DEFINITIVE PROOF 5 : CARTAN EVANS IDENTITY

This is an example of the Cartan Bianchi dual identity and is constructed simply by taking the Hodge dual term of the result of Definitive Proof 2. I named it to distinguish it from the Cartan Bianchi identity.

Proof.

Consider the action of the commutator of covariant derivatives on the vector V^ρ in any spacetime:

$$[\, D_\mu \, , D_\nu \,]\, V^\rho = R^\rho_{\sigma\mu\nu}\, V^\sigma - T^\lambda_{\mu\nu}\, D_\lambda V^\rho \quad (1)$$

Take the Hodge dual term by term and lower indices term by term to find:

$$[D_\alpha \, , D_\beta]_{\mathbf{HD}}\, V^\rho \; := \tilde{R}^\rho_{\sigma\alpha\beta}\, V^\sigma \; - \tilde{T}^\lambda_{\alpha\beta}\, D_\lambda\, V^\rho \quad (2)$$

Where the tilde denotes Hodge dual. In a four dimensional spacetime the Hodge dual of an antisymmetric tensor (i.e. a differential two-form) , is another differential two-form. For example, indices 01 are transformed to 23. So the resulting is an example of the original tensor, but with different indices. By definition:

$$\widetilde{T}^\lambda_{\alpha\beta} = \; (\, \Gamma^\lambda_{\alpha\beta} - \Gamma^\lambda_{\beta\alpha} \,)_{\mathbf{HD}} \quad (3)$$

and

$$\tilde{R}^\rho_{\sigma\alpha\beta} = (\, \partial_\beta\, \Gamma^\rho_{\sigma\alpha} - \partial_\sigma\, \Gamma^\rho_{\beta\alpha} + \Gamma^\rho_{\beta\lambda}\Gamma^\lambda_{\sigma\alpha} - \Gamma^\rho_{\sigma\lambda}\, \Gamma^\lambda_{\beta\alpha})_{\mathbf{HD}} \quad (4)$$

A cyclic sum of the definitions (4) is identically equal to the same cyclic sum of the definitions of the same tensors. This is an example of the Cartan Bianchi identity proven in Definitive Proof 4. Therefore it follows that:

$$D_\mu\, \widetilde{T}^a_{\sigma\rho} + D_\rho\, \widetilde{T}^a_{\mu\nu} + D_\nu\, \widetilde{T}^a_{\rho\mu} \; := \tilde{R}^a_{\mu\nu\rho} + \tilde{R}^a_{\rho\mu\nu} + \tilde{R}^a_{\nu\rho\mu} \quad (5)$$

which can be written as the Cartan Evans dual identity:

$$D_\mu T^{a\mu\nu} \; := \; R^{a\mu\nu}_\mu \quad (6)$$

In the base manifold, eq.(6) is:

$$D_\mu T^{\kappa\mu\nu} \; := \; R^{\kappa\mu\nu}_\mu \quad (7)$$

and shows that the covariant derivative of torsion is a well defined curvature. The original Cartan Bianchi identity is:

$$D_\mu \, \tilde{T}^{\kappa\mu\nu} := \tilde{R}_\mu^{\kappa\mu\nu} \tag{8}$$

and equations (7) and (8) are invariant under Hodge dual transformations.

Some more details of Proof 5

The Dual Identity

This is an important proof because the flaw in the Einstein field equation shows up in the dual identity in its tensorial expression:

$$D_\mu \, T^{\kappa\mu\nu} \; := \; R_\mu^{\kappa\mu\nu} \tag{1}$$

The neglect of torsion in the standard model means that this fundamental equation of geometry is not obeyed. The proof of the dual identity depends on the fact that the Hodge dual of a two-form in 4-D space is another two-form, with different indices. So the Hodge dual of the commutator operator is:

$$[D^\alpha \, , \, D^\beta \,] = \tfrac{1}{2} \, \|g\|^{1/2} \, \epsilon^{\alpha\beta\mu\nu} [D_\mu \, , \, D_\nu \,] \tag{2}$$

Where $\epsilon^{\alpha\beta\mu\nu}$ is the totally antisymmetric unit tensor of flat spacetime by definition. Here, $\|g\|$ is the determinant of the metric $g_{\mu\nu}$ of non-flat spacetime. This is a matter of definition or convention. An example of Eq. (2) is:

$$[D^0 \, , \, D^1 \,] = \|g\|^{1/2} \, [D_2 \, , \, D_3] \tag{3}$$

where we have worked out the summation over repeated μ and ν indices in Eq. (2). So apart from a factor $\|g\|^{1/2}$ (a number) the Hodge dual transform produces $[D^0 \, , \, D^1 \,]$ from $[D_2 \, , \, D_3]$. The original indices 2 and 3 are changed to 0 and 1.

