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Chapter 1. Electromagnetism and General Relativity

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1.1 Introductory Essay

In this opening chapter of Vol. 4 some concepts of general relativity are summarized in the context of electromagnetism, which is usually regarded as a theory of special relativity [1—3]. Vacuum longitudinal solutions of the field equations of electromagnetism have recently emerged, however, from several directions independently [4—18], and in this volume a unified theory of electromagnetism and gravitation is suggested by these recent discoveries and developed in later chapters.

Although general relativity grew out of special relativity in about 1916, with Einstein's first paper on the subject, there remain to this day profound differences of philosophy between the two theories, differences which are summarized in a lucid recent essay by Sachs [19]. Contemporary general relativity is a theory which is able to describe data to very high precision [20] using the equivalence principle between curvilinear geometry and the gravitational field. There are no particles in the field continuum, only singularities caused by curvature of space-time. The gravitational field is space-time itself, and carries its own source [21], so that the theory of general relativity, the geometrical theory of the gravitational field, is non-linear and non-Abelian in group structure [22]. The Einstein field equations contain within them the equations of motion of a "test particle" in the field, but point mass does not appear in the originals themselves. The canonical energy-momentum tensor contains a scalar part which is related to mass *density*, i.e., mass per unit volume, and the Einstein equations reduce to the Newton equations only in a linear approximation [23].

In contrast, the quantum theory of light [24—31] is a linear probabilistic calculus [19] in which the quantized electromagnetic field is not regarded as space-time curvature. To Faraday and Maxwell [32] the electromagnetic field is a physical entity which is described using the coordinates of space, but is distinct from the coordinate frame. Thus Maxwell's field equations are linear equations using gradient, divergence and curl of vector quantities. They are not covariant under Galilean transformation, and this, together with

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the crisis of the Michelson-Morley experiment, led slowly to special relativity through the ideas of Fitzgerald, Lorentz, Poincaré, Einstein and others [32]. Action at a distance was ruled out of this development because the speed of light was found to be finite, and was postulated by Einstein to be invariant under Lorentz transformation, as are the Maxwell equations themselves in vacuo (shown by Lorentz) and in matter (shown by Poincaré). Einstein in 1905 postulated that the laws of physics be covariant under the Lorentz transformation and that c be a universal constant of the theory of special relativity as it became known. There is no concept here of field *being* space-time, as in general relativity, which is still regarded as a theory of gravitation only.

The challenge faced by contemporaries is that both general relativity and the quantum theory of the electromagnetic field have been developed with formidable precision [20], but remain philosophically apart. In this volume an attempt is made to use the equivalence principle of general relativity in the context of electromagnetism. The classical electromagnetic field strength tensor, denoted $G_{\mu\nu}$, is made equivalent to a space-time curvature, necessarily an antisymmetric tensor of rank two, denoted $R^{(A)}_{\mu\nu}$. The latter is derived from a novel index contraction of the Riemann curvature tensor of curvilinear geometry, a contraction which consists of setting two indices equal in the Riemann tensor. Electromagnetism therefore becomes a Riemannian theory, i.e., one of curvilinear geometry. The same Riemann tensor, when contracted in another way, produces the Einstein tensor used in his theory of general relativity. The gravitational and electromagnetic fields become describable by the symmetric and antisymmetric parts, respectively, of a rank two Ricci tensor derived from the same Riemann tensor and therefore from the same space-time continuum — the curved frame of reference in four dimensions.

This procedure is made possible by the development in the first three volumes of the theory of the $B^{(3)}$ field, and by recent independent confirmations [15—18] that there exist in vacuo longitudinal solutions of the field equations of electromagnetism, whose linear limit is the Maxwellian limit. The central theorem of those volumes is the minimal prescription applied to the free photon momentum, $\hbar\kappa$, which is equated to $eA^{(0)}$. Here $\hbar$ is Dirac's constant, κ is the magnitude of the wavevector, e is the elementary quantum of charge and $A^{(0)}$ is the scalar magnitude of the vector potential in vacuo of the classical electromagnetic field. The central theorem means that the scaling factor e is present in vacuum electromagnetism, and although there are no point charges in vacuo the field is a non-localized $\hat{C}$ negative space-time continuum (i.e., non-localized charge). As developed in Vol. 3 [6] the $\hat{C}$ negative, scalar quantum of charge, e, is present in the definition of field vectors in vacuo, and is used there in the radiation theory of the $B^{(3)}$ field. In analogy, there are no point masses in vacuo, and the gravitational field is a $\hat{C}$ positive space-time continuum containing non-localized mass, the gravitational field. In this view, point charge is found where there is point mass, for example an electron, or positron, and when point mass is absent, so is point charge. The gravitational field is caused by and causes point mass; the electromagnetic field is caused by and causes point charge. The gravitational field is symmetric in the sense that it is not handed, i.e., is achiral, being the symmetric contraction of the Riemann tensor. The electromagnetic field is left or right handed, i.e., chiral, being the antisymmetric contraction of the same Riemann tensor. Thus for one type of elementary charge, e, there are two types of electromagnetic field, left and right circularly polarized, and this follows from the fact that $R^{(A)}_{\mu\nu}$ is an axial rank two tensor, i.e., space-time in this view is inherently chiral whenever there is an electromagnetic field present, and this chirality must be represented by the nature of the product eA_{μ}.

In the received view [24—31], $B^{(3)}$ is not considered, the Maxwell equations have only transverse solutions, and there is no central theorem. In consequence electromagnetism is Abelian and linear in structure, and its sector symmetry [24] is U(1). In the new view the sector symmetry becomes that [4—14] of the Poincaré group, electromagnetism becomes non-linear and non-Abelian in vacuo, and there are cyclically symmetric angular momentum relations between field components [4], relations which are not present in the received view. It has recently been suggested by Chubykalo and Smirnov-Rueda [15], using the Maxwell equations, that the longitudinal element in the vacuum electromagnetic field indicates action at a distance, for which firm experimental evidence is now available in experiments on two photon entangled states [33]. The transverse solutions are components of a complete solution, components which represent the Faraday-Maxwell field propagating at c. Since $B^{(3)}$ can be formed from conjugate products of advanced and retarded solutions of the wave equation it can be interpreted as a phase free representative of action at a distance in electromagnetism. Múnera and Guzmán [16] have recently established that the Maxwell equations can be solved in vacuo to give a class of longitudinal solutions which are not considered in the received view. The work of Hunter and Wadlinger [17], Ahluwalia and Ernst [18], Dvoeglazov [34] and others seem to support the new philosophy emerging for the vacuum electromagnetic field, that it is fully four dimensional and that there are relations between its field components. It is but a short step to a new equivalence principle in which the four dimensionality of the field becomes the four dimensionality of antisymmetric space-time. The experimental evidence for the new philosophy ranges from data on finite photon radius, obtained at microwave frequencies [17], to magneto-optics and the optical Aharonov-Bohm effect [35] as discussed in earlier volumes [6] and in the source literature [35].

The field equations of electromagnetism are therefore non-linear and non-Abelian, contain in the new view covariant derivatives, and are equations in which the electromagnetic field is antisymmetrically curved space-time, right or left handed. The electromagnetic field in vacuo has scalar curvature $R = \kappa^2$ [36], and since κ^2 is non-zero and positive, so is R. In the received view, there is no consideration of scalar curvature, although it is easily derived in elementary differential geometry from the equation of the plane wave in vacuo [36]. So in the new view the philosophy of electromagnetism converges with that of gravitation, both fields becoming aspects of Mach's philosophy as described by Sachs [19]. Ultimately, Sachs [19] expects that the paradigm of particulate matter will be replaced by one based on space-time continuum, a holistic philosophy ranging over all scales of space and time, but one which includes as a corner stone the action-reaction law of Newton, and one which is intrinsically closed, or non-linear. The non-linearity of the theory must be reconciled with the linearity of probabilistic quantum mechanics, which has been highly accurate in its descriptions and predictions of nature. Some recent independent developments [15—18] show now that there exist in the vacuum longitudinal as well as transverse solutions of the electromagnetic field equations, and point in the direction of unifying field theory within general relativity. Thus charge as well as mass becomes a manifestation of curvature of space-time and therefore charge may be *local* or *non-local* depending on the extent to which the space-time of electromagnetism is curved. There is no real distinction between charge and field, just as there is none between mass and field in gravitation. Similarly the equation of motion of charge, traditionally the Lorentz equation, comes out of the new non-linear equations of electromagnetism in its curved space-time. The new view of the Lorentz equation is therefore based on the mixing of symmetrically and antisymmetrically curved space-time continua when electromagnetism meets matter.

This holistic view of nature, in which all is curvature of space-time, would appear at present to cure the contemporary obscurities of wave-particle dualism, in that there would be no particle. Sachs [19] and Hunter and Wadlinger [17] have reviewed and developed the arguments for the general relativistic and pure wave theories of matter respectively, but there remains the problem of correctly reducing the non-linear general theory to the powerful linear probabilistic calculus known as quantum mechanics. It is suggested in this volume that angular momentum theory may provide a route to further development in this important area of fundamental natural philosophy, because angular momentum commutator relations are non-Abelian and non-linear as described in previous volumes, but are still relations of quantum mechanics because angular momentum satisfies the principle of superposition [37] as is well known. Furthermore, angular momentum theory is rigorously covariant, and is well-developed both in special and in general relativity [38]. Atkins [37], for example, shows that the commutators of angular momentum can be used to develop almost the whole of quantum theory. Previous volumes have shown that in the basis ((1),(2),(3)) [4—6], there exist cyclically symmetric relations between the three space components $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, and $\boldsymbol{B}^{(3)}$ of the vector $\boldsymbol{B}$ in free space, a magnetic flux density vector whose magnitude is $B^{(0)}$. These are relations between angular momentum commutators [4], and are also therefore covariant relations between rotation generators in four dimensions. It is therefore convenient to begin this volume by demonstrating that the B cyclics can be derived from the definition in the Poincaré group of the Pauli-Lubanski four-vector [4—6], whose square is the second (spin) Casimir invariant of the Poincaré group, the ten parameter inhomogeneous Lorentz group. The first (mass) invariant is the square of the four-vector P_μ, the space-time translation generator, or within a factor $\hbar$, the energy-momentum four-vector p_μ. The B cyclics are therefore shown to be rigorously compatible with relativity and are both non-linear,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \text{ et cyclicum,} \tag{1.1}$$

and linear,

$$\boldsymbol{B} = \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \boldsymbol{B}^{(3)}, \tag{1.2}$$

$$|\boldsymbol{B}^{(1)}| = |\boldsymbol{B}^{(2)}| = |\boldsymbol{B}^{(3)}| = B^{(0)}, \tag{1.2a}$$

in nature. They are the basis of the general relativistic treatment of electromagnetism developed in this volume, and are also quantized as angular momentum commutators. The B cyclics are important therefore because they appear to provide a foot-bridge from general relativity to linear quantum mechanics. They are the first geometrical relations discovered between three space components of vacuum electromagnetism. Tautologically, the received view has no such result, because there are only two components, $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$.

For one photon define the elementary fluxon $\phi^{(0)} = \hbar/e$ in weber. Then the elementary flux density magnitude $B^{(0)}$ (tesla = weber/m^2) for one photon is $\phi^{(0)}/(\kappa V)$ where κ^{-1} is the Thompson radius [6] and V the volume occupied by a single photon. Thus κV is a mean photon cross section, or area. In general, as demonstrated by Hunter and Wadlinger [17], the volume occupied by one photon is an ellipsoid. With these definitions it follows that for one photon the field,

$$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} = i\frac{B^{(0)}}{\sqrt{2}}\boldsymbol{e}^{(1)}e^{i\phi}, \tag{1.3}$$

is a solution of Maxwell's equations, which are linear approximations of the novel field equations to be developed in this volume. Here

$$\phi = \omega t - \kappa Z := x^\mu \kappa_\mu , \tag{1.4}$$

is a phase of the electromagnetic field, and $\boldsymbol{e}^{(1)}$ is a unit vector if the basis ((1), (2), (3)) defined in previous volumes [4—6]. The phase is defined through the angular frequency ω at an instant t and the wavevector κ at point Z; and is relativistically invariant as discussed for example by Jackson [39]. From Eq. (1.3), the non-linear B cyclics (1.1) follow geometrically, as required in a theory of general relativity, and within a factor $\hbar$ the rotation generators used in the geometry are angular momentum operators as discussed for example by Ryder [24]. It is possible for one photon to work the theory through in terms of magnetic flux, whose quantum is the fluxon $\hbar/e$, the ratio of two fundamental constants. In terms of flux,

$$\boldsymbol{\Phi}^{(1)} \times \boldsymbol{\Phi}^{(2)} = i\,\Phi^{(0)}\,\boldsymbol{\Phi}^{(3)*}, \tag{1.5}$$

and these cyclic relations (phi cyclics) are independent of any consideration of volume occupied by the photon. The phi cyclics depend only on the existence of the fluxon $\hbar/e$ and on the geometry, in space that of the O(3) group, in space-time that of the Poincaré group. In the received view the phi cyclics do not exist because only transverse components are allowed. Recent work [4—18,34] has established the presence of longitudinal solutions which lead to the phi and B cyclics through geometry. Apart from the factor $\hbar/e$ therefore the cyclics are angular momentum relations, and are therefore central in unified field theory as argued already. The original photon of 1926 was defined as the quantum of energy, $\hbar\omega$, which is $\hat{C}$ positive, but as shown in Vol. 3 [6] the energy quantum is proportional to the square of e. The concomitant fields are always $\hat{C}$ negative as discussed in Vol. 1, and it is clearly not enough to define the photon as its own anti-particle because application of $\hat{C}$ reverses the sign of the concomitant fields by reversing the sign of e. Maxwell's equations are invariant under $\hat{C}$ only because each component field is reversed in sign, together with charge and current densities. Equations (1.1) and (1.2) conserve $\hat{C}$ trivially for the same reason, and the existence of $\boldsymbol{B}^{(3)}$ does not violate $\hat{C}$ symmetry. To extend this theory of one photon to a photon gas [40,41] it is necessary to use well-developed statistical methods, and to account for phenomena of photon-photon interaction, development taking place on the basis that for each photon there exist cyclic field relations which are angular momentum relations. The photon in this view is an element of curved, $\hat{C}$ negative, space-time with intrinsic spin. The quantum of charge, e, makes the complete electromagnetic space-time $\hat{C}$ negative, otherwise the space-time would be that of $\hat{C}$ positive gravitation. It appears therefore that the space-time of the unified field is chiral in $\hat{C}$, i.e., must simultaneously support $\hat{C}$ negative and $\hat{C}$ positive elements of the same curvature tensor. The $\hat{C}$ negative part is charged and the $\hat{C}$ positive part is uncharged.

Therefore there exists a longitudinal field component for one photon, produced experimentally for example by parametric down conversion [42]. For one photon there are three space-time components of its magnetic flux density $\boldsymbol{B}$, labeled $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, and $\boldsymbol{B}^{(3)}$ in the ((1),(2),(3)) frame, and for one photon there is one mode, one frequency and one wavenumber. The presence of three components per photon rather than two means that the $\hat{C}$ negative scalar e appears in the equations of the field in vacuo, and the relation of the field to point charge becomes analogous to that of the field to point mass in gravitation. Field quantities such as $\boldsymbol{E}$ and $\boldsymbol{B}$ are linear in e, regarded not as a point charge but as a $\hat{C}$ negative scaling factor, and particulate properties such as $\hbar\omega$ are quadratic in e and are therefore $\hat{C}$ positive. The photon is therefore not its own anti-particle because $\hat{C}$ leaves $\hbar\omega$ invariant but changes the sign of the photon's magnetic dipole moment, the longitudinal component of which is $\boldsymbol{m}^{(3)} = V\boldsymbol{B}^{(3)}$. The received view becomes self-inconsistent when dealing with $\boldsymbol{B}^{(3)}$. There is considerable confusion in the critical literature [43—48] which variously asserts that: a) $\boldsymbol{B}^{(3)} =? 0$ by $\hat{C}$ symmetry [43]; b) $\boldsymbol{B}^{(3)}$ is not zero but is not a magnetic field [44]; c) $\boldsymbol{B}^{(3)}$ is unknowable [45]; d) $\boldsymbol{B}^{(3)}$ is not fundamental [46]; e) $\boldsymbol{B}^{(3)}$ violates special relativity [47]; f) $\boldsymbol{B}^{(3)}$ is equal

to the operator $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} / \left(iB^{(0)}\right)$ but is not a magnetic field [48]. These criticisms are adequately discussed in the first three volumes [4—6], but points a) to f) illustrate that the B cyclics are new to electrodynamics and cannot be understood within the framework of a two component (transverse) vacuum field theory. Proponents of the $\boldsymbol{B}^{(3)}$ theory [15—18,34] accept it as the most fundamental representation of spin in electrodynamics [34]; and, importantly, as a fundamental magnetic field or permanent magnetic dipole moment of one photon.

A reasoned evaluation of the cyclic field relations has many potential advantages for electromagnetic theory. For example it makes the very strong equivalence principle [49] easier to accept because it allows the structure of the field strength tensor, $G_{\mu\nu}$, to be understood as that of a contracted Riemann tensor, in the simplest case the rank two antisymmetric tensor $R_{\mu\nu}^{(A)}$ to be developed in this volume. This makes the affine parameters (or connection coefficients) rotation generators of the Poincaré group and allows one possible simple system of field unification. Some experimental predictions of this unified field system are given later in this volume. In order to demonstrate experimentally the existence of the flux density $\boldsymbol{B}^{(3)} = \mu_0 \boldsymbol{m}^{(3)}/V$ it is necessary only to demonstrate that of the photon's spin angular momentum $\boldsymbol{S}^{(3)}$ [50] or that of everyday atomic absorption of the quantity $\pm\hbar$. In other words it is necessary only to recognize that the photon possesses a non-zero gyromagnetic ratio g_p,

$$\boldsymbol{m}^{(3)} = g_p \boldsymbol{S}^{(3)}, \qquad g_p = \frac{ec}{\hbar\kappa}, \tag{1.6}$$

which combines its particulate and electromagnetic natures even though mass does not enter
into the definition. (In pure wave theories, for example that of Hunter and Wadlinger [17], the gyromagnetic ratio g_p would combine the photon's electromagnetic and wave natures, which we seek ultimately to be representations of curved space-time.) Therefore the available experimental evidence for $\boldsymbol{B}^{(3)}$, reviewed in the first three volumes, is strongly indicative of a novel unified field structure such as the simplest one developed in this volume.

