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ON THE RELATIVISTIC THEORY OF CHARGE

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The concept of charge is developed relativistically by assuming that there is a linear relation between point charge (e) and point mass (m) of the type:

$$e = \zeta m \tag{1}$$

where ζ is a scalar parameter which is unchanged in all reference frames. The theory shows that charge, in a relativistic development based on this hypothesis, depends in general on the velocity of the particle carrying the charge, and the latter vanishes at the speed of light. The hypothesis (1) also implies that charge depends on the scalar and vector potentials of the electromagnetic field. These conclusions are in qualitative agreement with experimental observation.

1. Introduction

The two fundamental concepts of point charge (e) and point mass (m) are usually assumed to be inherent properties of a particle, for example an elementary particle of matter. Point mass (m) is positive or zero in all particles (there is no contemporary evidence for negative mass) but point charge can be positive, zero and negative. The photon and neutrino, which travel at the speed of light, c, are taken to have zero mass, because in special relativity[1–3] kinetic energy becomes infinite at c unless $m = 0$. In a "particle" with no mass, it is observed experimentally that there is also no charge (e). In a particle with positive point mass (m), it is observed experimentally that charge (e) can be positive, zero, or negative. Thus, the photon and neutrino each have zero mass and zero charge, for example, the proton, neutron, and electron are particles with positive m and different e. Extending these elementary considerations to other fundamental particles known in contemporary theory and experimentation, it is found in every case that the above considerations apply.

It is possible to deduce from elementary considerations that charge, e, on a particle travelling at c must be zero, because in special relativity, no particle and no field can propagate at greater than c. Thus, the electric field radiated in the rest frame from a charge e moving at c must vanish in the rest frame, because the field cannot travel at greater than c and cannot escape the point charge e. The latter is effectively zero.

On these fundamentals, we make the hypothesis in this Letter that

$$e = \zeta m \tag{2}$$

where ζ is a parameter which is the same in all frames of reference, but which depends on the elementary nature of the particle under consideration. Thus, ζ is positive for a proton, zero for a neutron, and negative for an electron. In a photon, m is zero independently of ζ, so e is also zero. Similarly for a neutrino. When a massive particle such as an electron is accelerated to c, our hypothesis implies that its charge will vanish with its mass as a consequence of the law of conservation of energy. Thus, an electron travelling at c will no longer interact with an electromagnetic field.

In Sec. 1, the hypothesis (2) is used in the theory of special relativity to produce an expression for e in terms of the Hamiltonian, H, and scalar potential, ϕ, of a point particle in an electromagnetic field. A "point particle" has point mass and point charge. This produces an expression for e which shows that for constant H (law of conservation of energy), the point charge e vanishes at c for all ζ. In Sec. 2, the simple theory of Sec. 1 is extended using a four-dimensional development to show that e can also be expressed in terms of the vector potential $\mathbf{A}$ of the electromagnetic field, leading to an expression for the parameter ζ in terms of the scalar and vector potentials of the field, the constant total momentum $\mathbf{P}$ and the constant Hamiltonian H. A discussion is given finally in terms of the fact that charge e and mass m are interchangeable elementary concepts, both being, ultimately an expression of energy. These conclusions are reached by a simple application of the principles of special relativity and the law of conservation of energy.

1. An Expression for e in Terms of the Scalar Potential ϕ

Consider a point particle in the presence of an electromagnetic field. It is well known[1–3] that the Hamiltonian of the system (the constant sum of kinetic and potential energy) is

$$H = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} + e\phi\,. \tag{3}$$

Here v is the velocity of the particle in the rest frame, and ϕ the scalar potential of the electromagnetic field in the rest frame. Thus $e\phi$ is the potential energy and the other term on the right-hand side is the kinetic energy. The sum of kinetic and potential energies is a constant, H, the Hamiltonian. It is worth recalling that v is defined in Minkowski space in the rest frame as

$$\mathbf{v}^2 = (dx^2 + dy^2 + dz^2)/dt^2 \tag{4}$$

i.e., through the existence in Minkowski space of an interval between two events in spacetime. There is no non-relativistic meaning to Eq. (3) because in the Galilean limit there is no rest energy mc^2. It is emphasised that Eq. (3) is in the rest frame of the observer, and all quantities are defined in that frame.

From Eqs. (2) and (3) it follows immediately that:

$$e = \frac{H\left(1 - \frac{v^2}{c^2}\right)^{1/2}}{\frac{c^2}{\zeta} + \left(1 - \frac{v^2}{c^2}\right)^{1/2}\phi} \tag{5}$$

which is a relativistic definition of point charge given the hypothesis (2) that mass m is proportional to e. Since H is constant by the law of conservation of energy, it is clear that at c, e vanishes for all ζ and all ϕ.

From this relativistic definition it is observed that:

(1) Charge is an expression of energy through the rest energy mc^2.

(2) From experimental observation of empirical laws of force between two point masses, and between two point charges, both being the well known inverse square laws of gravitational and electromagnetic field theories, it is reasonable to assume that charge and mass are linearly related, as in Eq. (2).