The indices of $[D^0 \, , \, D^1 \,]$ are lowered to $[D_0 \, , \, D_1]$ as follows:

$$[D_0 \, , \, D_1] = g_{00} \, g_{11} \, [D^0 \, , \, D^1 \,] \tag{4}$$

$$[D_0 \, , \, D_1] = g_{00} \, g_{11} \|g\|^{1/2} [D_2 \, , \, D_3] \tag{5}$$

Therefore, $[D_0\ ,\ D_1]$ is $[D_2\ ,\ D_3]$ multiplied by a number, $g_{00}\ g_{11}\|g\|^{1/2}$.

Both $[D_0\ ,\ D_1]$ and $[D_2\ ,\ D_3]$ are examples of $[D_\mu\ ,\ D_\nu]$, respectively, with

$$\mu = 0\ ,\quad \nu = 1 \tag{6}$$

and
$$\mu = 2\ ,\quad \nu = 3 \tag{7}$$

It follows that the following is true. If

$$[D_2\ ,\ D_3]\, V^\rho \ = \ R^\rho_{\sigma 23}\ V^\sigma\ -\ T^\lambda_{23} D_\lambda V^\rho \tag{8}$$

then:
$$[D_0\ ,\ D_1]\, V^\rho \ = \ R^\rho_{\sigma 01}\ V^\sigma\ -\ T^\lambda_{01} D_\lambda V^\rho \tag{9}$$

because $g_{00}\ g_{11}\|g\|^{1/2}$ cancels out. Eqs. (8) and (9) are both examples of

$$[D_\mu\ ,\ D_\nu]\, V^\rho \ = \ R^\rho_{\sigma\mu\nu}\ V^\sigma\ -\ T^\lambda_{\mu\nu} D_\lambda V^\rho \tag{10}$$

It may be shown that Eq. (10) implies Eq. (1).

Q.E.D.

<u>Example</u>

If the original entity is for:

$$\nu = 3 \tag{11}$$

Then the dual identity is for:

$$\nu = 1 \tag{12}$$

From Eq. (1) and (11) the original identity is:

$$D_\mu\ T^{\kappa\mu 3}\ :=\ R^{\kappa\mu 3}_\mu \tag{13}$$

i.e. $$D_0\ T^{\kappa 03}\ +\ D_1\ T^{\kappa 13} + D_2\ T^{\kappa 23} = R^{\kappa 03}_0 + R^{\kappa 13}_1 + R^{\kappa 23}_2 \tag{14}$$

From Eqs.(1) and (12) the dual identity is:

$$D_\mu \, T^{\kappa\mu 1} \;:=\; R_\mu^{\kappa\mu 1} \tag{15}$$

i.e. $$D_0 \, T^{\kappa 01} + D_2 \, T^{\kappa 21} + D_3 \, T^{\kappa 31} = R_0^{\kappa 01} + R_2^{\kappa 21} + R_3^{\kappa 31} \tag{16}$$

The computer algebra shows that the Einstein field equation does not obey <u>both</u> Eqs. (14) and (16) as it should.

More details of Proof Five

The Hodge dual of the commutator is defined by:

$$[D^\alpha \,,\, D^\beta\,] = \tfrac{1}{2}\, \|g\|^{1/2}\, \epsilon^{\alpha\beta\mu\nu}[D_\mu,\; D_\nu\,] \tag{1}$$

and there is a double summation over μ and ν. The antisymmetric unit tensor is defined by:

$$\epsilon^{0123} = -\,\epsilon^{0132} = 1 \tag{2}$$

and so on. So for example:

$$[D^0 \,,\, D^1\,] = \tfrac{1}{2}\, \|g\|^{1/2}\, (\epsilon^{0123}[D_2 \,,\, D_3\,] + \epsilon^{0132}[D_3 \,,\, D_2\,]\,) \tag{3}$$

use: $$[D_3 \,,\, D_2\,] = -\,[D_2 \,,\, D_3\,] \tag{4}$$

to find: $$[D^0 \,,\, D^1\,] = \|g\|^{1/2}[D_2 \,,\, D_3\,] \tag{5}$$

The $\|g\|^{1/2}$ is a number. It is the square root of the positive value of $|\,g\,|$, which denotes the determinant of the metric:

$$|\,g\,| = \begin{vmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{vmatrix} \tag{6}$$