$\boldsymbol{B}^{(3)}$ and related longitudinal field structures such as that proposed by Chubykalo and Smirnov-Rueda [15] indicate also the development of theories in electromagnetism to account for the increasing amount of direct experimental evidence for non-locality [51] and action at a distance in electrodynamics. Cosmological data also indicate that [49] quasars 3C 179, 273, 279 and 345; and galaxy 3C 120 expand at several times c so that tachyonic theories may develop around $\boldsymbol{B}^{(3)}$ in which longitudinal solutions in vacuo of the field equations of electrodynamics indicate instantaneous action at a distance. The best known of these is the longitudinal Coulomb field discussed by Dirac [15]. A holistic view of electromagnetism and gravitation would then imply that action at a distance occurs in a similar way in gravitation. It is notable that Mozart's fortieth symphony has been broadcast at 4.7c [52]. More generally, Mészáros and Molnár [49] have discussed in a rigorously reasoned manner the various fundamental anomalies that perplex the contemporary standard model of cosmology, and recently, Mészáros [53] has indicated that the thermodynamics of the adiabatically expanding photon gas are also anomalous to such an extent that a radical revision of the paradigm of photon thermodynamics is necessary, a revision which he suggests could well be based on the emergence of $\boldsymbol{B}^{(3)}$ and related longitudinal solutions in vacuo [15—18,34].

In Vol. 3 [6] it has been argued that the charge quantum, e, is present in the electromagnetic field, and is non-localized, i.e., is present as a charge density, charge per unit volume. In analogy the gravitational field equations of Einstein contain mass density in the scalar part of the Einstein tensor. A holistic view of the unified field would therefore lead to the expectation of a mass quantum, a minimum amount of mass which occurs in the quantization of the gravitational field, and ultimately, it is expected that the quantum of charge and mass will be unified within a new concept which allows two forms of charge and one form of mass. If the existence of negative mass is indicated experimentally at some stage in the future, the unified field theory should give two forms of mass and two forms of charge. Confining attention to the electromagnetic field, Vol. 3 has shown that the presence of e is sufficient to define and observe $\boldsymbol{B}^{(3)}$ each time atomic absorption takes place in the laboratory, because $\boldsymbol{B}^{(3)}$ is directly proportional to the spin angular momentum $\boldsymbol{S}^{(3)}$ whose eigenvalues are $\pm\hbar$ if the photon is massless and 0, $\pm\hbar$ otherwise [4]. The relation between $\boldsymbol{B}^{(3)}$ and $\boldsymbol{S}^{(3)}$ is

$$\boldsymbol{B}^{(3)} = \frac{\mu_0}{V} g_p \boldsymbol{S}^{(3)}. \tag{1.7}$$

Atomic absorption is therefore due to the magnetic dipole moment of the photon.

There is no doubt that the electromagnetic field is distinguishable experimentally from the gravitational field and a unified field theory must successfully reduce to either component by allowing for their different $\hat{C}$ symmetries. As argued, the space-time for the unified field must be $\hat{C}$ chiral, i.e., must support $\hat{C}$ negative and $\hat{C}$ positive components of the same fundamental curvature tensor. It is generally accepted that the electromagnetic sector of the unified field must transmit energy from point charge-mass to point charge-mass across the vacuum, for example from one oscillating and radiating electron to a receiving electron which is made to oscillate. The field must be $\hat{C}$ negative in order to do this, and this much is clear from the fact that the $\hat{C}$ positive gravitational sector does not cause oscillation in a receiving electron. As argued in Vol. 1, it follows that the potential four-vector A_μ is negative under $\hat{C}$; $\hat{C}(A_\mu) = -A_\mu$. The photon as quantum of energy is however $\hat{C}$ positive, because as shown in Vol. 3 it is proportional to the square of e. Therefore energy is proportional to e squared but is linear in rest mass (rest energy = mc^2) as discussed in Vol. 3. Therefore the photon does not carry point

charge, but nonetheless e, the quantum of charge, is present in its definition [6]. This provides one way of showing that although point charge is absent from the electromagnetic field, non-localized charge is present. The latter must be so if A_μ is to be negative under $\hat{C}$. Charge in any form is absent from the gravitational field, and the latter is $\hat{C}$ positive.

Therefore the photon, if it is a particle, is not its own anti-particle as frequently claimed in the literature [43], because there exists a distinct and physically distinguishable anti-photon whose energy, momentum and angular momentum are unchanged by $\hat{C}$, but whose concomitant field properties (A_μ, $\boldsymbol{E}$, $\boldsymbol{B}$) are reversed by $\hat{C}$. The photon is emitted by the oscillating electron in matter; the anti-photon by the oscillating positron in anti-matter. In order to demonstrate the existence of $\boldsymbol{B}^{(3)}$ in vacuo, and to provide a mechanism for unified field theory as developed in this volume, it is sufficient to use Eq. (1.7), and this leaves no doubt as to the existence of $\boldsymbol{B}^{(3)}$ provided that e is present in the electromagnetic field, as indicated for example by the fine structure constant and second quantum of action [17]. The existence of $\boldsymbol{B}^{(3)}$ in vacuo therefore depends on that of e, which must be interpreted as a fundamental $\hat{C}$ negative quantum, and obviously not as the charge on the electron. A holistic view of the unified field is therefore made up of non-local charge and non-local mass. A holistic view of matter is made up of localized curved space-time representing localized mass and charge arranged in such a way that some massive particles are negatively charged, some are positively charged, some are uncharged. Similar considerations extend to baryon number, lepton number, charm, and so forth. At each level of micro-matter (atomic, sub atomic, etc.) cyclic relations such as those discussed in earlier volumes may allow, as in this volume, a unified concept to emerge.

The photon's gyromagnetic ratio, g_p, is an example of a concept that in the received view is undeveloped, in that view it would be concluded that it is zero. Yet, in the new theory of longitudinal field components in vacuo, it is straightforward to derive g_p from the conventional [54] definition of photon spin angular momentum,

$$\boldsymbol{S}^{(3)} = -\frac{1}{2}\frac{\epsilon_0}{\omega}\int \boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} dV, \tag{1.8}$$

where ϵ_0 is the vacuum permittivity and $\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*}$ are conjugates of the transverse electric field strength of the electromagnetic radiation. Here V is the volume occupied by the radiation, which can be considered as being made up of a monochromatic, single photon beam, accessible experimentally with a parametric downconverter [55]. In the $\boldsymbol{B}^{(3)}$ theory the conventional definition simplifies, giving the energy per photon,

$$En = \omega|\boldsymbol{S}^{(3)}| = \frac{1}{\mu_0}B^{(0)}|\boldsymbol{B}^{(3)*}|\bar{V}, \tag{1.9}$$

where $B^{(0)}$ is the magnitude of $\boldsymbol{B}^{(3)}$. The permanent magnetic dipole moment of the photon is deduced using the relation [6],

$$\boldsymbol{B}^{(3)} = \left(\frac{\omega\mu_0}{\bar{V}B^{(0)}}\right)\boldsymbol{S}^{(3)}, \tag{1.10}$$

and is defined by

$$\boldsymbol{m}^{(3)} = \frac{\bar{V}}{\mu_0}\boldsymbol{B}^{(3)} = \left(\frac{\omega}{B^{(0)}}\right)\boldsymbol{S}^{(3)}. \tag{1.11}$$

In the received view the photon is asserted to be uncharged and the magnetic dipole moment is asserted not to exist, but even in an *uncharged* photon, Eq. (1.11) shows that $\boldsymbol{m}^{(3)}$ is directly proportional to $\boldsymbol{S}^{(3)}$ through a quantity which is $\hat{C}$ negative (because $B^{(0)}$ is $\hat{C}$ negative) but *uncharged* in the sense that e does not at first sight appear. However, Vol. 3 [6] has shown that $B^{(0)}$ *itself* must be defined in terms of e, and the ubiquitous and non-localized charge appears once more. A short step away is the inference that the space-time of electromagnetism is *itself* $\hat{C}$ negative, and by analogy, the space-time of gravitation is $\hat{C}$ positive. The role of the symbol e therefore becomes that of ensuring that the curved space-time to which electromagnetism is equivalent is $\hat{C}$ negative as required. Such an inference is an example of the extended Mach Principle [19], or very strong equivalence principle [41], in which all fields become equivalent to curved space-time. Applying $\hat{C}$ to the space-time of the photon results in that of the anti-photon in which the sign of e is reversed but in which all dynamical quantities remain the same. Dynamical quantities are defined in the space-time of gravitation, which in the linear approximation, becomes that of Newtonian dynamics.

Equation (1.10) reduces to Eq. (1.7) by using the definition of $\boldsymbol{B}^{(3)}$ developed in Vol. 3 from the Dirac equation, or equivalently from the central theorem, the minimal prescription for the free photon momentum. These theorems give the relation,

$$B^{(0)} = \frac{e}{\hbar}A^{(0)2}, \tag{1.12}$$

between the magnitudes $B^{(0)}$ and $A^{(0)}$ in vacuo. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are plane waves we also have the relations,

$$B^{(0)} = \frac{\omega}{c}A^{(0)}, \qquad \hbar\omega = ecA^{(0)} = \frac{1}{\mu_0}B^{(0)2}\bar{V}, \tag{1.13}$$

and using Eqs. (1.12) and (1.13) reduces Eq. (1.10) to Eq. (1.7) in which the photon's gyromagnetic ratio is transparently the ratio of charge to a quantity with the units of mass, i.e., the photon's momentum $\hbar\kappa$ divided by the photon's velocity c. In massive photon theories such a relation would become relativistic itself, because c would no longer be the speed of light [4—6] but a postulated universal constant. So using these arguments, if $\boldsymbol{S}^{(3)}$ is a photon spin angular momentum then g_p is a gyromagnetic ratio as usually defined for other particles, i.e., a ratio of charge and mass, and $\boldsymbol{m}^{(3)}$ is a magnetic dipole moment in an equally consistent way. The physics of the photon becomes indistinguishable from that of a particle with mass if the mass is defined as $\hbar\kappa/c$. Finally, using [6],

$$B^{(0)} = \frac{e\mu_0 c}{\kappa\bar{V}}, \tag{1.14}$$

one obtains the result proposed in the first paper on $\boldsymbol{B}^{(3)}$ [7] in 1992,

$$B^{(3)} = B^{(0)}\frac{\boldsymbol{S}^{(3)}}{\hbar} = \frac{\mu_0 c}{\kappa\bar{V}}\left(\frac{e}{\hbar}\boldsymbol{S}^{(3)}\right), \tag{1.15}$$

showing that $\boldsymbol{B}^{(3)}$ is the rotation generator $\boldsymbol{S}^{(3)}/\hbar$ within a $\hat{C}$ negative magnitude $B^{(0)}$, which, as we have seen, incorporates the elementary $\hat{C}$ negative symbol e. If we begin to think of curved space-time as being $\hat{C}$ negative when equivalent in principle to electromagnetism, then the quantity $e\boldsymbol{S}^{(3)}/\hbar$ would become *a* $\hat{C}$ *negative rotation generator*, which is simply spin angular momentum within a $\hat{C}$ negative universal factor $e/\hbar$. This is the inverse of the elementary fluxon. The development of the theory of electrodynamics as a theory of general relativity is then scaled by $e/\hbar$, and this is precisely the factor that converts the spin angular momentum of one photon, $\boldsymbol{S}^{(3)}$, to $\boldsymbol{B}^{(3)}$, the elementary photomagneton [4]. Charge is something that makes anti-symmetric space-time $\hat{C}$ negative and can be regarded as the elementary $\hat{C}$ negative influence rather than anything confined to a particle. There is no reason to assert that the equivalence principle must always apply to $\hat{C}$ positive space-time, as in gravitation, and the emergence of $\boldsymbol{B}^{(3)}$ and related longitudinal solutions indicates that the flat-world U(1) sector symmetry is self inconsistent [4—6]. The space-time of electromagnetism is four dimensional in the view developed in this volume, and because of the presence of e, is $\hat{C}$ negative. If we accept the de Broglie guidance principle, Eq. (1) of Vol. 1, then to every energy $\hbar\omega$ is associated an energy mc^2 in the rest frame of *any* particle, and if $\hbar\omega$ is proportional to the square of charge, so is rest mass. This means that the space-time of

Einsteinian dynamics is in a sense derived from a more fundamental space-time in which can occur two senses of charge, positive and negative. The existence of mass (i.e., mass-energy) cannot experimentally distinguish between these two signs of charge because mass is quadratic in charge, and for this reason, the gravitational space-time is $\hat{C}$ positive. All this is indicated strongly by the existence of $\boldsymbol{B}^{(3)}$ which is supported experimentally by atomic absorption, and less well known but precisely demonstrated phenomena in magneto-optics [4—6].

The one photon Beth effect [56], in which the spin angular momentum $\boldsymbol{S}^{(3)}$ can be observed directly, indicates in the new theory the existence of $\boldsymbol{B}^{(3)}$ through equations such as (1.7). In the received view the Beth effect indicates only the existence of $\boldsymbol{S}^{(3)}$ [57], which is unrelated to $\boldsymbol{m}^{(3)}$. Similarly, in the received view, the inverse Faraday effect [4—6] demonstrates $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ but this is unrelated to $iB^{(0)}\boldsymbol{B}^{(3)*}$. In both cases the received view fails because the sector of electromagnetism is asserted to be U(1), which has no longitudinal dimension. Such a view is philosophically remote from the curved 4-D of gravitation, and in view of the B cyclics (1.1) or phi cyclics (1.5), geometrically flawed. At radio frequencies the inverse Faraday effect leads to an induction profile proportional to the square root of intensity, I (watt m^{-2}), and in the new theory this effect is due to the $\boldsymbol{B}^{(3)}$ field acting at first order. In this limit the received view must interpret the square root intensity profile [4—6] as being one in the second order conjugate product divided by $iB^{(0)}$. This quotient cannot be identified with an elementary photonic magnetic field, $\boldsymbol{B}^{(3)}$, even though the experimental effect of $\boldsymbol{B}^{(3)}$ is indistinguishable from that of a magnetic field. At this point the received view becomes diametrically self-inconsistent in that a quantity which has the physical effect of a magnetic field is not a magnetic field. This is a basic paradox which requires a new paradigm in field theory. These volumes have suggested some ways of developing such a paradigm. It is well known that there are other fundamental paradoxes in classical electrodynamics, discussed for example by Chubykalo and Smirnov-Rueda [15] and by Hunter and Wadlinger [17]. Paradoxes in the standard models of cosmology and photon thermodynamics are discussed rigorously by Mészáros and Molnár [41] and by Mészáros [58]. The solution suggested in Ref. 15 succeeds in removing century old paradoxes such as infinite electron self-energy, and is based, significantly, on the use of longitudinal solutions in vacuo of the Maxwell field equations. In the view suggested in this volume, the Maxwell equations become linear limits of a more general structure written in curved electrodynamic ($\hat{C}$ negative) space-time. If it is accepted that $e\boldsymbol{S}^{(3)}/\hbar$ can be interpreted as a $\hat{C}$ negative rotation generator of $\hat{C}$ negative space-time itself, then the existence of $\boldsymbol{B}^{(3)}$ becomes the existence of a fundamental space-time current defined through

$$\boldsymbol{j}^{(3)} = \frac{c}{V}\left(\frac{e}{\hbar}\boldsymbol{S}^{(3)}\right), \tag{1.16}$$

which is just the $\hat{C}$ negative coordinate multiplied by c/V. This current is again the result of the principle that e is the $\hat{C}$ negative scalar that makes the electromagnetic field curved space-time. The principle of equivalence between electromagnetism and curved Riemannian space-time is therefore

$$R_{\mu\nu}^{(A)} = e\frac{G_{\mu\nu}}{\hbar}, \tag{1.17}$$

where $R_{\mu\nu}^{(A)}$ is a novel antisymmetric Ricci tensor derived from the Riemann tensor by index contraction and $G_{\mu\nu}$ is the field strength tensor. The affine connections in the tensor $R_{\mu\nu}^{(A)}$ become proportional to the potential four-vector A_μ which is therefore a rotation generator of the local Poincaré group and the metric of the $\hat{C}$ negative curved space-time known as the electromagnetic field is a 4 x 4 antisymmetric tensor. The field strength tensor $G_{\mu\nu}$ includes $\boldsymbol{B}^{(3)}$, which is self-consistently defined within it. From Eq. (1.17) the $G_{\mu\nu}$ tensor is the elementary fluxon $\hbar/e$ multiplied by $R_{\mu\nu}^{(A)}$, and the product is a $\hat{C}$ negative tensor of antisymmetrically curved space-time. The equivalence principle (1.17) is a straightforward consequence of the existence of $\boldsymbol{B}^{(3)}$, which is an observable, and so Eq. (1.17) is based on experimental evidence such as atomic absorption, inverse Faraday induction and the Beth effect.

Finally in this introductory essay, we consider the effect of the new philosophy of electromagnetism on thermodynamics and the radiation laws. Mészáros [58] has pointed out that the Planck distribution, Wien's law, and the Rayleigh-Jeans law are not valid without modification for an adiabatically expanding photon gas, in which case a new type of ultraviolet catastrophe appears. The cause of these contradictions in the received view can be traced [58] to the role of a varying $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$. The piezotropic-auto-barotropic equation of state for an ideal photon gas can be written [58] as $P = U/3$, where P is pressure, V is volume and U is energy density En/V; but this cannot be valid simultaneously with the equations of state suggested by Mészáros and Molnár [41],

$$PV^{4/3} = \text{constant}, \qquad TV^{1/3} = \text{constant},$$
$$T^{-4/3}P^{1/3} = \text{constant}, \tag{1.18}$$

The equilibrium state of the ideal photon gas contains in its description, therefore, a basic inherent thermodynamic paradox which originates in the electrodynamical origin of the equation $P = U/3$. This is equivalent to the fact that the Maxwell equations are temperature independent, as pointed out by Mészáros [58]. This paradox leads to several other contradictions and brings into doubt the validity of the basic radiation laws themselves when applied to an adiabatically changing photon gas. It may well turn out that the implementation of a varying and quantized $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$ is the *only* way out of this paradox. The related paradox in cosmology has been described in rigorous detail by Mészáros and Molnár [41] in a recent essay.