(3) In the limit $v \ll c$:

$$e \xrightarrow[v \ll c]{} \frac{H}{c^2/\zeta + \phi} \tag{6}$$

and if, in this limit, $\phi = 0$, we regain the original hypothesis (2):

$$e \xrightarrow[v \ll c]{\phi=0} \frac{\zeta H}{c^2} = \zeta m\,. \tag{7}$$

Thus in the limit $v \ll c$, charge becomes proportional to the rest energy $H = mc^2$ when there is no electromagnetic field present.

(4) If we do **not** make the assumption (2) then

$$e = \frac{1}{\phi}\left[H - \frac{mc^2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}}\right] \tag{8}$$

and it is not immediately clear that $e \to 0$ at the speed of light, because e and m remain independent of each other, i.e., remain independently postulated elementary properties of a point particle.

Therefore, as a direct consequence of Eq. (2), there is no charge in a point particle travelling at the speed of light, and e becomes dependent on the velocity of the point particle. Therefore there cannot be electric and magnetic fields independent of time in the rest frame of the observer. This conclusion is of course consistent with special relativity, in which there is no Galilean action at a distance. The maximum speed of propagation of all fields is c, and if a field does not propagate (i.e. has zero velocity) there would not be action between two point particles, in contradiction with the observation of, for example, gravitational and electromagnetic forces. Since no electric and magnetic field can be independent of time, then strictly speaking there are no electrostatic and magnetostatic fields, in contradiction with everyday experience, in which distances are small and propagation at c effectively instantaneous, for example, between capacitor plates a centimetre apart. The non-existence of electrostatic and magnetostatic fields is consistent with Maxwell's equations, which assert phenomenologically that all electromagnetic fields are time varying. Recently[4] it has been shown that Maxwell's equations allow LONGITUDINAL solutions which consist of E and B fields moving at c through regions of free space in which there are no charges and no currents.

2. Four-Dimensional Analysis

In Minkowski space the equations of motion of a point charge in an electromagnetic field can be expressed, as is well known,[1-3] in the four-dimensional form:

$$mc\frac{du_i}{ds} = \frac{e}{c}F_{ik}u^k \tag{9}$$

where

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}; \tag{10}$$

Here

$$A_i \equiv (\phi, -\mathbf{A}) \tag{11}$$

is the fourth vector of the electromagnetic field, incorporating the scalar and vector potentials, as usual[1-3] and u_i is the fourth velocity. Equation (10) is derived from the variation properties of the action function as shown, for example, in Landau and Lifshitz, pp. 60 ff. In three dimensions, Eq. (10) reduces for $i = 0$ to the work equation, and for $i = 1$, 2 and 3 to the Lorentz equation. The development of special relativity from Eq. (10) leads to the definition of the fourth vector of generalised momentum,[1-3] defined by

$$P^i \equiv \left(\frac{1}{c}(En_{\text{kin}} + e\phi),\ \mathbf{P} + \frac{e}{c}\mathbf{A}\right) \tag{12}$$

where En_{kin} is the kinetic energy and $\mathbf{P}$ the three-dimensional linear momentum in the rest frame. From the scalar part of this fourth vector we recover Eq. (5) of Sec. 1 of this Letter, and from the vector part we obtain, with Eq. (2), our basic hypothesis, the new relation:

$$e = \frac{\mathbf{P}\left(1 - \frac{v^2}{c^2}\right)^{1/2}}{\frac{\mathbf{v}}{\zeta} + \frac{\mathbf{A}}{c}\left(1 - \frac{v^2}{c^2}\right)^{1/2}}. \tag{13}$$

In this case, the charge e is defined in terms of the generalised momentum $\mathbf{P}$, which from the law of conservation of momentum, is a constant quantity for the motion of the point charge in the electromagnetic field.

Clearly, the elementary point charge e, if proportional to point mass through Eq. (2), becomes expressible in special relativity in terms of both Eqs. (5) and (13). From these equations, at the speed of light, we obtain:

$$\mathbf{P} = H\,\frac{\mathbf{v}}{c^2}\,; \quad |\mathbf{v}| = c \tag{14}$$

which is the same in structure as the relation between momentum and kinetic energy **for a free particle**, for example Eq. (9.8) of Landau and Lifshitz.[1] This result, based on the hypothesis (2), is consistent with the deduction that at c there is no charge and no mass, so that for a point particle moving at c, there is no interaction possible with an electromagnetic field. In consequence, the point particle behaves as a free particle travelling at c in the absence of an electromagnetic field, i.e., through an equation of the type (14).

