^a = \partial_\mu q_\lambda^a + \omega_{\mu b}^a q_\lambda^b - \Gamma_{\mu\lambda}^\nu q_\nu^a\ = 0 \tag{14}$$

in which the covariant derivative acts on a rank two mixed index tensor, the tetrad.

Additional notes for Proof 3, Equation 9:

Equation (9) was (changing σ for ν in the original equation) :

$$D\,V = q_a^\sigma\ (q_\nu^a\ \partial_\mu V^\nu + V^\nu \partial_\mu q_\nu^a + \omega_{\mu b}^a q_\lambda^b V^\lambda\ )\ \mathrm{d}x^\mu \otimes \partial_\sigma \tag{15}$$

$$= q_a^\sigma\ q_\nu^a\ \partial_\mu V^\nu + q_a^\sigma\ (V^\nu \partial_\mu q_\nu^a + \omega_{\mu b}^a q_\lambda^b V^\lambda\ )\ \mathrm{d}x^\mu \otimes \partial_\sigma \tag{16}$$

The Cartan convention is:

$$q^{\sigma}_{a}\ q^{a}_{\nu} = \delta^{\sigma}_{\nu} \tag{17}$$

where:

$$\left.\begin{array}{lll} \delta^{\sigma}_{\nu} = 1 & \text{if} & \sigma = \nu \\ \delta^{\sigma}_{\nu} = 0 & \text{if} & \sigma \neq \nu \end{array}\right\} \tag{18}$$

Equation (1) is:

$$D\ V = q^{\sigma}_{a}\ q^{a}_{\nu}\ \partial_{\mu} V^{\nu}\ \mathrm{d}x^{\mu} \otimes \partial_{\sigma} + \ldots.$$

$$= \ \delta^{\sigma}_{\nu}\ \ \partial_{\mu} V^{\nu}\ \mathrm{d}x^{\mu} \otimes \partial_{\sigma} + \ldots.$$

$$= \ \delta^{0}_{0}\ \partial_{\mu} V^{0}\ \mathrm{d}x^{\mu} \otimes \partial_{0} + \delta^{1}_{1}\ \partial_{\mu} V^{1}\ \mathrm{d}x^{\mu} \otimes \partial_{1} +$$

$$\delta^{2}_{2}\ \partial_{\mu} V^{2}\ \mathrm{d}x^{\mu} \otimes \partial_{2} + \delta^{3}_{3}\ \partial_{\mu} V^{3}\ \mathrm{d}x^{\mu} \otimes \partial_{3} + \ldots.$$

$$= \ \partial_{\mu} V^{0}\ \mathrm{d}x^{\mu} \otimes \partial_{0} + \ \partial_{\mu} V^{1}\ \mathrm{d}x^{\mu} \otimes \partial_{1} + \ldots.$$

$$\partial_{\mu} V^{2}\ \mathrm{d}x^{\mu} \otimes \partial_{2} + \ \partial_{\mu} V^{3}\ \mathrm{d}x^{\mu} \otimes \partial_{3} + \ldots.$$

$$= \ \ \partial_{\mu} V^{\nu}\ \mathrm{d}x^{\mu} \otimes \partial_{\nu} + \ldots.$$

Q.E.D.