1.2 The Origin of the B Cyclics in the Invariant Relations of the Poincaré Group

Before embarking on the development of the theory of electromagnetism in general relativity it is shown that the B and phi cyclics are relations of a sub group, O(3), of the ten parameter inhomogeneous Lorentz group in special relativity — the Poincaré group [4—6]. The B and phi cyclics are thus shown to be rigorously valid in special relativity, as required in field theory; and the Poincaré group replaces U(1) as the sector group for vacuum electromagnetism. This is essential for the validity of the equivalence principle (1.17) between the field tensor $G_{\mu\nu}$ and the contracted Riemann tensor $R_{\mu\nu}^{(A)}$.

The Casimir invariants of the Poincaré group [4—6] are the mass invariant and spin invariant, proportional respectively to $P_\mu P^\mu$ and $W_\mu W^\mu$. Here P_μ is the generator of space-time translation [25] which is the energy momentum four-vector within a factor $\hbar$, and W_μ is the Pauli-Lubanski four-vector. The Poincaré group invariants are connected by the well known equation,

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}\boldsymbol{p}^\sigma, \tag{1.19}$$

where $J^{\nu\rho}$ is a representation of the rotation and boost generators [4]. In Chap. 11 of Vol. 1 it was demonstrated that the four-vector corresponding to $\boldsymbol{B}^{(3)}$ in vacuo is a Pauli-Lubanski vector, a demonstration that shows that $\boldsymbol{B}^{(3)}$ is a vector of O(3), the little group of the Poincaré group [4—6]. Therefore $\boldsymbol{B}^{(3)}$ is compatible with special relativity, and is invariant under Lorentz transformation [4—6]. In this section it is demonstrated that the cyclic relations characterizing the new paradigm of vacuum electrodynamics developed in these volumes [4—6] are space relations generated by the fundamental relation, Eq. (1.19), between the W_μ and P_μ vectors of the Poincaré group, the group of special relativity.

The method used in this section is based on an extension of the central theorem,

$$\hbar\kappa = eA^{(0)} \Rightarrow \boldsymbol{P}_\mu = \frac{e}{\hbar}A_\mu, \tag{1.20}$$

an extension which recognizes that the P_μ vector is proportional to the A_μ vector; and that the complete field tensor $G_{\mu\nu}$ must include non-zero longitudinal components in vacuo. It is also recognized that the W_μ vector is proportional to the ***vector*** dual of the tensor $G_{\mu\nu}$, a dual which is denoted G_μ. Therefore Eq. (1.19) is replaced by

$$A^0 G_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\nu\rho}A^\sigma, \tag{1.21}$$

where A^0 is a scalar. Equation (1.21) already contains within it the cyclic equations in space epitomized by the first to be discovered of this type, the B cyclics, Eq. (1.1). This is deduced by choosing indices, and using the properties of the antisymmetric 4-D tensor, i.e., $\epsilon_{0123} = -1$; and all other values are obtained by successive pair permutations, each of which changes the sign of ϵ_{0123}. Thus $\epsilon_{0132} = 1$; $\epsilon_{0312} = -1$; and so on. The 4-D symbol $\epsilon_{\mu\nu\rho\sigma}$ reduces to the 3-D symbol ϵ_{ijk} in space. The latter is defined by $\epsilon_{123} = 1$; $\epsilon_{132} = -1$; and so on.

Two types of potential four-vector can be defined for electromagnetism in the vacuum, describing respectively light-like translational motion,

$$A^\mu_{(T)} := \left(A^0,\ 0,\ 0,\ A^3\right), \tag{1.22}$$

and light-like rotational motion,

$$A^\mu_{(R)} = \left(0,\ A^1,\ A^2,\ 0\right), \tag{1.23}$$

in which the components A^1 and A^2 are in general phase dependent. For a plane wave, for example, their vector form is given by

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} = \left(A^1\boldsymbol{i} + A^2\boldsymbol{j}\right)e^{i\phi}. \tag{1.24}$$

The B cyclics are derived using the rotational part of A^μ in Eq. (1.21), and the translational part of A^μ gives as follows the well known space (O(3)) relation between an axial vector and an axial tensor. The rotational and translational parts of A^μ are orthogonal in the vacuum,

$$A^\mu_{(T)}A^{(R)}_\mu = 0, \tag{1.25}$$

and so share the properties in the vacuum of the two Casimir invariants for the photon, regarded as a hypothetically massless particle,

$$\boldsymbol{P}^\mu\boldsymbol{P}_\mu = W^\mu W_\mu = 0. \tag{1.26}$$

This indicates that the electromagnetic field has rotational and translational character as it propagates through the vacuum, a result which is important in the generally relativistic theory because the geodesic in space of the charge quantum becomes a helix.

If we consider the translational $A^{\mu}_{(T)}$; $\mu = 3$ and $\sigma = 0$ in Eq. (1.21) we obtain

$$A^0 G_3 = -\frac{1}{2}\epsilon_{3\nu\rho 0} G^{\nu\rho} A^0, \quad (1.27)$$

which reduces in space (O(3) sub-group of the Poincaré group) to

$$G_3 = -\frac{1}{2}\left(\epsilon_{321} G^{21} + \epsilon_{312} G^{12}\right), \quad (1.28)$$

thus identifying G_3 as $cB_Z = cB^{(3)}$ [4—6] as demonstrated in Chap. 11 of Vol. 1. Equation (1.28) is clearly the relation between an axial vector and its equivalent rank two antisymmetric tensor in 3-D space. In 4-D space-time this relation becomes Eq. (1.27). It turns out that for electromagnetism to be developed into a theory of general relativity it is essential to use the fact that there exists a non-zero G_3. In the received view this is not accepted, but in the new paradigm there is evidence for G_3 from independent sources using different theoretical approaches, some based on the original Maxwell equations as described in Sec. 1.1. We conclude that the relation between $\boldsymbol{B}^{(3)}$ as an axial vector and as an axial tensor is a special case of the general B cyclics (1.1), a relation derived from a consideration of the translational part of A^{μ}.

In order to derive the B cyclical structure itself the rotational $A^{\mu}_{(R)}$ is used on the right hand side of Eq. (1.21), and therefore we fix $\mu = 3$ and consider $\sigma = 1$ and *2* in Eq. (1.21) to give the equations,

$$A^0 G_3 = -\frac{1}{2}\left(\epsilon_{3\nu\rho 1} G^{\nu\rho} A^1 + \epsilon_{3\nu\rho 2} G^{\nu\rho} A^2\right), \quad (1.29)$$

a sum which gives the Pauli-Lubanski vector G_3. This can be identified from the foregoing analysis as $cB_3 = cB_Z = cB^{(3)}$ in the vacuum. The space part of the right hand side of Eq. (1.29) reduces (Appendix A) to a vector cross product expressed in tensor notation,

$$A_0 G_k = -\frac{1}{2}\epsilon_{kij} E_i A_j\,. \quad (1.30)$$

This equation already has the necessary cyclical structure in space, and to reduce it to the form of the B cyclics, Eq. (1.1), we use the free space relations [4—6]:

$$G_3 = cB_3\,, \qquad A_0 = \frac{B_0}{\kappa}\,, \qquad E_0 = cB_0\,, \quad (1.31)$$

and transform from Cartesian (X, Y, Z) to ((1), (2), (3)). Using the plane waves [4—6],

$$\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}}\left(\boldsymbol{i} - i\boldsymbol{j}\right)e^{i\phi}\,, \quad (1.31a)$$

$$\boldsymbol{B}^{(1)} = \kappa \boldsymbol{A}^{(1)} = \boldsymbol{B}^{(2)*} = \frac{B^{(0)}}{\sqrt{2}}\left(i\boldsymbol{i} + \boldsymbol{j}\right)e^{i\phi}, \quad (1.31b)$$

Eq. (1.30) becomes (Appendix A)

$$iB^{(0)}\boldsymbol{B}^{(3)*} = \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}, \quad (1.32)$$

and the other equations of the B cyclics are obtained by cyclic permutation.

The cyclical relations (1.1) are therefore relations of special relativity, because Eq. (1.21) is a relation between the W_{μ} and P^{μ} vectors of the Poincaré group of special relativity. The cyclical equations (1.1) are sub-equations in space of Eq. (1.21), although they were not discovered until 1992 [4—14]. They imply and are implied by the central theorem $\hbar\kappa = eA^{(0)}$, which shows the presence in the vacuum of e, the fundamental $\hat{C}$ negative influence. In the development in this volume the relation of electromagnetic field to point charge becomes analogous exactly to the relation of the gravitational field to point mass, and this is an aspect of the Mach Principle. Cyclical relations similar to Eq. (1.1) may also exist for the weak and strong fields, depending on the symmetry. The rotational $A^{\mu}_{(R)}$ used in this section may itself be given a rotation generator interpretation using [6],

$$A_{\mu}^{(R)} = \frac{\kappa}{e}J_{\mu} = \frac{\hbar\kappa}{e}M_{\mu}, \quad (1.33)$$

where J_{μ} is the angular energy-angular momentum four-vector defined in Vol. 3's introduction [6] of the rotational Poynting theorem based on $\boldsymbol{B}^{(3)}$ in the vacuum. The energy of one photon becomes in this representation the familiar $\hbar\omega$ because $\hbar$ is the quantum, or minimum amount, of angular momentum. The latter also has the units of action as is well known. The angular momentum J_{μ} is the rotation generator M_{μ} within the factor $\hbar$, and so M_{μ} is dimensionless,

$$M_{\mu} = \frac{J_{\mu}}{\hbar}, \quad (1.34)$$

being the rotation generator of the Poincaré group. The magnitude of the product κJ_{μ}

is equal to the magnitude of the energy-linear momentum four-vector $p_\mu = eA_\mu$. In the development given in this volume the interpretation of the rotational A_μ as a rotation generator of the Poincaré group becomes the basis for the definition of the affine connection for electromagnetism as a theory of general relativity. There are therefore three rotation generators for A_μ, two transverse and one *longitudinal*, signalling the existence of a rotational $A^{(3)}$ component defined by the A cyclics,

$$A^{(1)} \times A^{(2)} = iA^{(0)}A^{(3)*}. \tag{1.35}$$

Using the relations $B^{(1)} = \kappa A^{(1)}$; $B^{(2)} = \kappa A^{(2)}$; $B^{(3)} = \kappa A^{(3)}$, the A cyclics become angular momentum relations, i.e., rotation generator relations of the Poincaré group, whose space sub group is O(3), the rotation group.

1.3 The $B^{(3)}$ Field in Riemannian Space-time

In order to develop a Riemannian theory of classical vacuum electromagnetism it is convenient to consider a curve corresponding to a plane wave [59],

$$f(Z) = (i - ij)e^{i\phi}, \tag{1.36}$$

where i and j are Cartesian unit vectors and ϕ is the electromagnetic phase (1.4) where ω is the angular frequency at instant t, and κ the wave vector at point Z. In terms of the retarded time $[t] = t - Z/c$ [6], the phase ϕ is $\omega[t]$. The concept of retarded time means that the instant t is replaced by the instant $t - t_0$, where $t_0 = Z/c$. Similarly, the retarded distance can be defined as $[Z] = Z - Z_0 = c[t]$, where the point $[Z]$ is calculated at the instant $[t]$. The electromagnetic wave propagates, or moves, along the Z axis, and the trajectory of its real part is

$$f_R(Z) := Ref(Z) = (\cos\phi, \sin\phi, \phi), \tag{1.37}$$

which is a circular helix in this simple representation. The unitless term $\phi = \kappa[Z]$ is non-zero for this very reason, if it were zero, the curve (1.37) would be a circle, and there would be no propagation.

The curve (1.37) is a function of Z, with Z_0 regarded as a constant in partial differentiation of $f(Z)$ with respect to Z. More generally, a Z independent phase angle,

$$\Phi = \tan^{-1}\frac{Y}{X}, \tag{1.38}$$

must be incorporated in $f_R(Z)$, which becomes [61]

$$f_R(Z) = \big(\cos(\kappa(Z-Z_0)+\Phi),\ \sin(\kappa(Z-Z_0)+\Phi),\ \kappa(Z-Z_0)+\Phi\big), \tag{1.39}$$

Differentiating, Frenet's tangent vector (T) is obtained [61],

$$\frac{\partial f_R}{\partial Z}(Z) = \kappa T = (-\kappa\sin\phi, \kappa\cos\phi, \kappa) = \kappa(-\sin\phi, \cos\phi, 1). \tag{1.40}$$

In elementary differential geometry, therefore, the electromagnetic helix produces a non-zero T, and tangent vectors are characteristic of curved space-time [61] in general relativity. The scalar curvature in elementary differential geometry is

$$R = \left|\frac{\partial^2 f_R}{\partial Z^2}(Z)\right| = |\kappa^2(\cos\phi, -\sin\phi, 0)| = \kappa^2. \tag{1.41}$$

$R = \kappa^2$ is also the scalar curvature of the electromagnetic wave in general relativity, i.e., is the scalar curvature of Riemann's tensor, obtained by suitable antisymmetric index contraction [62].

The metric coefficient [60] in the theory of gravitation is locally diagonal, but in order to develop a metric coefficient of vacuum electromagnetism, the antisymmetry of the field must be taken into consideration. The electromagnetic field strength tensor $G_{\mu\nu}$ is essentially an angular momentum tensor in 4-D, made up of rotation and boost generators [4—6]. An ordinary axial vector in 3-D space can always be expressed as the sum of cross products of unit vectors,

$$I = i \times j + j \times k + k \times i, \tag{1.42}$$

a sum which can be expressed as the metric,

$$g = g^{(A)}_{\mu\nu} i^\mu j^\nu, \tag{1.43}$$

where the $g^{(A)}_{\mu\nu}$ coefficient in 3-D is the fully antisymmetric 3 x 3 unit matrix,

$$g_{\mu\nu}^{(A)} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}_{3\text{-}D} \rightarrow \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix}, \tag{1.44}$$

which becomes the right hand side in 4-D. In the language of differential geometry, the field tensor becomes the Faraday 2-form [63],

$$F = \frac{1}{2} F_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta}, \tag{1.45}$$

where the wedge product $dx^{\alpha} \wedge dx^{\beta}$ between differential forms is an exterior product. Equation (1.45) translates in tensor notation into

$$F = F_{\alpha\beta}\, dx^{\alpha} \otimes dx^{\beta}, \tag{1.46}$$

and the field tensor $F_{\alpha\beta}$ plays the role of the metric coefficient, which is thus anti-symmetric as argued already in elementary language.

In the Riemannian theory of vacuum electromagnetism developed here the existence of the physical vacuum $\boldsymbol{B}^{(3)}$ field implies that the conventional Faraday 2-form F must be replaced by a geometrically correct 2-form G which is defined by [64]

$$\begin{aligned} G = {} & E_X\, dX \wedge dt + E_Y\, dY \wedge dt + E_Z\, dZ \wedge dt \\ & + B_X\, dY \wedge dZ + B_Y\, dZ \wedge dX + B_Z\, dX \wedge dY, \end{aligned} \tag{1.47}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are electric and magnetic components of the vacuum field, and where the wedge products are formed between space-time components in a local Poincaré group, not the conventional *flat-world* group [25] U(1) = O(2). The plane wave 2-form [64] is defined by only two out of six wedge products,

$$F = F_{01}\, dt \wedge dX + F_{31}\, dZ \wedge dX, \tag{1.48}$$

but the general 2-form in four dimensions (three space and one time) consists in differential geometry of six distinct wedge products as in Eq. (1.47).

The plane wave representation of electromagnetism is not therefore a rigorous geometrical representation, but an approximation, one which arbitrarily discards the third dimension, and one in which the observable $\boldsymbol{B}^{(3)}$ [4—6] is undefined. This much is clear from the fact that for a plane wave, the wedge product containing $\boldsymbol{B}^{(3)}$ (the product $B_Z dX \wedge dY$ in Eq. (1.47)) is missing from Eq. (1.48). We therefore adopt the

rigorously correct 2-form G for the development in this volume of a geometrical theory of electromagnetism. Through the principle of equivalence, the field 2-form G must be proportional to a frame 2-form R_2 which is a non-inertial frame of reference. In tensor language, the antisymmetric $G_{\mu\nu}$ tensor must be proportional to an antisymmetric frame tensor which we henceforth denote $R_{\mu\nu}^{(A)}$. The latter is derived from the Riemann curvature tensor, which is the fundamental curvature tensor in curved space-time. We conclude that

$$R_{\mu\nu}^{(A)} = R^{\lambda}_{\lambda\mu\nu}, \tag{1.49}$$

i.e., $R_{\mu\nu}^{(A)}$ is a novel antisymmetric Ricci tensor obtained by the index contraction $\kappa = \lambda$ from the Riemann curvature tensor. Further contraction of $R_{\mu\nu}^{(A)}$ must lead to the scalar curvature R which for electromagnetism is κ^2 from Sec. 1. The contraction must be

$$R = \frac{1}{12} g_{\mu\nu}^{(A)} R_{\mu\nu}^{(A)}, \tag{1.50}$$

because $R_{\mu\nu}^{(A)}$ is antisymmetric and R is a scalar.

The proportionality constant between the Ricci 2-form R_2 and the field 2-form G can be deduced to be $e/\hbar$, a frame invariant and fundamental ratio of charge and action quanta. This is easily seen from the scalar form of Eq. (1.17), which is

$$eG^{(0)} = \hbar R, \tag{1.51}$$

where $G^{(0)}$ is a scalar field amplitude and $R = \kappa^2$ is the scalar curvature of vacuum electromagnetism. Equation (1.51) is the minimal prescription for the free photon momentum [4—6], a minimal prescription which is the central theorem,

$$eA^{(0)} = \hbar\kappa, \tag{1.52}$$

of previous volumes. In other words the photon momentum in vacuo can be written either as $\hbar\kappa$ or $eA^{(0)}$. Thus Eq. (1.51) is a rigorously correct equivalent of Eq. (1.52). The amplitude $G^{(0)}$ is thereby defined as

$$G^{(0)} = \kappa A^{(0)} = B^{(0)} = \frac{E^{(0)}}{c}, \tag{1.53}$$

i.e., is the scalar amplitude of magnetic flux density, $B^{(0)}$ ($= E^{(0)}/c$ where $E^{(0)}$ is the electric field strength amplitude in vacuo and c the speed of light). Therefore,

$$B^{(0)} = \frac{\hbar}{e}R = \Phi^{(0)}R, \tag{1.54}$$

and so the scalar curvature R is directly proportional to $B^{(0)}$ in the Riemannian theory of vacuum electromagnetism. Since $\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{e}^{(3)}$, it is rigorously non-zero in curved space-time, and is defined by a rigorously non-zero component of the Riemann tensor, the component quadratic in the affine connection [65]. The latter, furthermore, is proportional to the Frenet unit tangent vector [66]: unit tangent vectors are signatures of curvilinear geometry.

