

DEFINITIVE PROOF 4 : THE CARTAN BIANCHI IDENTITY

The Cartan Bianchi identity was first introduced by Cartan in about 1925 and has been a standard feature of Cartan geometry ever since. It is a rigorously correct identity based on the definitions of the curvature and torsion tensors introduced in Definitive Proof 2. As always in Cartan geometry it relies on the tetrad postulate without loss of generality. It links together torsion and curvature and torsion, and is the basis of the homogeneous field equations of both dynamics and electrodynamics in ECE theory.

Proof.

In standard differential form notation (see Carroll, chapter 3) the identity is:

$$d\wedge T^a + \omega_b^a \wedge T^b := R_b^a \wedge q^b \quad (1)$$

which translates into tensor notation as follows:

$$\begin{aligned} \partial_\mu T_{\nu\rho}^a + \partial_\rho T_{\mu\nu}^a + \partial_\nu T_{\rho\mu}^a + \omega_{\mu b}^a T_{\nu\rho}^b + \omega_{\rho b}^a T_{\mu\nu}^b + \omega_{\nu b}^a T_{\rho\mu}^b := \\ (R_{\mu\nu\rho}^\lambda + R_{\rho\mu\nu}^\lambda + R_{\nu\rho\mu}^\lambda) q_\lambda^a \end{aligned} \quad (2)$$

where:

$$T_{\nu\rho}^a = (\Gamma_{\nu\rho}^\lambda - \Gamma_{\rho\nu}^\lambda) q_\lambda^a \quad (3)$$

and so on. Using Leibniz Theorem:

$$\partial_\mu T_{\nu\rho}^a = (\partial_\mu \Gamma_{\nu\rho}^\lambda - \partial_\mu \Gamma_{\rho\nu}^\lambda) q_\lambda^a + (\Gamma_{\nu\rho}^\lambda - \Gamma_{\rho\nu}^\lambda) \partial_\mu q_\lambda^a \quad (4)$$

and so on. Therefore, Eq. (2) becomes:

$$\begin{aligned} (\partial_\mu \Gamma_{\nu\rho}^\lambda - \partial_\mu \Gamma_{\rho\nu}^\lambda) q_\lambda^a + (\Gamma_{\nu\rho}^\lambda - \Gamma_{\rho\nu}^\lambda) (\partial_\mu q_\lambda^a + \omega_{\mu b}^a q_\lambda^b) + \dots := \\ (R_{\mu\nu\rho}^\lambda + R_{\rho\mu\nu}^\lambda + R_{\nu\rho\mu}^\lambda) q_\lambda^a \end{aligned} \quad (5)$$

Now relabel repeated summation indices (dummy indices) as follows:

$$\lambda \rightarrow \sigma \quad (6)$$

to obtain:

$$\begin{aligned}
& (\partial_\mu \Gamma_{\nu\rho}^\lambda - \partial_\mu \Gamma_{\rho\nu}^\lambda) q_\lambda^a + (\Gamma_{\nu\rho}^\sigma - \Gamma_{\rho\nu}^\sigma) (\partial_\mu q_\sigma^a + \omega_{\mu b}^a q_\sigma^b) + \dots := \\
& \qquad \qquad \qquad (R_{\mu\nu\rho}^\lambda + R_{\rho\mu\nu}^\lambda + R_{\nu\rho\mu}^\lambda) q_\lambda^a \qquad (7)
\end{aligned}$$

Use the tetrad postulate to obtain:

$$\begin{aligned}
& \partial_\mu \Gamma_{\nu\rho}^\lambda - \partial_\mu \Gamma_{\rho\nu}^\lambda + \Gamma_{\mu\sigma}^\lambda (\Gamma_{\nu\rho}^\sigma - \Gamma_{\rho\nu}^\sigma) + \\
& \partial_\rho \Gamma_{\mu\nu}^\lambda - \partial_\rho \Gamma_{\nu\mu}^\lambda + \Gamma_{\rho\sigma}^\lambda (\Gamma_{\mu\nu}^\sigma - \Gamma_{\nu\mu}^\sigma) + \\
& \partial_\nu \Gamma_{\rho\mu}^\lambda - \partial_\nu \Gamma_{\mu\rho}^\lambda + \Gamma_{\nu\sigma}^\lambda (\Gamma_{\rho\mu}^\sigma - \Gamma_{\mu\rho}^\sigma) := R_{\mu\nu\rho}^\lambda + R_{\rho\mu\nu}^\lambda + R_{\nu\rho\mu}^\lambda \qquad (8)
\end{aligned}$$

and re-arrange:

$$\begin{aligned}
& R_{\rho\mu\nu}^\lambda + R_{\mu\nu\rho}^\lambda + R_{\nu\rho\mu}^\lambda := \\
& \partial_\mu \Gamma_{\nu\rho}^\lambda - \partial_\nu \Gamma_{\mu\rho}^\lambda + \Gamma_{\mu\sigma}^\lambda \Gamma_{\nu\rho}^\sigma - \Gamma_{\nu\sigma}^\lambda \Gamma_{\mu\rho}^\sigma + \\
& \partial_\nu \Gamma_{\rho\mu}^\lambda - \partial_\rho \Gamma_{\nu\mu}^\lambda + \Gamma_{\nu\sigma}^\lambda \Gamma_{\rho\mu}^\sigma - \Gamma_{\rho\sigma}^\lambda \Gamma_{\nu\mu}^\sigma + \\
& \partial_\rho \Gamma_{\mu\nu}^\lambda - \partial_\mu \Gamma_{\rho\nu}^\lambda + \Gamma_{\rho\sigma}^\lambda \Gamma_{\mu\nu}^\sigma - \Gamma_{\mu\sigma}^\lambda \Gamma_{\rho\nu}^\sigma \qquad (9)
\end{aligned}$$

This is an exact identity because the curvature tensors on the left hand side are defined by:

$$\begin{aligned}
R_{\rho\mu\nu}^\lambda &= \partial_\nu \Gamma_{\rho\mu}^\lambda - \partial_\rho \Gamma_{\nu\mu}^\lambda + \Gamma_{\nu\sigma}^\lambda \Gamma_{\rho\mu}^\sigma - \Gamma_{\rho\sigma}^\lambda \Gamma_{\nu\mu}^\sigma \\
R_{\mu\nu\rho}^\lambda &= \partial_\rho \Gamma_{\mu\nu}^\lambda - \partial_\mu \Gamma_{\rho\nu}^\lambda + \Gamma_{\rho\sigma}^\lambda \Gamma_{\mu\nu}^\sigma - \Gamma_{\mu\sigma}^\lambda \Gamma_{\rho\nu}^\sigma \\
R_{\nu\rho\mu}^\lambda &= \partial_\mu \Gamma_{\nu\rho}^\lambda - \partial_\nu \Gamma_{\mu\rho}^\lambda + \Gamma_{\mu\sigma}^\lambda \Gamma_{\nu\rho}^\sigma - \Gamma_{\nu\sigma}^\lambda \Gamma_{\mu\rho}^\sigma \qquad (10)
\end{aligned}$$

Quod erat demonstrandum.