The general rule for lowering the indices of a rank two tensor or tensor operator is:

$$[D_{\mu} \ , D_{\nu} \ ] \ = \ g_{\mu\alpha} \ g_{\nu\beta} \ [D^{\alpha} \ , D^{\beta} \ ] \tag{7}$$

i.e. the metric is applied twice to lower the indices.

Consider the example:

$$\mu = 0 \ , \ \nu = 1 \ , \ \alpha = 0 \ , \beta = 1 \tag{8}$$

Therefore:

$$[D_0 \ , D_1 \ ] \ = g_{00} \ g_{11} \ [D^0 \ , D^1 \ ]$$

$$[D_0 \ , D_1 \ ] \ = g_{00} \ g_{11} \ \|g\|^{1/2} [D_2 \ , D_3 \ ] \tag{9}$$

Similarly:

$$R^{\rho}_{\sigma 01} \ = g_{00} \ g_{11} \ \|g\|^{1/2} \ R^{\rho}_{\sigma 23} \tag{10}$$

$$T^{\lambda}_{01} \ = g_{00} \ g_{11} \ \|g\|^{1/2} \ T^{\lambda}_{23} \tag{11}$$

It is seen that the Hodge transformed quantities are examples of the original quantities, but with 23 index transformed to 01 index. The result is summarized in the following flow chart:

Flowchart I for Proof Five

$$[D_2 \ , D_3 \ ] \ V^{\rho} = R^{\rho}_{\sigma 23} \ V^{\sigma} - \ T^{\lambda}_{23} \ D_{\lambda} \ V^{\rho} \qquad \mathrm{I}$$

$$\downarrow$$

$$g_{00} \ g_{11} \ \|g\|^{1/2} \ [D_2 \ , D_3 \ ] \ V^{\rho} = g_{00} \ g_{11} \ \|g\|^{1/2} \ (R^{\rho}_{\sigma 23} \ V^{\sigma} - \ T^{\lambda}_{23} \ D_{\lambda} \ V^{\rho})$$

$$\downarrow$$

$$[D_0 \ , D_1 \ ] \ V^{\rho} = R^{\rho}_{\sigma 01} \ V^{\sigma} - T^{\lambda}_{01} \ D_{\lambda} \ V^{\rho} \qquad \mathrm{II}$$

I = Original

II = Hodge dual

Both are examples of:

$$[D_{\mu} \ , D_{\nu} \ ] \ V^{\rho} = R^{\rho}_{\sigma \mu\nu} \ V^{\sigma} - T^{\lambda}_{\mu\nu} \ D_{\lambda} \ V^{\rho}$$

$$\downarrow$$

$$D_{\mu} \ T^{\kappa\mu\nu} := R^{\kappa\mu\nu}_{\mu},$$

$$D_{\mu} \ \tilde{T}^{\kappa\mu\nu} := \tilde{R}^{\kappa\mu\nu}_{\mu}$$

Flowchart II for Proof Five

Any two-form in four dimensions is itself the Hodge dual of another two-form.

↓

Any two-form in four dimensions can be written with a tilde accent above it, the tilde denotes “Hodge dual”.

↓

If $D \wedge T := R \wedge \mathrm{q}$

then $D \wedge \tilde{T} := \tilde{R} \wedge \mathrm{q}$

↓

$$D_\mu \, \tilde{T}^{\kappa\mu\nu} := \tilde{R}_\mu^{\kappa\mu\nu}$$

$$D_\mu \, T^{\kappa\mu\nu} := R_\mu^{\kappa\mu\nu}$$

↓

$R_\mu^{\kappa\mu\nu}$ can be computed from the Einstein Field Equation.

↓

In general, $R_\mu^{\kappa\mu\nu}$ is not zero, Einstein equation fails.