The flat-world theory of vacuum electromagnetism, the received view [25], therefore arbitrarily discards a rigorously non-zero component of the Riemann tensor. This geometrically incorrect procedure leads to the paradox described in the opening section of this volume. This volume therefore develops the following ideas:

(1) Vacuum electromagnetism is the geodesic of the charge quantum e in curved space-time, described by Riemannian space-time.

(2) The metric coefficient $g_{\mu\nu}^{(A)}$ of vacuum electromagnetism is antisymmetric.

(3) The central theorem becomes $eG_{\mu\nu} = \hbar R_{\mu\nu}^{(A)}$.

(4) The scalar curvature of vacuum electromagnetism is $R = \kappa^2$.

(5) The Ricci 2-form R_2 is proportional to the geometrically correct field 2-form G, and this is a principle of equivalence.

(6) The $\boldsymbol{B}^{(3)}$ field is defined by the scalar curvature R through $\boldsymbol{B}^{(3)} = (\hbar/e)R\boldsymbol{e}^{(3)} = \Phi^{(0)}R\boldsymbol{e}^{(3)}$.

(7) Vacuum electromagnetism is the antisymmetric Ricci 2-form; gravitation is the symmetric Ricci 2-form.

1.4 The Geodesic Equations for the Electrodynamic Sector in Riemannian Space-time

In this section the Planck-Einstein and de Broglie postulates are shown to originate in space-time curvature, and the general relativistic equations of vacuum electromagnetism are shown to be

$$p^{\mu} = \hbar\Gamma^{\mu}, \tag{1.55}$$

where p^{μ} is the energy-momentum, $\hbar$ is Dirac's constant, and Γ^{μ} is the contracted form of an antisymmetric affine connection, proportional directly to the rotational part of A^{μ}

discussed in Sec. 1.2. Geodesic equations are derived, and the theory of electromagnetism shown to be a general theory of relativity, suggesting a holistic framework for field theory.

Elementary differential geometry (Sec. 1.3) shows that $R = \kappa^2$, and that the vacuum plane wave has non-zero scalar curvature. In this section, geodesic equations are derived which confirm this result, and which show that electromagnetism can be interpreted using Riemannian space-time, whose local group is the Poincaré group. This self consistently produces $\boldsymbol{B}^{(3)}$ through a rigorously non-zero part of the fundamental Riemann tensor itself, and suggests one simple way in which electromagnetism and gravitation can be unified. In the received view $\boldsymbol{B}^{(3)}$ and R are unconsidered, although they are present in radiation of all frequencies. In the holistic view suggested in this volume, $\boldsymbol{B}^{(3)}$ and R are well and naturally defined within Riemannian geometry, and in consequence the Planck-Einstein and de Broglie postulates originate in four dimensional curvature. It turns out that in so doing, the contracted (single index) affine connection Γ^{μ} becomes the wave four-vector κ^{μ} itself through the central theorem mentioned in earlier sections, a theorem which relates p^{μ} to A^{μ}. The quantized p^{μ} is proportional to κ^{μ} through the Dirac constant $\hbar$. The idea of an affine connection is unconsidered in the special relativistic theory of electromagnetism, a theory which uses *flat space-time* in which curvature and affine connections are zero [67]. However, $R = \kappa^2$ is non-zero for all κ, and so refinement of the received view becomes logically necessary. In special relativity, the fundamental quantum postulate of radiation,

$$p^{\mu} = \hbar\kappa^{\mu}, \tag{1.56}$$

is unidentified with space-time curvature, fields are distinct from the frame of reference. In general relativity the field is identified with the frame and vice-versa, and Eq. (1.56) is related to the equivalence principle (1.17) suggested earlier. The symmetric contraction of the Riemann tensor produces the Einstein tensor in the gravitational sector of the unified field, whose scalar component is mass density. The antisymmetric contraction of the same Riemann tensor produces the electromagnetic field strength tensor $G_{\mu\nu}$ in a novel form [4—6], one which contains $\boldsymbol{B}^{(3)}$ through the antisymmetric product of affine connections. The contraction of $R_{\mu\nu}^{(A)}$, using the antisymmetric metric of Eq. (1.44), produces $R = \kappa^2$ self consistently. In this section the electromagnetic sector in curved space-time is developed using fiducial equations which reduce to well known wave equations (of d'Alembert and Proca) in well defined limits.

1.4.1 Geodesic Equations of the Plane Wave in Vacuo

The geodesic equation in this section is such that the charge quantum e moves so that its worldline is a geodesic line, and such that $R = \kappa^2$ is non-zero. In the presence of the electromagnetic field, space-time is not flat, or Galilean, as in the received view, but the development is therefore one in general relativity. Accordingly, covariant derivatives are used to describe the electromagnetic field, and the starting point is [26]

$$D\kappa^{\mu} = \frac{d\kappa^{\mu}}{d\lambda} + \Gamma^{\mu}_{\nu\sigma}\kappa^{\nu}\kappa^{\sigma} = 0\,, \tag{1.57}$$

where $\kappa^{\mu} = dx^{\mu}/d\lambda$ is the wave four-vector and $\Gamma^{\mu}_{\nu\sigma}$ the three index affine connection used in the Riemann tensor [26]. In the received view of the electromagnetic sector [26], Eq. (1.57) is one of flat (or Galilean) space-time,

$$d\kappa^{\mu} = 0\,, \tag{1.58}$$

an equation which shows that in the propagation of electromagnetism in vacuo, the wave-vector does not vary along its path [26]. In geometrical optics the propagation of a light ray is determined by the wave vector tangent to the ray [26]. In Eq. (1.57) therefore, λ is a parameter that varies along the ray. It will be shown in this section that Eq. (1.57) is a geodesic equation and that the worldline of e is not rectilinear, its space part is a helix as shown by Hunter and Wadlinger [17].

A relation is first established between κ^{μ} and the A^{μ} four-vector,

$$p^{\mu} = eA^{\mu} = \hbar\kappa^{\mu}, \tag{1.59}$$

a relation which is the minimal prescription applied to the momentum-energy $\hbar\kappa^{\mu}$. Equation (1.59) shows that electromagnetism has charge-energy in the same way that gravitation has mass-energy [26]. Electromagnetism carries its own source in the same way as gravitation, and the electromagnetic field is a vacuum four-current. The self consistency of the central theorem used to construct Eq. (1.59) has been checked in several ways [4—6]. The central theorem is consistent with gauge theory in the Poincaré group [25] and with the existence of the second quantum of action [6,17] $e^2/(4\pi\epsilon_0 c)$ observable in the quantum Hall effect [17]. Any electromagnetic field (Sec. 1.1) is $\hat{C}$ negative and proportional in consequence to e. The quantum of electromagnetic energy $\hbar\omega$ is therefore proportional to e^2 [6] as in Eq. (1.5) which constructs energy by multiplying the $\hat{C}$ negative A^{μ} with e, i.e., energy is quadratic in e. Using Eq. (1.59) in Eq. (1.57) gives

$$\frac{dA^{\mu}}{d\lambda} + \frac{e}{\hbar}\Gamma^{\mu}_{\nu\sigma}A^{\nu}A^{\sigma} = \frac{dA^{\mu}}{d\lambda} + \frac{e}{\hbar}A^2\Gamma^{\mu} = 0\,, \tag{1.60}$$

where A is a scalar. The contracted affine connection Γ^{μ} is proportional to A^{μ} [25] in general gauge theory and we adopt this rule here to give

$$\Gamma^{\mu} = \frac{e}{\hbar}A^{\mu}. \tag{1.61}$$

Thus A^{μ} is the fluxon $\hbar/e$ multiplied by the contracted affine connection. Equation (1.61) is an equivalence principle between field and frame properties, between A^{μ} and Γ^{μ} respectively. Such an equivalence does not appear in the received view, in which any affine connection is zero.

Using $\kappa = eA/\hbar$ changes Eq. (1.61) to

$$\frac{dA^{\mu}}{d\lambda} + \kappa^2 A^{\mu} = 0, \tag{1.62}$$

in which $R = \kappa^2$ is the scalar curvature of the geodesic equation [26] if A^{μ} is taken to be a plane wave, such as in Eq. (1.31b) of Sec. 1.2. The dimensionality of λ is therefore that of the inverse of κ^2, i.e., that of the Thompson area of the photon [17], and if $\lambda = Z^2/2$ Eq. (1.62) becomes

$$\frac{d^2A^{\mu}}{dZ^2} + RA^{\mu} = 0. \tag{1.63}$$

This has the form of a geodesic equation [26]. It is easily checked that Eq. (1.63) is obeyed for the plane wave of Eq. (1.31b), i.e., for

$$A = \frac{A^{(0)}}{\sqrt{2}}(i\mathbf{i} + \mathbf{j})e^{i(\omega t - \kappa Z)}\,, \tag{1.64}$$

in which the signal velocity is $c = \omega/\kappa$. Similarly, we obtain

$$\frac{1}{c^2}\frac{d^2A^{\mu}}{dt^2} + RA^{\mu} = 0, \tag{1.65}$$

and this is also obeyed by Eq. (1.64). Now subtract Eq. (1.63) from Eq. (1.65) to give the d'Alembert wave equation,

$$\Box A^{\mu} := \left(\frac{1}{c^2} \frac{d^2}{dt^2} - \frac{d^2}{dZ^2} \right) A^{\mu} = (R - R) A^{\mu} = 0 \,. \qquad (1.66)$$

Writing this as

$$\Box A^{\mu} = (R - R) A^{\mu}, \qquad (1.67)$$

it becomes a Proca wave equation whose right hand side happens to be zero because of our previous use of Eq. (1.64). Equation (1.67) is however an equation of curved space-time in which the scalar curvature R is non-zero, i.e., from Eq. (1.64), $R = \kappa^2$ using the methods of differential geometry explained in Sec. 1.2.

The special theory of relativity, which gives Eq. (1.67), but with R =? 0, is self-inconsistent when applied to the electromagnetic sector in general relativity. Equations (1.63) and (1.65) become incorrect if R =? 0, but Eq. (1.67) appears still to be fortuitously correct if R =? 0. The vacuum eigenvalue of the familiar d'Alembertian $\Box$ is $R - R$, i.e., the difference of equal and opposite curvatures which are individually non-zero in general relativity but individually zero in special relativity. In both special and general relativity the d'Alembert equation (and therefore the Maxwell equations to which it is equivalent) appear to be correct. We conclude that the Maxwell equations are equations of general relativity, because they produce Eq. (1.64), whose scalar curvature $R = \kappa^2$ is not zero.

If there is a massive source of radiation present in the equations, additional scalar curvature R_S is imparted to the right hand side of Eq. (1.67) from symmetric contraction of the Riemann tensor to give the inhomogeneous wave equation,

$$\Box A^{\mu} = R_S A^{\mu} := \frac{J^{\mu}}{\epsilon_0}. \qquad (1.68)$$

The matter four-current is derived in this view from the scalar curvature R_S, in which appears mass density as in Einstein's gravitational theory. If the photon itself has mass, then Eq. (1.68) is a Proca equation derived in general relativity, not in special relativity as is usual [4].

1.4.2 Geodesic Equations from the Definition of the Riemann Tensor

Equation (1.63) and (1.65) are special cases of the usual definition of the Riemann tensor in curvilinear geometry [26],

$$A_{\mu;\nu;\kappa} - A_{\mu;\kappa;\nu} := R^{\lambda}_{\mu\nu\kappa} A_{\lambda} \,, \qquad (1.69)$$

where A_{λ} a general four-vector field [27]. Equation (1.69) can be written in D notation [25] as

$$(D_{\nu} D_{\kappa} - D_{\kappa} D_{\nu}) A_{\mu} + R^{\lambda}_{\mu\nu\kappa} A_{\lambda} = 0 \,, \qquad (1.70)$$

and this is a geodesic equation. Multiply Eq. (1.70) by the antisymmetric metric coefficient defined in Eq. (1.44),

$$g^{\nu\kappa}_{(A)} (D_{\nu} D_{\kappa} - D_{\kappa} D_{\nu}) + g^{\nu\kappa}_{(A)} R^{\mu}_{\mu\nu\kappa} A_{\mu} = 0 \,, \qquad (1.71)$$

and identify

$$R := g^{\nu\kappa}_{(A)} R^{\mu}_{\mu\nu\kappa}, \qquad \frac{d^2}{dZ^2} := g^{\nu\kappa}_{(A)} (D_{\nu} D_{\kappa} - D_{\kappa} D_{\nu}) \,. \qquad (1.72)$$

This procedure reduces Eq. (1.69) to Eqs. (1.63) and (1.65), which are special cases obtained by tensor contraction.

1.4.3 The Planck-Einstein and De Broglie Postulates

The vacuum minimal prescription (central theorem [4—6]) defines the affine connection,

$$\Gamma^{\mu} = \kappa^{\mu} = \frac{e}{\hbar} A^{\mu} \,, \qquad (1.73)$$

in contracted form. Therefore the affine connection of curved space-time is always non-zero
in the electromagnetic sector because it is directly proportional to the energy momentum four-vector,

$$p^{\mu} = \hbar \Gamma^{\mu}, \qquad (1.74)$$

and this is the general relativistic form of the Planck-Einstein and de Broglie postulates. The special relativistic theory applied to electromagnetism in the received view does not recognize that there is curvature inherent in the field, a curvature whose scalar or Gaussian form is $R = \kappa^2$. In general relativity this field curvature is also the curvature of the space-time, a space-time which is antisymmetric, and derivable from the Riemann tensor (1.69). The same Riemann tensor describes the gravitational field by a different index contraction. In special relativity the fundamental quantum postulate of radiation, $p^{\mu} = \hbar\kappa^{\mu}$, is one unconnected with curvature of space-time itself, because the space-time in special relativity is Galilean, a space-time whose curvature is zero. This is self-contradictory in the general relativistic theory developed in this volume because $R = \kappa^2$ is also the curvature of space-time. We have the choice at this point of either accepting electromagnetism as a theory of general relativity, making it easier to unify this theory with that of gravitation, or of rejecting a priori the equivalence principle (1.17). Since $\boldsymbol{B}^{(3)}$, however, emerges from the structure of $R_{\mu\nu}^{(4)}$, this rejection is experimentally unacceptable because $\boldsymbol{B}^{(3)}$ is a physical field. In general philosophical terms the rejection of a field frame equivalence principle for electromagnetism on the one hand and its acceptance for gravitation on the other means that there will remain two different natural philosophies of fields. In the holistic view developed here, all fields are frame equivalent in some way, and $R = \kappa^2$ is a property of the curved frame. The $\boldsymbol{B}^{(3)}$ is non-zero [4—6] and there exists a class of novel longitudinal solutions of the field equations in vacuo of electromagnetism, regarded holistically as a sector of the unified field. This class exists even in the Maxwellian view [15], which we take as a linear approximation to a non-linear field equation structure to be derived in Chap. 2. Equation (1.73) is also a type of equivalence principle in the holistic view, and $\hbar$ is the proportionality between energy-momentum and the irremovable curvature of space-time through which electromagnetism in vacuo is described. This curvature is a property of the vacuum itself, as is space-time curvature in the theory of the gravitational sector [26]. This analysis suggests the origin in curvilinear geometry of the fundamental postulates of quantum mechanics in special relativity, made originally by Planck, Einstein and de Broglie.

References

[1] A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Dover, New York, 1980).

[2] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill Book Co., New York, 1980).

[3] K. Huang, *Quarks, Leptons and Gauge Fields* (World Scientific, Singapore, 1992).

[4] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $\boldsymbol{B}^{(3)}$ (Kluwer Academic, Dordrecht, 1994).

[5] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).

[6] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).

[7] M. W. Evans, *Physica B* **182**, 227, 137 (1992); **183**, 103 (1993); *Physica B* **190**, 310 (1993); *Physica A* **214**, 605 (1995).

[8] M. W. Evans and S. Kielich, eds., *Modern Nonlinear Optics*, Vol. 85(2) of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, eds. (Wiley Interscience, New York, 1993).

[9] M. W. Evans, *The Photon's Magnetic Field* (World Scientific, Singapore, 1992); A. A. Hasanein and M. W. Evans, *The Photomagneton in Quantum Field Theory* (World Scientific, Singapore, 1994).

[10] M. W. Evans, *Found. Phys.* **24**, 892, 1519, 1671 (1994); **25**, 175, 383 (1995); M. W. Evans and S. Roy, *Found. Phys.*, in press.

[11] M. W. Evans, *Found. Phys. Lett.* 7, 67, 209, 379, 437, 577, 591 (1994); ibid. **8**, 83, 187, 253, 279 (1995).

[12] F. Farahi, Y. Aktas, and M. W. Evans, *J. Mol. Structure* **285**, 47 (1993); M. W. Evans and F. Farahi, ibid., **300**, 435 (1993); M. W. Evans, *Mod. Phys. Lett.* **B**7, 1247 (1993); M. W. Evans, S. Roy and S. Jeffers, *Il Nuovo Cim. D* (1995).

[13] M. W. Evans in A. van der Merwe and A. Garuccio, eds., *Waves and Particles in Light and Matter* (Plenum, New York, 1994).

[14] M. W. Evans, *Found. Phys. Lett.* and *Found. Phys.*, in press (1996 to 1997); M. W. Evans and S. Roy, *Found. Phys.*, submitted.

[15] A. Chubykalo and R. Smirnov-Rueda, *Phys. Rev. E*, in press (1996); also V. Dvoeglazov, *Phys. Rev. D*, submitted for publication (Feb., 1996).

[16] H. Múnera and O. Guzmán, *Found. Phys. Lett.*, submitted (Feb. 1996).

[17] G. Hunter and R. P. L. Wadlinger, *Phys. Essays* **2(2)**, 158 (1989).

[18] D. V. Ahluwalia and D. J. Ernst, *Mod. Phys. Lett.* **A7**, 1967 (1992); also D. V. Ahluwalia, M. B. Johnson, and T. Goldman, *Phys. Lett.* **B316**, 102 (1993); D. V. Ahluwalia and T. Goldman, *Mod. Phys. Lett.* **A8**, 2623 (1993).

[19] M. Sachs, *Apeiron* **2(4)**, 104 (1995).

[20] J. B. B. Filho, *Apeiron* **2(4)**, 103 (1995).

[21] H. Yilmaz, *Annals N. Y. Acad. Sci.* **755**, 476 (1995), Wheeler Honorarium.

