NEW GENERAL CONDITION FOR ANY METRIC

Start with the definition of the tetrad:

$$V^a = q^a_\mu \, V^\mu \quad (1)$$

A particular case of this is:

$$x^\kappa = g^\kappa_\mu \, x^\mu \quad (2)$$

where g^κ_μ is the metric.

Consider: $$D_\mu \, V^\nu = \partial_\mu \, V^\nu + \Gamma^\nu_{\mu\lambda} \, V^\lambda \quad (3)$$

and $$D_\mu \, V^a = \partial_\mu \, V^a + \omega^a_{\mu b} \, V^b \quad (4)$$

Eqs. (3) and (4) imply the tetrad postulate:

$$\partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda = 0 \quad (5)$$

i.e. $$\Gamma^\lambda_{\mu\nu} = q^\lambda_a \, (\partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu \,) \quad (6)$$

using $$q^a_\lambda \, q^\alpha_a = \delta^\alpha_\lambda \quad (7)$$

The special case of Eq.(2) implies that Eq. (4) becomes:

$$\partial_\mu g^\kappa_\nu + \Gamma^\kappa_{\mu\lambda} g^\lambda_\nu - \Gamma^\lambda_{\mu\nu} g^\kappa_\lambda = 0 \quad (8)$$

i.e. a is replaced by κ , b by λ, and ω by Γ . In Eq. (8):

$$\Gamma^\kappa_{\mu\lambda} g^\lambda_\nu = \Gamma^\kappa_{\mu\nu} \quad (9)$$

and

$$\Gamma^\lambda_{\mu\nu} g^\kappa_\lambda = \Gamma^\kappa_{\mu\nu} \quad (10)$$

Therefore:

$$\boxed{\partial_\mu g^\kappa_\nu = 0} \tag{11}$$

This is an important new fundamental equation for the metric:

$$g^\kappa_\nu = g^{\kappa\alpha}\ g_{\alpha\nu} \tag{12}$$

Any Riemannian metric obeys Eq. (11), and in general any metric in any spacetime of any dimension, in general a spacetime with torsion and curvature.

Diagonal metric

In this case, off-diagonals are zero, so Eq. (11) produces:

$$\partial_\mu\ (\ g^{00}\ g_{00}\ ) = 0 \tag{13}$$

$$\partial_\mu\ (\ g^{11}\ g_{11}\ ) = 0 \tag{14}$$

$$\partial_\mu\ (\ g^{22}\ g_{22}\ ) = 0 \tag{15}$$

$$\partial_\mu\ (\ g^{33}\ g_{33}\ ) = 0 \tag{16}$$

for any μ. Eqs (13) – (16) are true because

$$g^{00}\ g_{00} = g^{11}\ g_{11} = g^{22}\ g_{22} = g^{33}\ g_{33} = 1 \tag{17}$$

In general, in four dimensions:

$$\partial_\mu\ (\ g^{\kappa\alpha}\ g_{\alpha\nu}\ ) = 0 \tag{18}$$

where:

$$g^{\kappa\alpha}\ g_{\alpha\nu} = g^{\kappa 0}\ g_{0\nu} + g^{\kappa 1}\ g_{1\nu} + g^{\kappa 2}\ g_{2\nu} + g^{\kappa 3}\ g_{3\nu} \tag{19}$$

and where the metric has diagonal and off-diagonal elements.

The metric of the orbital theorem of Paper 111 obeys Eq. (11) because it is a diagonal metric.

Computer Test.

It is possible now to test metrics with off-diagonal elements by using computer algebra with Eq. (18).