By comparing Eqs. (5) and (13) for the point charge, e, the hypothesis (2) leads to a relation between the scalar and vector potential of the electromagnetic field:

$$H\left(\frac{\mathbf{v}}{\zeta} + \frac{\mathbf{A}}{c}\left(1 - \frac{v^2}{c^2}\right)^{1/2}\right) = \mathbf{P}\left(\frac{c^2}{\zeta} + \phi\left(1 - \frac{v^2}{c^2}\right)^{1/2}\right) \tag{15}$$

and to a definition of the parameter ζ of Eq. (2):

$$\zeta = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}\left(\frac{H\mathbf{v} - \mathbf{P}c^2}{\mathbf{P}\phi - H\,\dfrac{\mathbf{A}}{c}}\right) \tag{16}$$

showing it to depend on ϕ and $\mathbf{A}$, and on the velocity of the point particle.

In an uncharged point particle, it is clear that

$$H\mathbf{v} = c^2\mathbf{P} \tag{17}$$

and this is the same as Eq. (14), obtained by setting $v = c$ individually in Eqs. (5) and (13). This can be interpreted as meaning that in an uncharged body, there is no interaction with the electromagnetic field for finite m because there is a relation (17) between the constant Hamiltonian H (law of conservation of energy) and the constant total momentum $\mathbf{P}$ (law of conservation of momentum), a relation which implies that a point particle whose trajectory is not affected by an electromagnetic field does not have charge. Note that this deduction does not presuppose that e must be 0 in such a point particle, but deduces that e is zero from the relation (17). The latter is of course a consequence of the fundamental hypothesis (2).

Clearly, this reasoning can also be extended to a macroscopic scale, where point particle of mass is replaced by a massive object which may or may not be charged, depending on its chemical constitution.

In the case $v \ll c$, Eq. (16) reduces to

$$\zeta \xrightarrow[v \ll c]{} \frac{H\mathbf{v} - c^2\mathbf{P}}{\phi\mathbf{P} - \frac{H}{c}\mathbf{A}} \tag{18}$$

so that charge e is defined in terms of mass, m, and the constants of motion H and $\mathbf{P}$ of a point particle in an electromagnetic field, characterised by the scalar and vector potentials ϕ and $\mathbf{A}$ respectively.

3. Discussion

The hypothesis (2) introduced in this communication assumes that point charge e is proportional to point mass m, and on this basis we have shown that the coefficient of proportionality ζ is given by Eq. (16) in the rest frame.

This is a radical departure from conventional theory,[1–4] which assumes that e and m are not related, but are independent elementary properties of a point particle. Equation (2) of this communication is a step towards the unification of gravitational and electromagnetic theory, because the force between two point charges e and between two point masses m becomes expressible with Eq. (2) in terms of the same inverse square law in the rest frame. Although we hvae made use of special relativity, we have deduced Eq. (18) for the coefficient ζ in the "non-relativistic" limit $v \ll c$. Equation (16) provides a link between the scalar and vector potentials of an electromagnetic field as follows.

From Eq. (16) at the speed of light:

$$\mathbf{P}\phi - H\frac{\mathbf{A}}{c} = \frac{e}{m}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}(H\mathbf{v} - c^2\mathbf{P}) \tag{19}$$

but we know from Eq. (17) that the right-hand side is zero, so:

$$\mathbf{P}\phi = H\frac{\mathbf{A}}{c} \tag{20}$$

where at c:

$$\mathbf{P} = \mathbf{P} = m\mathbf{v}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$H = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{21}$$

Therefore, with $v = c$, from (21) in (20):

$$\phi = |\mathbf{A}| \quad \text{at} \quad c\,. \tag{22}$$

Since

$$\phi^i = (\phi,\, \mathbf{A})$$

and

$$x^i = (ct,\, \mathbf{r})$$

are four vectors, the result (22) clearly parallels the condition for the four vector x^i at c:

$$ct = |\mathbf{r}| = (x^2 + y^2 + z^2)^{1/2} \quad \text{at} \quad c\,. \tag{23}$$

Furthermore, with the use of the well known four-D Lorentz condition:

$$\frac{\partial A^k}{\partial x^k} = 0 \tag{24}$$

which, in three D, becomes

$$\frac{1}{c}\frac{\partial \phi}{\partial t} = -\nabla \cdot \mathbf{A} \tag{25}$$

we obtain, with Eq. (22), the two equations:

$$\frac{1}{c}\frac{\partial |\mathbf{A}|}{\partial t} = -\nabla \cdot \mathbf{A} \tag{26}$$

and

$$\frac{1}{c}\frac{\partial \phi}{\partial t} = -\nabla \cdot \phi \tag{27}$$

where $|\phi| = \phi = |\mathbf{A}|$, which are satisfied by LONGITUDINAL solutions in terms of delta functions similar to those obtained recently for the Maxwell equations by the present authors.[4] Clearly, the novel conditions (26) and (27) provide constraints on the scalar and vector potentials of the well known transverse solutions for electromagnetic field at the speed of light in free space, and were obtained on the basis of our fundamental hypothesis (2).

4. Conclusions

The hypothesis (2) is essentially empirically based on the properties of point particles, essentially and fundamentally the property that mass is positive, and that charge is positive, negative, or zero for finite mass. It leads to several conclusions in special relativity, and relates the scalar and vector potentials of the electromagnetic field propagating at the speed of light.

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