[22] C. O. Alley, *Annals N. Y. Acad. Sci.* 755, **464** (1995), Wheeler Honorarium.

[23] H. Weyl, *Space, Time and Matter* (Dover, New York, 1952).

[24] R. P. Feynman, *Quantum Electrodynamics* (Addison-Wesley, Reading, 1961).

[25] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).

[26] P. W. Atkins, *Molecular Quantum Mechanics* (Oxford University Press, Oxford, 1983). For a relativistic treatment see L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th edn. (Pergamon, Oxford, 1975).

[27] N. N. Bogolyubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Wiley, New York, 1980).

[28] P. A. M. Dirac, in H. Hora and J. R. Shepanski, eds., *Directions in Physics* (Wiley, New York, 1978).

[29] P. A. M. Dirac, *Quantum Mechanics*, 4th edn. (Oxford University Press, Oxford, 1974).

[30] G. H. Wagnière, *Linear and Non-Linear Optical Properties of Molecules* (VCH, Basel, 1993).

[31] S. Weinberg, *Phys. Rev.* **B134**, 882 (1964); **B138**, 988 (1965).

[32] N. Calder, *Einstein's Universe* (Viking, New York, 1979).

[33] G. Hunter, S. Jeffers, S. Roy, and J.-P. Vigier, eds., *Present Status of the Quantum Theory of Light*, Proc. First de Broglie Vigier Symposium, York University, Toronto, 1995, (Kluwer, Dordrecht, 1996).
[34] V. Dvoeglazov, *Majorana-Like Models in the Physics of Neutral Particles.* XXX Escuela Latino Americana de Fisica, Mexico City (1995); Conference on the Theory of the Electron, ICTE 95, Cuautitlan, Mexico (Sept. 27—29, 1995); XXXVIII Congreso Nacional de la SMF, Zacatecas, Mexico (1995); *Phys. Rev. D*, submitted for publication.
[35] M. W. Evans and S. Roy, *Found. Phys.*, in press (1996/1997).
[36] M. W. Evans, *Apeiron*, submitted for publication.
[37] Atkins (Ref. 26a) gives an excellent account of quantized angular momentum.
[38] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[39] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).
[40] M. Mészáros and P. Molnár, *Ann. Phys.* **45**, 155 (1988); **46**, 153 (1989).
[41] M. Mészáros and P. Molnár, *Phys. Essays* **3**, 284 (1990).
[42] see articles in Refs. 13 and 33.
[43] L. D. Barron, *Physica B* **190**, 307 (1993); A. D. Buckingham, *Science* **266**, 665 (1994); A. D. Buckingham and L. C. Parlett, *Science* **264**, 1748 (1994).
[44] A. Lakhtakia, *Physica B* **191**, 367 (1993).
[45] A. Lakhtakia, *Found. Phys. Lett.* **8**, 183 (1995).
[46] D. M. Grimes, *Physica B* **191**, 367 (1993).
[47] E. Cornay, communications to Found. Phys. Lett.
[48] A. D. Buckingham, personal communications.
[49] reviewed by Mészáros and Molnár, in Ref. 41; also S. Weinberg, *The First Three Minutes* (Basic Books, New York, 1977).
[50] S. J. van Enk and G. Nienhuis, *J. Mod. Opt.* **41**, 963 (1994); R. A. Beth, *Phys. Rev.* **50**, 115 (1936).
[51] Non-locality in two photon entangled states has been demonstrated experimentally, see Ref. 33.
[52] E. Recami, email communication to V. Dvoeglazov, (1996).
[53] M. Mészáros, *Found. Phys. Lett.*, submitted for publication.
[54] S. J. van Enk and G. Nienhuis, *Opt. Commun.* **94**, 147 (1992).
[55] e. g. several papers on one photon, electron, neutron and atom effects in Refs. 13 and 33.
[56] One photon limit of the Beth effect, see Refs. 26 and 50.
[57] S. J. van Enk, e mail communications based on Ref. 50a.
[58] M. Mészáros, submitted for publication, and e mail communications.
[59] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms* (Wiley, New York, 1989).
[60] S. K. Bose, *An Introduction to General Relativity* (Wiley Eastern Ltd., New Delhi, 1980).

[61] A. Trautman, *Foundations and Current Problems of General Relativity.* in *Lectures on General Relativity*, Vol. 1 (Prentice-Hall, New York, 1964).
[62] M. W. Evans, *Found. Phys. Lett.* and *Found. Phys.*, submitted for publication.
[63] See Refs. 25 and 38 for discussions of geometrical forms.
[64] This discussion is based on Ref. 38.
[65] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[66] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Clarendon, Oxford, 1934).
[67] P. T. Landsberg and D. A. Evans, *Mathematical Cosmology* (Clarendon, Oxford, 1978).
[68] Ref. 25, Chaps. 1—3.

Chapter 2. Field Equations of the Electromagnetic Sector

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2.1 Geometry

In this chapter, novel and general electromagnetic field equations are derived from a consideration of the $B^{(3)}$ field in magneto-optics and cosmology. In the notation of differential geometry [1] they are:

$$D^{*}G = 0\,, \quad DG = \frac{J}{\epsilon_0}\,, \quad G = DA = \frac{\hbar}{e}R\,, \tag{2.1}$$

where D is the differential form corresponding to the covariant derivative in the Poincaré group, G is the 2-form corresponding to the field strength tensor in the Poincaré group, $^{*}G$ is the dual of G; J is the charge-current 3-form; A is the 1-form corresponding to the potential four-vector; ϵ_0 is the vacuum permittivity; and R is the 2-form corresponding to an antisymmetric Ricci tensor. The ratio $e/\hbar$ is the inverse of the elementary fluxon in weber (Chap. 1) and is a fundamental constant. These equations are geometrical in origin and use the language of forms [2] developed by Wheeler *et al.* [1]. They treat the electromagnetic field as being equivalent to the helical propagation [3] of e in curved space-time. They reduce to the Maxwell equations under well-defined conditions, i.e., when the Poincaré group is replaced by the U(1) group [2] of the received view.

In the elegant notation provided by differential geometry [1], electromagnetism in the received view is described by the Maxwell equations,

$$d^{*}F = 0\,, \quad dF = \frac{J}{\epsilon_0}\,, \quad F = dA\,, \tag{2.2}$$

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which are essentially linear approximations of Eqs. (2.1). In Eqs. (2.2) D is approximated by the d form corresponding to the ordinary four-derivative $\partial_\mu := \partial/\partial x^\mu$, so that the affine connections are zero in Eqs. (2.2), non-zero in Eqs. (2.1). The 2-form G of Eqs. (2.1) is approximated in Eqs. (2.2) by the Faraday 2-form [1] F corresponding to the field strength tensor $F_{\mu\nu}$ in the *flat-world* group U(1) = O(2). In Eqs. (2.2), two out of the six components of G are set to zero (the longitudinal components), allowing the non-linear (2.1) to reduce to the linear (2.2), the Maxwell equations of 1865 in form notation. In the received view it is thought that electromagnetic radiation in vacuo translates identically at c, which is the constant speed of light of the received theory. This implies that the particulate photon is identically massless, and that the concomitant electromagnetic field is transverse [2] with infinite range. In consequence, the electromagnetic sector of field theory is thought to be described by the flat-world U(1) symmetry [2].

While unreservedly admiring the many profound achievements of this famous theory of electromagnetism, it is recognized in this chapter that it has serious inherent difficulties which nonetheless can be addressed by replacing the U(1) group by the 4-D Poincaré group. In this way a simple mechanism for a unified theoretical description of electromagnetism and gravitation can be developed. In this view, electromagnetism turns out to be described by the antisymmetric part of the Ricci tensor $R^{(A)}_{\mu\nu}$, introduced in Chap. 1. Gravitation is accepted as being derived from Einstein's description of general relativity, although experimental evidence now points towards acceptance of Yilmaz's extension [4] of the original Einstein theory, because the latter has developed serious inherent difficulties summarized for example by Alley [5]. These difficulties in general relativity are present despite its seeming accuracy [6], and similarly there are severe conceptual difficulties (infinities) in the most accurate seeming theory of all, quantum electrodynamics [2], so we are entering an era in field theory where paradigm shifts are to be expected. These developments are well summarized by Mészáros and Molnár [7].

Therefore in the new field equations to be developed in this chapter, the field strength tensor, $G_{\mu\nu}$, or 2-form G, of electromagnetism is proportional to the antisymmetric part of a rank two Ricci tensor; and the canonical energy-momentum tensor of gravitation is proportional to the symmetric part, so that in this view, the electromagnetic and gravitational fields are both derived from the same Riemann curvature tensor of space-time, through rotational and translational equivalence principles. This view is based on the experimental evidence for $B^{(3)}$ available in well defined magneto-optical effects [8]; and $B^{(3)}$ is the catalyst for change. The Maxwell equations cannot describe its existence fully self-consistently. The phenomenological approach to magneto-optics of the received view constructs the conjugate product from solutions of the Maxwell equations, a vector product of, for example, complex conjugate plane waves of magnetic flux density in vacuo [9]. Equation (1.1) of Chap. 1 has O(3) symmetry which is disallowed by the flat-world group U(1), a group which allows no fields perpendicular to the plane defined by U(1).

Equations (2.1) contain the 2-form which corresponds to the antisymmetric Ricci tensor derived from the Riemann curvature tensor [1],

$$R^{(A)}_{\mu\nu} := R^{\lambda}_{\lambda\mu\nu}, \tag{2.3}$$

and $e/\hbar$ is the universally constant ratio of charge and action quanta. The basic hypothesis is that the geodesic, or least path, of the charge quantum is a helix; and that the vacuum electromagnetic field is equivalent to a non-inertial frame of Riemannian space-time. This is a rotational equivalence principle akin to the original translational equivalence principle of Einstein's theory of gravitation. Thus, translational acceleration is due to gravitation; rotational acceleration is equivalent to electromagnetism. In order to convert the Maxwellian theory to the geometrical theory of electromagnetism, replace ∂_μ by D_μ, the covariant derivative in the Poincaré group; and replace $F_{\mu\nu}$ by $G_{\mu\nu}$. The geometrical theory developed in this chapter therefore enlarges the two dimensional flat-world of U(1) to the four dimensional space-time of the Poincaré group. This implies the replacement of Eqs. (2.2) by Eqs. (2.1), in which appears the Ricci 2-form that betrays the geometrical origin of the new theory of electromagnetism. This symmetry enlargement allows several new solutions to appear of the field equations of vacuum electromagnetism.

2.2 $B^{(3)}$ the Catalyst for Change

It has been shown in Chap. 1, Sec. 1.2 that the Pauli-Lubanski formalism produces the B cyclics within special relativity. Experimentally, the observable $B^{(1)} \times B^{(2)}$ implies the existence of that of $iB^{(0)}B^{(3)*}$, and therefore of $B^{(3)}$. If the dual of $G^{\nu\rho}$ is defined by $^*G^{\nu\rho}$, then,

$$A^{(0)}\Gamma_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Gamma^{\nu\rho}A^\sigma, \quad \Gamma^{\nu\rho} = G^{\nu\rho} + {}^*G^{\nu\rho}, \tag{2.4}$$

defines the invariant vectors G_μ and $^*G_\mu$ within an overall Poincaré group symmetry. The extension of the Lorentz group to the Poincaré group is achieved by making the space-time translation generator proportional to A^σ. Equation (2.4) shows that

$$\Gamma^\mu A_\mu = 0, \tag{2.5}$$

i.e., A_μ is orthogonal to G^μ (and to $^*G^\mu$). The first (mass) Casimir invariant is defined by

$$A^{\mu}A_{\mu} := A^{(0)2} - \boldsymbol{A}\cdot\boldsymbol{A}\,, \tag{2.6}$$

and the second (spin) Casimir invariant by

$$\Gamma^{\mu}\Gamma_{\mu} := \Gamma^{(0)2} - \boldsymbol{\Gamma}\cdot\boldsymbol{\Gamma} = -\boldsymbol{\Gamma}\cdot\boldsymbol{\Gamma}\,. \tag{2.7}$$

Neither invariant is zero in general, if we allow for the possible existence of non-zero photon mass, but the orthogonality condition (2.5) is always true by definition. Therefore,

$$A_{\mu} := \left(A^{(0)}, \boldsymbol{A}\right), \quad \Gamma_{\mu} := \left(0,\ c\boldsymbol{B} + i\boldsymbol{E}\right), \tag{2.8}$$

and so A_{μ} is not proportional to Γ_{μ} in general, meaning that the photon, if particulate, can have more than two helicities, i.e., is a massive boson with helicities -1, 0, +1. Furthermore, using the space-like vector $(0,\ \boldsymbol{E} + ic\boldsymbol{B})$, whose square is also an invariant of special relativity, the Maxwell equations can be written in the form of vacuum neutrino equations [10],

$$\beta_{\nu}\frac{\partial\Gamma_{\mu}}{\partial x_{\nu}} = 0\,, \tag{2.9}$$

where β_{ν} are 4 x 4 Hermitian matrices. Therefore Eq. (2.9) has the form of a Dirac and/or Kemmer equation [10] involving cyclic relations between spinors. Therefore the B cyclics have their equivalents in neutrino physics, and similarly, indicate a non-zero neutrino mass. The cyclic relations for the β_{ν} vectors [10] are

$$\beta_i\beta_j = \delta_{ij} - \epsilon_{ijk}\beta_k\,, \tag{2.10}$$

which is a structure similar to that of Eq. (1.1). The vector Γ_{μ} that generates these cyclics is given by the *master* Eq. (2.4), which in a sense is both a photon and neutrino equation. For both particles, experimental data to date put an upper limit on the mass of both particles [11—13]. Eq. (2.4) also produces self-consistently the field invariant of the Lorentz group (part of the Poincaré group), and the two Casimir invariants of the Poincaré group itself. The vacuum minimal prescription (1.20) is an expression of gauge invariance, the gauge group being the Poincaré group, not the flat-world U(1) in which $\boldsymbol{B}^{(3)}$ is not defined. It turns out that $\boldsymbol{B}^{(3)}$ is defined by a part of the Riemann curvature tensor of helical space-time, as detailed later in this chapter. Insight to the nature of the invariant vector is given by Landau and Lifshitz [14b], who show that the Lorentz transformation along Z of this vector is a rotation in the (Z, t) plane in 4-space equivalent to a rotation in 3-space through *three* complex angles. In other words the six angles of rotation in 4-D reduce to three complex angles of rotation in 3-D. However, if it is asserted, as in the received view, that $\boldsymbol{B}^{(3)}$ and its dual in vacuo $\left(i\boldsymbol{E}^{(3)}\right)/c$ [11] are zero, then there can be only one complex angle in space because we have reduced

space to a plane, represented by the flat-world group U(1). This is a flaw in the received view. The existence of the magneto-optical observable $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ implies that this view is incomplete [11], and in the language of general relativity, the received view asserts that a rigorously non-zero part of the Riemann curvature tensor is zero.

Hunter and Wadlinger [3] have shown that the invariant vector Γ_{μ} is always an eigenfunction of the angular momentum operator $\hat{L} := (\hbar/i)(\partial/\partial\phi)$ in cylindrical coordinates, with eigenvalues $\pm\hbar$. To this view, we add the eigenvalue 0 corresponding to the (3) component in vacuo of the complete invariant vector Γ [11—13], so that all six components of $G_{\mu\nu}$ are properly considered. As argued elsewhere, $\boldsymbol{B}^{(3)}$ is observable, therefore real and physical, while its dual $\left(i\boldsymbol{E}^{(3)}\right)/c$ is not an observable, because there are no data equivalent to magneto-optic data for $\boldsymbol{B}^{(3)}$ [11—13] that show the effect of a putative $\boldsymbol{E}^{(3)}$ at first order. However Coulombic effects due to a longitudinal $\boldsymbol{E}^{(3)}$ are present in the self-consistent solutions of Maxwell's equations, as demonstrated recently by Chubykalo and Smirnov-Rueda [15]. In the cyclic equations, which are angular momentum equations as explained in Chap. 1, $-i\boldsymbol{E}^{(3)}/c$ is pure imaginary with magnetic symmetry. For this reason it is unphysical at first order but its square modulus in this calculus is pure real and physical. Similarly, the dual of the transverse, complex, field is $i\left(\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*}\right)/c$, but both $\boldsymbol{B}^{(1)}$ and $\boldsymbol{E}^{(1)}/c$ have physical and real parts because $\boldsymbol{B}^{(1)}$ and $\boldsymbol{E}^{(1)}$ are both complex, i.e., each has both real and imaginary parts. As discussed by Hunter and Wadlinger [3], Γ in general are the only valid eigenfunctions of the electromagnetic field in vacuo, regarded as a general solution of the wave equation. The plane wave is a mathematical limit of the general solution, which is a wavicle [3], with three finite space dimensions, not two. Jackson [16] has also discussed briefly the appearance of a longitudinal wave component in space if the electromagnetic field is constrained along the transverse axes, so that it is no longer a plane wave. Our analysis shows [11] that the conjugate product of the plane wave also produces the longitudinal $\boldsymbol{B}^{(3)}$ in vacuo. All these solutions contradict the U(1) flat-world symmetry for vacuum electromagnetism, whose group is considered here to be the Poincaré group in the special relativistic limit (local group limit of general relativity).

2.3 Geometrical Equations of Electromagnetism and the Unified Field

Chapter 1 has shown that electromagnetism can be defined geometrically in curvilinear coordinates [14]. If electromagnetism and gravitation can both be described by local Poincaré group symmetry then unification of the theory is achieved, both sectors being derived from the Riemann tensor describing the geometrical curvature of the same space-time. Thus, the vacuum electromagnetic field has geometrical properties such as scalar curvature; metric coefficient; affine connection; and Ricci tensor. There is an equivalence principle between the electromagnetic field and angular acceleration.

The starting point for the geometrical theory of vacuum electromagnetism is Riemann's curvature tensor, a property of pure geometry independent of any coordinate system [1]. Gravitation is a particular manifestation of curvature, and electromagnetism in our view a different type of curvature. Gravitation is therefore taken to be described by the Einstein tensor (although see [4]) and so is the symmetric part of the Ricci tensor obtained by suitable index contraction from the Riemann tensor. In so doing the affine connections are symmetric. The Einstein field equations are accepted, and they assert that the Einstein tensor is proportional to the canonical energy-momentum tensor $T_{\mu\nu}$ through a universal constant containing the gravitational constant k. If electromagnetism is some manifestation of curvature, it is reasonable to propose

$$G_{\mu\nu} = \Phi^{(0)} R_{\mu\nu}^{(A)} = \frac{\hbar}{e} R_{\mu\nu}^{(A)} , \tag{2.11}$$

so that the *complete* electromagnetic field strength tensor $G_{\mu\nu}$ is directly proportional to the antisymmetric part of the Ricci tensor, defined by Eq. (2.3). The $R_{\mu\nu}^{(A)}$ tensor is related to the Weyl conformal tensor and is antisymmetric. The metric coefficient of vacuum electromagnetism is fully antisymmetric (Eq. (1.44)) as a result of the hypothesis (2.11), which makes use of the antisymmetry of the Riemann tensor in the indices μ and ν. In consequence of the hypothesis (2.11) the affine connections for electromagnetism must also be antisymmetric, and in a contracted form reduce to axial vectors such as A_μ of Eq. (1.35). In contrast, the metric for the gravitational field is diagonal, and the affine connections are symmetric.

In this view, the geodesic of the charge quantum e is a helix, and the d'Alembert and Proca equations are geodesic equations, as shown in Chap. 1. The $B^{(3)}$ field is obtained from the Riemann tensor in this view, specifically from the rigorously non-zero part of the Riemann tensor quadratic in the affine connection. Thus, if Eq. (2.11) is accepted, either in the simple form given in this chapter or more generally as a Weyl conformal tensor [1], then it follows in this field theory that $B^{(3)}$ must be non-zero. In the received view there is of course no connection between the electromagnetic field and the frame of space-time, and $B^{(3)}$ is assumed to be zero because of U(1) sector symmetry. The flat-lander's view of electromagnetism (received theory) therefore asserts that a part of space-time is missing and that the electromagnetic field is not a space-time curvature. In the new view, the quantity analogous to the universal constant k (Einstein constant) is $\hbar/e$, the fundamental fluxon in weber, also a universal constant. In this view therefore, gravitation is the warping of space-time, electromagnetism the twisting of space-time. In both cases the frame curvature is equivalent to a field; gravitation is equivalent to linear acceleration, electromagnetism to angular acceleration.

The use of curvilinear coordinates requires the replacement of a four-derivative ∂_μ by D_μ, defined through the affine connection, otherwise known as the Riemann-

Christoffel symbol [14b]. Identities such as the Bianchi identity must be expressed through D_μ and not ∂_μ. Similarly, in the geometrical theory of electromagnetism, ∂_μ must be replaced by D_μ, which becomes a covariant derivative within the Poincaré group, a derivative defined by the appropriate affine connection. This process ensures gauge invariance within the Poincaré group. The older, flat-world, theory manifests itself through Maxwell's equations, and through the ordinary derivative ∂_μ of the Maxwell vacuum equations,

$$\partial_\mu F^{\mu\nu} = 0 , \tag{2.12}$$

where $F_{\mu\nu}$ is the flat-world field strength tensor with, conventionally [2], the two longitudinal components missing. In the flat-world there can be no three dimensional cyclics such as Eqs. (1.1) of this volume, no properly defined $B^{(3)}$; and no scalar curvature. These assertions are geometrically incorrect if electromagnetism is regarded as a manifestation of curved space-time. The latter hypothesis is the essence of field unification as proposed in this chapter.

It is recognized that there are analogies between the flat-world theory of vacuum electromagnetism and Einstein's geometrical theory of gravitation. One of these is between the potential four-vector of the electromagnetic field and the affine connection in the gravitational field [2,10]. Within the Poincaré group, the analogy becomes fully understood because the affine connection becomes an expression in the rotation generators M_μ of the group [2],

$$\Gamma_{\mu\nu}^{\lambda} = \frac{e}{\hbar} M_\mu M_\nu A^\lambda . \tag{2.13}$$

The connection is therefore a product of two rotation generators with the potential four-vector A^λ and this is clearly something indicating a rotation (helical motion) of space-time itself, a motion fully equivalent to the electromagnetic field. Analogously, the warping of space-time is equivalent to the gravitational field, as in general relativity [1]. The field equation (2.11) is non-linear, but is consistent with the superposition principle for the purposes of quantization, because angular momentum commutators occur within quantum theory. The difference between Eq. (2.11) and the flat-world theory is that the latter is Abelian. Indeed, quantization is implied already in Eq. (2.11) by the presence of the quantum of angular momentum, or action, $\hbar$.

There is a more profound difference, however, between the geometrical and Maxwellian theories. Ryder [2], for example, states that: "*In Einstein's theory the gravitational field is manifested as curvature of space-time. In electrodynamics, the field is, as it were, an actor on the space-time stage, whereas in gravity the actor becomes the space-time stage itself.*" Equation (2.11) removes this conceptual objection to field unification by making $G_{\mu\nu}$ proportional to $R_{\mu\nu}^{(A)}$; essentially a novel principle of equivalence. Other objections to unification are well known in the literature, for example, the apparent lack of experimental evidence for gravitons [1], implying difficulties in observing the quantized gravitational field. However, gravitons are implied by several theories [1] and some experimental evidence may already be available [2]. In contrast, the magneto-optical evidence for $B^{(3)}$ is unequivocal, and incompatible with the older, flat-world theory. Equation (2.11) contains the ratio of charge and action quanta, $e/\hbar$, a universal constant akin to the Einstein gravitational constant k. The constant $e/\hbar$ therefore introduces universal angular acceleration into the equations of electromagnetism, just as k introduces universal linear acceleration into the equations of gravitation. Therefore the charge quantum e plays the role of k, and not mass, and the electromagnetic field in vacuo is the geodesic of e, a helix. As soon as the electromagnetic field interacts with material matter, mass enters into consideration, and there is a mixing between warped and helical space-time. In the geometrical theory presented here this is the equivalent of the Lorentz force equation. In the traditional point of view [2] the universality of $e/\hbar$ is not identified, and so it is conventionally asserted that there is no universal acceleration in electromagnetism. The origin of this statement is the appearance in the Lorentz force equation of the ratio e/m, charge quantum to unquantized mass. However, this occurs only in the interaction of electromagnetic and gravitational components of space-time. The ratio e/m would also become universal if there occurred in nature a mass quantum, a minimum and unchangeable amount of mass, and it is reasonable to equate this with the mass of the lightest particle, the photon. Analogously, the quantum of angular momentum $\hbar$ is the angular momentum of the photon. The mass quantum defined in this way is incompatible with the flat-world theory because the latter prohibits the existence of photon mass. Equation (2.11), however, allows photon mass [11—13], and so the interaction of electromagnetism with matter would contain a universal constant if we used in its theoretical description the ratio of the accepted charge quantum, e, to a proposed mass quantum m_q. Such a mechanism would remove nearly all known objections to unification of the gravitational and electromagnetic fields.

Significantly, Eq. (2.11) can be written as follows in precise analogy with the Einstein equation,

$$T^{(A)}_{\mu\nu} = \hbar\omega\left(\frac{R^{(A)}_{\mu\nu}}{R}\right), \tag{2.14}$$

where $T^{(A)}_{\mu\nu}$ is an antisymmetric electromagnetic energy-momentum tensor and $R = \kappa^2$ is the scalar curvature in electromagnetism. Equation (2.14), in which appears the quantum of electromagnetic energy $\hbar\omega$ (the photon), is therefore a rotational Einstein equation. It generalizes the Planck-Einstein hypothesis,

$$En = \hbar\omega, \tag{2.15}$$

and is related both to the well known de Broglie hypothesis and the vacuum equivalence principle,

$$eA^{(0)} = \hbar\kappa. \tag{2.16}$$

Equation (2.11) can in turn be regarded as a generalization of the vacuum equivalence condition. The scalar curvature in electromagnetism is defined through the antisymmetric metric coefficient ($g^{(A)}_{\mu\nu}$),

$$R = \kappa^2 = g^{\mu\nu}_{(A)} R^{(A)}_{\mu\nu}. \tag{2.17}$$

The analogous definition of scalar curvature in gravitation is given through the metric $g_{\mu\nu}$ [2,17] and the symmetric part of the Ricci tensor $R_{\mu\nu}$; i.e., through the equation

$$R(\text{grav}) = g^{\mu\nu}R^{(S)}_{\mu\nu}, \tag{2.18}$$

where the symmetric Ricci tensor is well known to be obtainable from the index contraction,

$$R^{(S)}_{\lambda\nu} = R^{\kappa}_{\lambda\kappa\nu}, \tag{2.19}$$

of the Riemann tensor. If electromagnetism and gravitation are both to be seen as phenomena of curved space-time, then both fields are derived ultimately from the *same* Riemann curvature tensor as follows:

$$T^{(A)}_{\mu\nu}(\text{e.m.}) = \hbar\omega\left(\frac{R^{(A)}_{\mu\nu}}{R}\right), \tag{2.20a}$$

$$T^{(S)}_{\mu\nu}(\text{grav.}) = \frac{c^4}{8\pi k}\left(R^{(S)}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right), \tag{2.20b}$$

$$R_{\mu\nu} = R^{(S)}_{\mu\nu} + R^{(A)}_{\mu\nu}. \tag{2.20c}$$

This is the basis of our hypothesis for the geometrical origin of the electromagnetic field, and for the unification of field theory through curvilinear geometry. Although our end results, Eqs. (2.1), look similar at first sight to the Maxwell equations (2.2), and when linearized reduce to them, they are derived on a different philosophical basis. Maxwell (although a contemporary of Riemann) did not make the electromagnetic field equivalent to curved space-time. Contemporary theories of gravitation in the presence of electromagnetism also rely on Maxwell's equations and deduce that it is sufficient to replace derivative by covariant derivative (comma by semicolon), a procedure that does not affect the flat-world tensor $F_{\mu\nu}$. Our point of view is radically different, but at the same time is based on experimental magneto-optics [11]. The flat-world's $F_{\mu\nu}$ is replaced by $G_{\mu\nu}$ in curved space-time, and $G_{\mu\nu}$ is proportional to a novel antisymmetric Ricci tensor $R^{(A)}_{\mu\nu}$, obtained by a novel index contraction of the curvature tensor of space-time. Curvature of space-time therefore gives rise both to gravitation and electromagnetism.

These concepts are summarized in Table 1 and rely essentially on the recognition and emergence of the $\boldsymbol{B}^{(3)}$ field in ground based magneto-optics. In cosmology, $\boldsymbol{B}^{(3)}$ is the relict field responsible for anisotropy in the 2.7 K background, and in general $\boldsymbol{B}^{(3)}$ is the fundamental magnetizing field. The essence of the unification hypothesis used in Table 1 is that the complete Ricci tensor, being a second rank tensor, is the sum of its symmetric $(R^{(S)}_{\mu\nu})$ and antisymmetric $(R^{(A)}_{\mu\nu})$ parts. This means that $R^{(A)}_{\mu\nu}$ must be proportional to $R^{(S)}_{\mu\nu}$ through a dimensionless constant, and from this it can be shown that the speed of light becomes expressible in terms of the Planck length L as follows

$$c = \lambda f = \left(\frac{4k}{c^2L^2}\right)\hbar, \tag{2.21}$$

where λ is electromagnetic wavelength and f is frequency. Equation (2.21) shows that the speed of light (electromagnetic) is linked to the gravitational constant k through the Planck length L. In a sense, c itself becomes quantized, because it is proportional to $\hbar$, and this means that space-time becomes quantized at dimensions commensurate with L [1]. These well known ideas about space-time quantization are produced self-consistently in our view merely by putting $R^{(A)}_{\mu\nu}$ proportional to $R^{(S)}_{\mu\nu}$, in the first instance equal to $R^{(S)}_{\mu\nu}$. Furthermore, in the rest frame, Eq. (2.21) is the de Broglie Guidance theorem [11—13], use
because

$$\hbar\omega_0 = mc^2 = \frac{4mk}{cL^2}\hbar, \tag{2.22}$$

thus identifying the rest frequency as

$$\omega_0 = \frac{4mk}{cL^2}, \tag{2.23}$$

TABLE 1

Some Concepts in the Unified Theory of Fields

Concept-Quantity	Gravitation	Electromagnetism
Riemann tensor	$R^{\kappa}_{\lambda\mu\nu}$	$R^{\kappa}_{\lambda\mu\nu}$
Ricci tensor	$R^{(S)}_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$	$R^{(A)}_{\mu\nu} = R^{\alpha}_{\alpha\mu\nu}$
metric coefficient	$g_{\mu\nu}$ (diagonal)	$g^{(A)}_{\mu\nu}$ (off-diagonal)
scalar curvature	$R = g^{\mu\nu}R^{(S)}_{\mu\nu}$	$R = g^{\mu\nu(A)}R^{(A)}_{\mu\nu} = \kappa^2$
Einstein tensor	$R^{(S)}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R := G^{(E)}_{\mu\nu}$	$R^{(A)}_{\mu\nu}$
field equation	$G^{(E)}_{\mu\nu} = \frac{8\pi k}{c^4}T^{(S)}_{\mu\nu}$	$R^{(A)}_{\mu\nu} = \frac{\kappa^2}{\hbar\omega}T^{(A)}_{\mu\nu}$
connection	$\Gamma^{\lambda}_{\mu\nu}$	$\Gamma^{\lambda}_{\mu\nu} = \frac{e}{\hbar}M_{\mu}M_{\nu}A^{\lambda}$
local group	Poincaré	Poincaré
group generator identity	Bianchi identity $D_{\rho}R^{\kappa}_{\lambda\mu\nu} + D_{\mu}R^{\kappa}_{\lambda\nu\rho} + D_{\nu}R^{\kappa}_{\lambda\rho\mu} = 0$	Becomes a field equation when $\kappa = \lambda$.
energy-momentum tensor	$T^{(S)}_{\mu\nu}$ (translational)	$T^{(A)}_{\mu\nu} = \omega J_{\mu\nu} = \frac{\hbar\omega}{R}R^{(A)}_{\mu\nu}$ (rotational)
$\boldsymbol{B}^{(3)}$ in vacuo	consistent with local Poincaré group	enables U(1) tensor $F_{\mu\nu}$ to be replaced by $G_{\mu\nu}$ in the Poincaré group

Concept-Quantity	Gravitation	Electromagnetism
equivalence principle	gravitation is a translating and accelerating non-inertial frame	electromagnetism is a rotating non-inertial frame
universal constant	k (Einstein's gravitational constant)	$\frac{e}{\hbar}$ (ratio of charge and action quanta)
linearization	produces gravitational plane waves	produces electromagnetic plane waves
wave equation	$D_\mu T^{\mu\nu(S)} = 0$	$D_\mu T^{\mu\nu(A)} = 0$
Lagrangian	calculated through extremum of [10], $I = \frac{1}{2}\int g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu d\lambda$	$\mathcal{L} = -\frac{1}{4}\epsilon_0\left(\frac{\hbar}{e}\right)^2 R^{(A)}_{\mu\nu} R^{(A)*}_{\mu\nu}$
linking equation	$c = \lambda f = \left(\frac{4k}{c^2L^2}\right)\hbar$ (see text) L = Planck length	same
de Broglie guidance theorem $\left(mc^2 = \hbar\omega_0\right)$	same	$mc^2 = \hbar\omega_0,\ \omega_0 = \frac{4mk}{cL^2},$ m = photon mass

and defining it in terms of the Planck length and gravitational constant. These are examples of results from a theory which interlinks the gravitational and electromagnetic fields, unifying them essentially through the concept of space-time curvature and the Riemann tensor [1]. In a nutshell, gravitation and electromagnetism are different parts of the Riemann tensor, i.e., are interlinked components of the curvilinear geometry of space-time.

Finally, the electromagnetic field equations can be deduced in this purely geometrical point of view by using Eq. (2.11) in the Bianchi identity with appropriate index contraction [2]. This gives the *homogeneous* field equation,

$$D_\rho G_{\mu\nu} + D_\nu G_{\rho\mu} + D_\mu G_{\nu\rho} = 0, \tag{2.24}$$

in which appears the covariant derivative D_μ in the Poincaré group and the complete field strength tensor $G_{\mu\nu}$. The homogeneous field equation (2.24) can be written as [2]

$$D_\nu {}^*G^{\mu\nu} = 0 . \tag{2.25}$$

In the presence of material matter with mass, the electromagnetic field (helical space-time) encounters warped space-time, and this encounter is represented symbolically by the equation

$$D_\nu G^{\mu\nu} = \frac{J^\mu}{\epsilon_0}, \tag{2.26}$$

where J^μ is a charge-current vector. There is therefore conservation of charge-current provided that we use covariant derivatives and the complete field tensor, and this is in conformity with Noether's theorem. In differential geometry, charge-current is a 3-form [2,10], thus giving the required equations (2.1). It is already clear that these have the structure of Maxwell's equations if D_μ were replaceable by ∂_μ and $G_{\mu\nu}$ by $F_{\mu\nu}$, and in the following section we discuss this *linearization*. The field equations (2.1) are derived, however, on a different philosophical basis from those of Maxwell. In our view the charge quantum is a singularity of helical space-time — the field being an expanding space-time helix emanating from a source.

2.3.1 Equations of the Unified Field

The reduction of the novel field equations (2.1) to the Maxwell equations (2.2) occurs either when the affine connection vanishes or when it is defined within an Abelian group symmetry such as U(1) = O(2). In both cases, the 4-D Poincaré group is collapsed onto the 2-D flat-world symmetry of U(1), the electromagnetic sector symmetry in the received view [2]. The transformation of Eqs. (2.1) to Eqs. (2.2) is therefore accomplished by replacing D_μ by ∂_μ and $G_{\mu\nu}$ by $F_{\mu\nu}$. For example, the covariant derivative of A_μ is [1]

$$D_\mu A_\nu := \partial_\mu A_\nu - \Gamma^\lambda_{\nu\mu} A_\lambda , \tag{2.27}$$

which becomes the ordinary derivative of A_μ if the affine connection is zero. If the group symmetry is U(1) = O(2), the covariant derivative introduces the electromagnetic field through the minimal prescription in the received view [2]. Prior to the discovery of $B^{(3)}$ [11—13] this view was accepted widely, both in electromagnetic and unified field theory [2]. However, the field tensor $G_{\mu\nu}$ is defined by [1]

$$G_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu - \left(\Gamma^\lambda_{\nu\mu} A_\lambda - \Gamma^\lambda_{\mu\nu} A_\lambda\right), \tag{2.28}$$

and if, in accordance with general gauge theory [2], the affine connection is defined by Eq. (2.13), Eq. (2.28) becomes

$$G_{\mu\nu} = F_{\mu\nu} + \frac{e}{\hbar} A^2 \left(M_\mu M_\nu - M_\nu M_\mu\right). \tag{2.29}$$

Thus $G_{\mu\nu} = F_{\mu\nu}$ in the U(1) = O(2) group because the rotation generators of this group commute [2]; and $F_{\mu\nu}$ is defined as being made up of only transverse components in vacuo, as is the received opinion. This is of course consistent with the fact that U(1) = O(2) is two dimensional — it cannot define a field orthogonal to the plane of definition, which is transverse to the direction of propagation [2]. In the Poincaré group, the rotation generators in Eq. (2.29) do not commute, and so Eq. (2.29) contains longitudinal solutions in vacuo for this reason. In general, A^2 contains the electromagnetic phase, and so in general the longitudinal solutions are phase dependent [15]. The $B^{(3)}$ solution is one generated by a conjugate product of phases, and so $B^{(3)}$ itself appears as phase independent [11]. As shown recently by Chubykalo and Smirnov-Rueda [15] there can be phase dependent longitudinal solutions even of the traditional Maxwell equations, but these again indicate the need for more general field equations such as those proposed in this chapter. The philosophical difference between Eqs. (2.1) and (2.2) is revealed through the fact that if the affine connection is zero, there can be no curvature, directly contradicting the fact that the curvature of electromagnetism is κ^2 from a plane wave.

Equation (2.29) is consistent with the novel equivalence principle (2.16) which is the basis for the field equations (2.1), and this in turn is consistent with the fundamental, geometrical, structure of the Riemann curvature tensor itself, the curvature tensor from which $R^{(A)}_{\mu\nu}$ is obtained from the contraction defined in Eq. (2.3). The equations of the unified field can also be written from the fundamental definition of the Riemann curvature tensor, Eq. (1.69), [2,14b,10] by defining the antisymmetric field tensor $G_{\mu\nu}$ using covariant derivatives of the Poincaré group,

$$G_{\mu\nu} := D_\mu A_\nu - D_\nu A_\mu . \tag{2.30}$$

We obtain without further assumption the Jacobi-Bianchi identity,

$$D_\kappa G_{\mu\nu} + D_\nu G_{\kappa\mu} + D_\mu G_{\nu\kappa} = R^\lambda_{\mu\nu\kappa} A_\lambda + R^\lambda_{\nu\kappa\mu} A_\lambda + R^\lambda_{\kappa\mu\nu} A_\lambda = D_\rho R^\kappa_{\lambda\mu\nu} + D_\mu R^\kappa_{\lambda\nu\rho} + D_\nu R^\kappa_{\lambda\rho\mu} = 0, \tag{2.31}$$

which can be written as the homogeneous field equation,

$$D_\mu {}^*G^{\mu\nu} = 0, \tag{2.32}$$

where ${}^*G^{\mu\nu}$ is the dual of $G_{\mu\nu}$ [2,10]. It is also the Bianchi identity in the theory of gravitation, because $G_{\mu\nu}$ is derived, as we have seen, from the antisymmetric part of the Riemann tensor, whose symmetric part can be contracted to the Einstein tensor. Therefore Eq. (2.32) is the homogeneous equation of the unified field.

Similarly, Eq. (2.31) can be developed into the inhomogeneous equation of the unified field. Firstly raise indices in the Riemann tensor and field tensor,

$$G^{\nu\kappa} = g^{\nu\rho} g^{\kappa\sigma} G_{\rho\sigma}, \qquad R_\mu^{\ \lambda\nu\kappa} = g^{\nu\rho} g^{\kappa\sigma} R^\lambda_{\ \mu\rho\sigma}. \tag{2.33}$$

Form the equivalence of $G_{\mu\nu}$ and $R^{(A)}_{\mu\nu}$ (Eq. (2.11)) individual terms in the identity (2.31) can be equated:

$$D_\kappa G^{\mu\nu} = R_\kappa^{\ \lambda\mu\nu} A_\lambda, \tag{2.34a}$$

$$D_\nu G^{\kappa\mu} = R_\nu^{\ \lambda\kappa\mu} A_\lambda, \tag{2.34b}$$

$$D_\mu G^{\nu\kappa} = R_\mu^{\ \lambda\nu\kappa} A_\lambda, \tag{2.34c}$$

Consider the antisymmetric part of the Riemann tensor in Eqs. (2.34) by suitable contraction. In Eq. (2.34) for example the contraction is $\lambda = \mu$. The result reduces to an inhomogeneous field equation by setting $\mu = \kappa$,

$$D_\mu G^{\nu\mu} = R_\mu^{\ \mu\nu\mu} A_\mu = \frac{J^\nu(\text{vac.})}{\epsilon_0}, \tag{2.35}$$

where the term

$$J^\nu(\text{vac.}) := \epsilon_0 R_\mu^{\ \mu\nu\mu} A_\mu, \tag{2.36}$$

is the charge current four vector of the electromagnetic sector itself. In other words, the electromagnetic field carries its own source in precise analogy with the gravitational field. Equation (2.35) is also a wave equation, because $R_\mu^{\mu\nu\mu}$ plays the role of the d'Alembertian in Maxwellian field theory.

The inhomogeneous field equation in vacuo is therefore,

$$D_\mu G^{\nu\mu} = \frac{J^\nu(\text{vac.})}{\epsilon_0}, \tag{2.37}$$

in which all quantities are consistent with general relativity and gauge invariance. In Eq. (2.37) appears the vacuum charge-current, which acts as a source for the field in vacuo, in precisely the same way as mass-energy acts as a source of the gravitational field. The vacuum charge-current does not appear in the Maxwell field equations because the latter are Abelian, i.e., are linear approximations in the U(1) group. Tiny, but well-observed, magneto-optic effects reveal the limits of this linear approximation.

If the electromagnetic field encounters material matter, by which we mean matter with mass, then in the unified theory of fields one part of the field mixes with another, i.e., electromagnetic and gravitational sectors interact. This is represented in Eq. (2.34c) by contracting $R^\lambda_{\mu\nu\kappa}$ with $\lambda = \nu$. The massive material is therefore represented by a symmetric Ricci (i.e., Einstein) tensor as demanded by general relativity [1]. To obtain a field equation we set $\kappa = \mu$,

$$D_\mu G^{\nu\mu} = R_\mu^{(S)\mu} A^\nu = RA^\nu = \frac{J^\nu(\text{matter})}{\epsilon_0}, \tag{2.38}$$

where J^ν (matter) is the charge-current vector of massive material matter. The inhomogeneous field-matter equation of the unified field is therefore

$$D_\mu G^{\nu\mu} = \frac{J^\nu(\text{matter})}{\epsilon_0}. \tag{2.39}$$

From Eqs. (2.37) and (2.39) follows the equation of charge conservation of the unified field,

$$J^\nu(\text{vac.}) = J^\nu(\text{matter}). \tag{2.40}$$

i.e.,

$$R_\mu^{\mu\nu\mu} A_\mu = RA^\nu. \tag{2.41}$$

Equation (2.40) is an expression of Noether's theorem [2] and shows that charge-current is conserved when there is a balance between sectors of the unified field. The charge quantum e is always conserved and unchanged in this theory, and this fact expresses itself as a balance between different parts of the Riemann tensor. The symmetric part represents mass-energy and the antisymmetric part field-energy. Finally, as in the geometrical theory of gravitation [1], the field equation contains the force equation (or equation of motion). If current is defined by Eq. (2.41), then the force equation of the unified field is

$$f_\mu = J^\nu G_{\mu\nu}, \tag{2.42}$$

in which, according to Eq. (2.39), J^ν may be either electromagnetic or gravitational in origin. If J^ν is electromagnetic then Eq. (2.42) is the unified form of the Lorentz force equation describing force generated in charged material matter (for example an electron) by the electromagnetic sector. In a purely gravitational context, Eq. (2.42) reduces to the Newton equation using well-known approximations [1] in general relativity.

Equations (2.32), (2.37), (2.39), (2.40) and (2.42) are equations unifying the gravitational and electromagnetic field. They are based on the assumption that all fields are equivalent to space-time curvature [1]. The first three reduce to Maxwell's field equations if $G_{\mu\nu}$ is replaced by $F_{\mu\nu}$; D_μ by ∂_μ; and J_μ (vacuum) =? 0 (self-inconsistently). Equation (2.40) illustrates charge conservation, and Eq. (2.42) reduces either to the Lorentz equation of Maxwellian theory or to Newton's equation of force with the product of mass and acceleration in the gravitational sector.

References

[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[2] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).

[3] G. Hunter and R. P. L. Wadlinger, *Phys. Essays* 2(2), 158 (1989).

[4] H. Yilmaz, *Annals N. Y. Acad. Sci.* **755**, 476 (1995), Wheeler Honorarium.

[5] C. O. Alley, *Annals N. Y. Acad. Sci.* **755**, 464 (1995), Wheeler Honorarium.

[6] J. B. B. Filho, *Apeiron* 2(4), 103 (1995).

[7] M. Mészáros and P. Molnar, *Phys. Essays* 3, 284 (1990).

[8] M. W. Evans and S. Kielich, eds., *Modern Nonlinear Optics*, Vol. 85(2) of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, eds. (Wiley Interscience, New York, 1993).

[9] G. H. Wagnière, *Linear and Non-Linear Optical Properties of Molecules* (VCH, Basel, 1993).
[10] A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Dover, New York, 1980).
[11] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $\boldsymbol{B}^{(3)}$ (Kluwer Academic, Dordrecht, 1994).
[12] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).
[13] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).
[14] P. W. Atkins, *Molecular Quantum Mechanics* (Oxford University Press, Oxford, 1983). For a relativistic treatment see L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th edn. (Pergamon, Oxford, 1975).
[15] A. Chubykalo and R. Smirnov-Rueda, *Phys. Rev. E*, **53**, 5373 (1996); also V. Dvoeglazov, *Phys. Rev. D*, submitted for publication (Feb., 1996).
[16] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).
[17] P. A. M. Dirac, in H. Hora and J. R. Shepanski, eds., *Directions in Physics* (Wiley, New York, 1978).

Chapter 3. The Fundamental Spin Field $\boldsymbol{B}^{(3)}$

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In this chapter we arrive at the structure of the unified field from the single axiom that the fundamental spin field of electromagnetism in the vacuo is the $\boldsymbol{B}^{(3)}$ field, whose very existence is taken to imply that it can be expanded in terms of transverse components. This is no more than a statement to the effect that three dimensional geometry is three dimensional, and that the electromagnetic field exists likewise in the space of O(3). Therefore the unification of the electromagnetic and gravitational fields can be achieved within Riemann's geometry using the newly inferred and observed longitudinal component of the electromagnetic sector described in these volumes. The conventional flat-lander's theory of electromagnetism is a two dimensional world which works within the familiar U(1) group, the group of rotations in a plane, with nothing defined perpendicular to that transverse plane. The discovery of the $\boldsymbol{B}^{(3)}$ field leads to U(1) being replaced by the group of rotations in the ordinary three dimensional world, the group O(3). In space-time, this is enlarged to the local Poincaré group, which becomes the symmetry group of electromagnetism. It follows that the electromagnetic field is a particular contraction of the Riemann tensor in which affine connections are antisymmetric. This contraction gives an antisymmetric Ricci tensor, which is proportional to an electromagnetic field strength tensor $G_{\mu\nu}$. The latter contains within it a longitudinal component in the vacuum, the $\boldsymbol{B}^{(3)}$ field. It is now understood that $\boldsymbol{B}^{(3)}$ is one of a set of longitudinal components of the propagating electromagnetic field. Chubykalo and Smirnov-Rueda [1] have argued that it could be interpreted as the longitudinal component responsible for action at a distance. The magnetic components of the electromagnetic field are inter-related in the vacuum through an O(3) cyclical symmetry governed in structure by the commutators of angular momentum, or equivalently, by the relation between infinitesimal rotation generators of O(3). This structure remains the same in the Poincaré group of space-time provided that 3 x 3 matrices are replaced by 4 x 4 equivalents. Therefore the electromagnetic field becomes space-time curved according to O(3) within the Poincaré group. The Ricci tensor that governs this type of space-time curvature is the antisymmetric component

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of the overall rank two tensor whose symmetric part appears in the Einstein equations for the gravitational field in the general theory of relativity. Since any rank two tensor is the sum of its symmetric and antisymmetric parts, we arrive at a simple, Riemannian, method of treating electromagnetism and gravitation as a unified field.

In the conventional point of view, the electromagnetic field is not curved space-time, it is a physical entity which is philosophically distinct from the frame of reference, as in physics prior to the equivalence principle. The difficulties of unifying the electromagnetic and gravitational fields in the conventional (*pre-*$B^{(3)}$) point of view spring largely from the fundamental philosophical difference between the concept of field as given by Faraday, Maxwell, and contemporaries, and the concept of field-frame equivalence as given by Einstein. The local group of gravitation in general relativity is the Poincaré group; that of electromagnetism, until recently, was thought to be U(1). These groups are mutually incompatible because they are dimensionally incompatible. The myth of transversality was founded in turn on the belief that in the vacuum, only transverse components existed in the electromagnetic field, whose particulate embodiment, the photon, was thought to be without mass and unlocalized. The transverse components were thought to propagate in the vacuum at the speed of light, the signal velocity known empirically with great precision. These transverse components were related within the linear field equations of Maxwell. The equations of gravitation in general relativity can be reduced to the special relativistic form of Maxwell's equations, but only in the linear, weak field approximation. Gravitation otherwise carries its own source in a non-linear mathematical structure. Electromagnetism in the vacuum was written within a linear field calculus and was considered to be source free. Above all there loomed the philosophical barrier between the idea of field as space-time curvature (gravitational theory) and that of the field as a separate physical entity (electromagnetic theory).

A look at the Riemann or Ricci tensors in gravitation, and at gauge theory in electromagnetism written for the field strength tensor in a group symmetry other than the flat-lander's U(1), reveals a close similarity of structure. In U(1), the structural similarity disappears with that part of the Riemann or field strength tensor quadratic in the affine connection. However, the Riemann tensor is written with covariant derivatives; the electromagnetic field strength tensor with ordinary ones in U(1) theory. There is an analogous and related similarity of structure between the Jacobi-Bianchi identity in gravitation and the homogeneous Maxwell equations in electromagnetism; but again, covariant is replaced by ordinary derivative. These seemingly fundamental dissimilarities in derivative operators disappear however if we accept longitudinal components for the vacuum electromagnetic field. If so, the latter's field strength tensor becomes an antisymmetric Ricci tensor, defined with covariant derivatives in which appear antisymmetric affine connections [2]. Similarly, the homogeneous Maxwell equations become a Jacobi-Bianchi identity within O(3), or more rigorously, within the Poincaré group. The metric coefficient for the electromagnetic field becomes totally antisymmetric (off-diagonal); and the longitudinal component of the electromagnetic field is incorporated self-consistently in a non-linear, non-Abelian structure. The philosophy of the electromagnetic field becomes compatible with that of the gravitational field, because the electromagnetic field

strength tensor becomes proportional to the antisymmetric Ricci tensor through the elementary fluxon $\hbar/e$. In order therefore to describe the electromagnetic field we require a space-time that is curved antisymmetrically, and we require the elementary fluxon. Similarly, in order to describe gravitation we need a space-time that is curved symmetrically, and we need a scaling constant proportional to the gravitational constant. The electromagnetic and gravitational fields, within given factors of proportionality, become respectively the antisymmetric and symmetric parts of the Ricci tensor. These parts are obtained from the same Riemann tensor by different types of index contraction [14], and the beauty of this is that both fields are Riemannian in nature.

This simple field unification rests on the empirical evidence for the existence of longitudinal components of the electromagnetic field in the vacuum; for example magneto-optic phenomena [4—6]. Transverse mythology requires a phenomenon of inverse induction, for example phase free magnetization by a ruby laser (the inverse Faraday effect), to be described through non-linear, quadratic, products of transverse field components. One of these, the conjugate product [22], is the vector product of the vector potential with its own complex conjugate. The conjugate product is longitudinal and axial as the result of ordinary vector algebra in three dimensions. There is an irreconcilable conflict between the flat-lander's U(1) group, which allows nothing to be defined outside the transverse U(1) plane. This is the point at which the older philosophy of the electromagnetic field becomes a general theory of relativity within Riemannian curvilinear geometry. Longitudinal, phase free, magnetization is produced by one of a family of new and extraordinary longitudinal solutions in the vacuum of the electromagnetic field [4—20,15]. This is the phase free component, the fundamental spin $B^{(3)}$. It becomes rational to look at vacuum electromagnetism from the longitudinal perspective; and to construct, or infer the existence of, transverse components by an expansion of the fundamental spin. In this way, it is possible to introduce the electromagnetic phase, containing the frequency and wavenumber; and to deduce that the electromagnetic field is defined completely in terms of [7] the Pauli-Lubanski axial four-vector,

$$B^{\mu} = \left(B^{(0)}, \boldsymbol{B}^{(3)}\right), \tag{3.1}$$

where $B^{(0)}$ is the magnitude of $\boldsymbol{B}^{(3)}$; and the energy momentum polar four-vector p^{μ}.

Coulombic action at a distance can be explained in terms of $\boldsymbol{F} = \boldsymbol{B}^{(3)}$, where **F** is the field introduced by Chubykalo and Smirnov-Rueda [1], and now identified with $\boldsymbol{B}^{(3)}$. Significantly, experimentally observable superluminal phenomena [23—25] cannot be explained from Maxwell's equations without longitudinal wave components being present in the vacuum. Instantaneous action at a distance surely means that there is some influence present that is transmitted through the intervening vacuum at speeds much greater than c: logically, infinitely greater than c, and action at a distance means a physical, longitudinal, transmission [23—25]. Such a theory allows as argued a simple, Riemannian, unification of the electromagnetic and gravitational fields as components of the same Ricci tensor; and if this is accepted, action at a distance also becomes logically possible in gravitation. General relativity itself may become [26] a logical outcome of action at a distance. Superluminal phenomena in the electromagnetic field have recently appeared in the popular literature [25] following the transmission of Mozart at greater than c: the music of the spheres?

3.1 Theorems

The following theorems provide a structural basis for the hypothesis that the electromagnetic field's fundamental spin is $B^{(3)}$, or in terms of flux, the fluxon multiplied by the unit vector $e^{(3)}$. No quantity other than $\hbar/e$ and $e^{(3)}$ is required to describe the electromagnetic field concomitant with the photon. The notion of electromagnetic field is thereby reduced to the helical geodesic of e, the quantum of primordial charge.

Theorem 1. A purely imaginary polar vector (e.g. $-iE^{(3)}$), appearing in O(3) group Lie algebra plays the role of a pure real axial vector (e.g. $B^{(3)}$). Therefore $-iE^{(3)}$ has axial symmetry and is identically non-zero.

Theorem 2. In an O(3) Lie algebra such as the B cyclics between the three components of an axial vector, one component is pure real and axial.

Theorem 3. In an O(3) Lie algebra between components of a polar vector one component is pure imaginary and identically non-zero.

Theorem 4. The classical electromagnetic field is completely defined by the vectors p^μ, where p^μ is the polar momentum four-vector and B^μ is the axial, Pauli-Lubanski, four-vector defined by: $B^\mu := \left(B^{(0)}, B^{(3)}\right)$.

Theorem 5. The wavefunction of one photon is completely defined by its four-momentum and by its concomitant B^μ field.

Theorem 6. The Casimir invariants for one photon are $p_\mu p^\mu$ and $B_\mu B^\mu$; respectively the mass and spin invariants which completely define the particle within the Poincaré group.

Theorem 7. Angular frequency (ω) is implied by the existence of these vectors, and can be introduced through the quantum postulate.

Theorem 8. The polar and axial four-vectors p^μ and B^μ completely define the interaction between field and matter.

Theorem 9. The relation between B^μ and p^μ is defined by the equation of Pauli and Lubanski; in which B^μ is the vector dual to the electromagnetic field strength tensor.

Theorem 10. The helicity of the field is defined relativistically by

$$\zeta := p^{(3)} \cdot \frac{B^{(3)*}}{\left(p^{(0)} B^{(0)}\right)} , \tag{3.2}$$

where $p^{(0)}$ is the modulus of $p^{(3)}$.

Theorem 11. The existence of transverse components such as $A^{(1)} = A^{(2)*}$ is implied by the existence of $B^{(3)}$; and so is that of the electromagnetic phase. For example,

$$B^{(3)*} = -i\frac{e}{\hbar} A^{(1)} \times A^{(2)} , \tag{3.3}$$

is a result of this theorem and also the result of the Dirac equation of the relativistic quantum field.

Theorem 12. The Lorentz condition expresses the relativistic orthogonality of p^μ and B^μ, i.e.,

$$p_\mu B^\mu = 0 , \tag{3.4}$$

In the quantum postulate $p_\mu = i\hbar\partial_\mu$.

Theorem 13. From theorem 12,

$$\partial_\mu B^\mu = 0, \tag{3.5}$$

is a Lorentz condition and is also the conservation of angular momentum, thus part of Noether's theorem.

Theorem 14. In general, ∂_μ is covariant, introducing general relativity through the antisymmetric Riemann-Christoffel symbol.

Theorem 15. If ∂_μ is used within U(1), theorem 13 gives Maxwell's equations.

This progression of theorems is an attempt at building the geometry of general relativity from p^μ and B^μ, the latter being basically *longitudinal* in nature. However, we know empirically that the electromagnetic field is wave-like, diffraction and spectroscopy providing a mass of everyday evidence for this, so the inference of B^μ as a fundamental spin field must be developed systematically to include the frequency and wavenumber within the phase,

$$\phi := \omega t - \kappa \cdot r . \tag{3.6}$$

This is a reversal of the historical development, in which $B^{(3)}$ is a distinct latecomer. The phase can be introduced through the expansion,

$$B^{(3)*} = -i\frac{e}{\hbar} A^{(0)2} e^{(1)} \times e^{(2)} e^{i\phi} e^{-i\phi} , \tag{3.7}$$

which is simply an expression of geometry in O(3). This equation already contains within it the minimal prescription for the free photon,

$$\frac{e}{\hbar} = \frac{\kappa}{A^{(0)}} . \tag{3.8}$$

Equation (3.7) can be interpreted as being equivalent to another expansion of the vector $B^{(3)}$ in space, one which uses the fact that $B^{(3)}$ has axial symmetry, and so is always the curl of a polar vector A, i.e., $B^{(3)} = \nabla \times A$. The equivalence of this well known equation to the new Eq. (3.7) leads to the optical Aharonov-Bohm effect as discussed in Sec. 3.1. At this stage the arbitrary product of plus and minus exponentials in Eq. (3.7) is justified empirically, through data which show that there exists an electromagnetic phase. Indeed the existence of $B^{(3)}$ (and therefore of B^{μ}) may be justified empirically through the fact that light magnetizes matter and that this phenomenon is empirically proportional to intensity I and thus to the square of the amplitude of magnetic field density. Therefore the inverse Faraday effect at visible frequencies is proportional empirically to a product $B^{(0)}B^{(3)}$ with axial vector symmetry, because the observed effect is a magnetization, in turn an axial vector. If the experiment were repeated under the right conditions we know from Vols. 1 to 3 [4—6] that the observed magnetization would gradually become proportional to the square root of intensity, and it would be inferred reasonably that this is due to $B^{(3)}$ because the magnetization would be phase free empirically. In other words, magnetization is caused by a magnetic field, $B^{(3)}$, contained within a circularly polarized light beam.

In Eq. (3.7), the phase ϕ is an arbitrary, real, unitless scalar, but $e^{-i\phi}$ and $e^{i\phi}$ must be relativistically invariant because $B^{(3)}$ is unchanged by Lorentz transformation. Therefore the scalar ϕ must be constructed in such a way as to allow this property. To investigate further the nature of ϕ, empirical evidence for electromagnetic waves of frequency ω leads to the inference that light, in addition to the magnetic field, $B^{(3)}$, is a carrier of traveling waves, because light propagates through the vacuum at a signal velocity, which we denote c. Therefore complex exponentials of the type $e^{-i\phi}$ and $e^{i\phi}$ can be described by a wave equation. It is reasonable to conclude that the phase ϕ must be of the form (3.6), as given by a traveling wave equation [4—6]. This can be expressed as the product of two four-vectors in pseudo-Euclidean geometry,

$$\phi = x^{\mu}\kappa_{\mu} = \omega t - \kappa \cdot r , \tag{3.9}$$

and this product is relativistically invariant. This latter property is inferred from the definition of the axial four-vector ($B^{(0)}$, $B^{(3)}$) as a Pauli-Lubanski four-vector, and so far we have used no equations of motion other than the general wave equation. Therefore, *without using Maxwell's equations*, the plane wave,

$$A^{(1)} = A^{(2)*} = A^{(0)}e^{(1)}e^{i\phi} , \tag{3.10}$$

has been deduced to exist in the vacuum *because* of the empirical existence of the fundamental vector $B^{(3)}$ and because of empirical evidence for frequency ω and signal velocity c.

The plane wave (3.10) is complex, and is a solution of a wave equation with the structure deduced by d'Alembert from linear relations between electric and magnetic fields proposed by Maxwell. How is it then possible to deduce the existence of electric fields as components of plane waves if the fundamental spin is the purely magnetic $B^{(3)}$? This question is answered through the fact that $B^{(3)}$ is dual to $-iE^{(3)}/c$ where $E^{(3)}$ is a polar vector in vacuo. If so then $E^{(3)}$ has the units of volt m^{-1} which is electric field strength. Therefore $-iE^{(3)}$ has the symmetry of a magnetic field, and $-iE^{(3)}$ is pure imaginary and unphysical. The duality theorem is a property of pseudo-Euclidean space-time. Therefore it is possible to get this far without knowing the structure of Maxwell's equations, which actually emerge from the fact that the axial four-vector B^{μ} is the vector dual of an antisymmetric four-tensor $\tilde{G}^{\mu\nu}$ [4]. From conservation of angular momentum, (if Eq. (3.5)),

$$\partial_{\mu}\tilde{G}^{\mu\nu} = 0 , \tag{3.11}$$

showing that the four-derivative of $\tilde{G}^{\mu\nu}$ is zero. This gives the inhomogeneous Maxwell equations in the vacuum, and the homogeneous equations can be deduced as a Jacobi identity [27] with ordinary derivatives replacing covariant derivatives,

$$\partial^{\lambda}G^{\mu\nu} + \partial^{\mu}G^{\nu\lambda} + \partial^{\nu}G^{\lambda\mu} = 0 . \tag{3.12}$$

It is now possible to proceed into a general relativistic theory using the fact that covariant derivatives can be replaced by ordinary ones only in a flat space-time, or flat space, such as that described by the flat-lander group U(1).

3.2 The Relativistic Helicity of the Classical Field

In this section we develop the theme that the electromagnetic field can be described in terms of longitudinal components, sometimes without any reference being necessary to the transverse components that are usually asserted to be the only solutions of the vacuum Maxwell equations. It proves convenient to use the polar vector potential defined [28] through

$$B = i\kappa \times A , \tag{3.13}$$

where $\boldsymbol{\kappa}$ is a polar wave vector and where $\boldsymbol{B}$ is an axial magnetic field vector. In the circular basis ((1), (2), (3)) we write

$$B^{(2)*} = -i\kappa^{(3)} \times A^{(1)}, \tag{3.14}$$

which relates the longitudinal $\kappa^{(3)}$ to the transverse $A^{(1)}$ and $B^{(2)}$. Using the quantum hypothesis (second quantum of action hypothesis [29], or minimal prescription),

$$p^{(3)} = eA^{(3)} = \hbar\kappa^{(3)}, \tag{3.15}$$

where $p^{(3)}$ is a quantized linear momentum along the (3) = Z axis of propagation of the field. Equation (3.14) becomes a cross product in ((1), (2), (3)) of two vector potentials, and this is a Grassman wedge product [30]. Therefore there emerges the cyclical symmetry

$$B^{(1)*} = -i\frac{e}{\hbar}A^{(2)} \times A^{(3)}, \quad B^{(2)*} = -i\frac{e}{\hbar}A^{(3)} \times A^{(1)},$$
$$B^{(3)*} = -i\frac{e}{\hbar}A^{(1)} \times A^{(2)}. \tag{3.16}$$

In these relations, the real parts of each quantity or product of quantities are physical and observable empirically; and the left hand side structures allow the A vectors to be either polar or axial. We have used polar A vectors, but could equally have chosen the axial A vectors identified with Stratton's V [28]. This is because the vector cross product of two axial vectors or two polar vectors gives an axial vector, $B^{(3)}$. The use of the polar $A^{(3)}$ vector means that $A^{(3)}/e$ is a polar linear momentum in (3) = Z.

Theorem 1 can be illustrated for polar A vectors as follows. Starting from the standard B cyclic theorem (a type of equivalence principle),

$$B^{(1)} \times B^{(2)} = iB^{(0)}B^{(3)*}, \tag{3.17}$$

we prove that the equivalent A cyclic theorem for polar A vectors is

$$A^{(1)} \times A^{(2)} = iA^{(0)}A^{(3)*}, \tag{3.18}$$

where $A^{(3)}$ must be *pure imaginary*. There cannot exist an O(3) cyclic symmetry between three real polar vectors; but there can be among three axial vectors (e.g. the usual $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$ et cyclicum of the Cartesian basis). This means that although there may exist [1] a physical $E^{(3)}$, it does not participate as a real vector in a cyclic theorem such

as (3.18). To prove Eq. (3.18) from Eq. (3.17) we first use $B^{(0)} = \kappa A^{(0)}$ on the right hand side and Eqs. (3.16) on the left hand side of Eq. (3.17). We then use the two vector identities [31],

$$F \times (G \times H) = G(F \cdot H) - H(F \cdot G), \tag{3.19}$$

and

$$F \cdot (G \times H) = G \cdot (H \times F) = H \cdot (F \times G) = (F \times G) \cdot H, \tag{3.20}$$

to give

$$A^{(0)2}A^{(1)} \times A^{(2)} = -A^{(3)}\left(A^{(3)} \cdot \left(A^{(1)} \times A^{(2)}\right)\right). \tag{3.21}$$

In this equation we try

$$A^{(1)} \times A^{(2)} =?\ iA^{(0)}A^{(3)*}, \tag{3.22}$$

to give

$$-iA^{(0)3}A^{(3)*} = A^{(3)}\left(A^{(3)} \cdot iA^{(0)}A^{(3)*}\right), \tag{3.23}$$

an equation which can be satisfied if and only if

$$A^{(3)} = -A^{(3)*}, \tag{3.24}$$

i.e., $A^{(3)}$ is pure imaginary. This proves Theorem 1. A pure imaginary polar vector obtained from the cross product of two polar vectors has *axial* symmetry. This result can be generalized within Clifford and Grassman algebra [30], and is in no sense mundane or trivial because the square root of minus one is a well defined operator in Clifford algebra. In the algebra $Cl^{+}(M)$ this i operator can be *identified* with the familiar square root of minus one [30]. The latter therefore plays a fundamental and ineluctable role in the cyclic equations of electrodynamics. These are inherently complex in nature, and cannot simply be replaced by a pure real algebra in which i is undefined. To illustrate this point, which recurs throughout these volumes, we note that the complex conjugate product is a Grassman wedge product [30],

$$B^{(3)*} = -i\frac{e}{\hbar}A^{(1)} \times A^{(2)} = -i\frac{e}{\hbar}A^{(1)} \wedge A^{(2)}, \tag{3.25}$$

and this has no counterpart in real algebra *unless* we re-introduce i [30] as follows,

$$a \times b = -\hat{i}(a \wedge b) . \tag{3.26}$$

As a simple illustration, if we try to replace $A^{(1)} \times A^{(2)}$ by a pure real $A \times A$, it disappears, but we know that $A^{(1)} \times A^{(2)}$ is an empirical observable [4—10]. The complex $A^{(1)}$ can be distinguished from $A^{(2)}$ precisely *because* it is complex, i.e., because of the presence of i in the appropriate Clifford algebra Cl⁺(M), that of the B cyclic theorem. This result has physical as well as mathematical significance, because $A^{(1)} \times A^{(2)}$ is an observable. This means that the equivalent of the B cyclics exists mathematically in other Clifford algebras, which are sometimes physical, e.g. Dirac algebra, opening the door to advances in the theory of all known field equations. This illustrates the significance of the seemingly mundane Theorems 1 and 2 of the Cl⁺(M) Clifford algebra.

To apply these considerations to the relativistic helicity of the classical electromagnetic field we use some results given recently by Afanasiev and Stepanovsky [28] in the conventional transverse formalism. The relativistically covariant helicity in the classical electromagnetic field is defined [28] as the time-like (scalar) part of the current four-vector,

$$j^{\mu} := \left(j^{(0)}, j\right) , \tag{3.27}$$

which is defined [28] to be proportional to a combination of classical field vectors as follows,

$$j^{\mu} \propto \left(A \cdot B,\ A^{(0)}B + \frac{1}{c} A \times E\right). \tag{3.28}$$

Here $A^{(0)}$ is the scalar part of the potential four-vector, defined to be a polar four-vector. If we consider the longitudinal components in Eq. (3.28) we arrive at

$$j^{\mu} \propto A^{(0)}\left(B^{(0)},\ B^{(3)}\right), \tag{3.29}$$

i.e., it is immediately clear that the relativistic helicity is proportional to the vector B^{μ}, a result first arrived at in Chap. 11 of Vol. 1 [4]. The relativistic helicity is simply a conserved current density, and so is a conserved, fundamental field property as expected from the particle interpretation [27].

In the conventional, transverse, formalism, this simple interpretation is not clear, because we are confronted with helicity being defined in terms of products such

as $B^{(1)} \cdot A^{(2)}$. The latter is remote from the interpretation of helicity in a massless particle as a scalar derived from the ratio of the Pauli Lubanski axial four-vector to the polar energy momentum four-vector. However, using Eq. (3.16), the obscure transverse interpretation clarifies easily as follows,

$$B^{(1)} \cdot A^{(2)} = -i\kappa^{(3)} \times A^{(1)} \cdot A^{(2)} . \tag{3.30}$$

Using the vector identity [31]

$$F \times G \cdot H = F \cdot G \times H , \tag{3.31}$$

reduces Eq. (3.30) to a clearly understandable dimensionless helicity,

$$\zeta := \frac{1}{A^{(0)}B^{(0)}} A^{(3)} \cdot B^{(3)*} , \tag{3.32}$$

which is simply the dot product of an axial and polar vector, a pseudo-scalar as required. This pseudo-scalar is inverted by parity. This product becomes that between the linear and angular momentum of one photon if we use the minimal prescription,

$$p^{(3)} = \hbar\kappa^{(3)} = eA^{(3)} , \tag{3.33}$$

and the definition [4] of the photomagneton,

$$J^{(3)} = \frac{\hbar}{B^{(0)}} B^{(3)} , \tag{3.34}$$

giving a clear result,

$$\zeta := \frac{1}{\hbar\kappa} \kappa^{(3)} \cdot J^{(3)*} . \tag{3.35}$$

This shows that the helicity of the photon is the dot product of its angular and linear momentum, and this is also the result obtained from the Wigner analysis [4,27] for a massless particle. Compare this, finally, with the obscurity represented by the dot product $A^{(1)} \cdot B^{(2)}$. Starting, indeed, from empirical evidence for $B^{(3)}$ and for ω, we would have no need of postulating the existence of transverse components of the electromagnetic field to give a satisfactory definition of relativistic helicity.

3.3 The Quantum of Charge as the Invariant Gyromagnetic Ratio

It has been argued throughout these volumes that electromagnetic radiation can magnetize material matter. This property implies the presence in the vacuum of the elementary quantum of magnetic flux (weber), the fluxon $\hbar e$ [27], which is the ratio of Dirac's quantum, $\hbar$, of angular momentum (or action) to the quantum of charge e. The empirical evidence for the existence of $\hbar/e$ in vacuo is alone enough to show that there is angular momentum, $\hbar$, present for each photon. The gyromagnetic ratio developed in this section is the proportionality factor between the magnetic dipole moment of the photon and its angular momentum. There is also present in high precision data the fine structure constant [27,31],

$$\alpha = \frac{e^2}{4\pi\epsilon_0 c\hbar}, \tag{3.36}$$

which can be regarded as an empirical constant which can be interpreted in terms of fundamental empirical quantities e, c, ϵ_0, and $\hbar$, where ϵ_0 is the vacuum permittivity. These empirical constants can be independently measured in other experiments. The elementary fluxon is therefore proportional to the quantum of charge,

$$\Phi^{(0)} = \frac{e}{4\pi\epsilon_0 c\alpha} := \beta e = 4110.8e. \tag{3.37}$$

Here β is a $\hat{C}$ positive constant. For each photon, defined as the quantum of energy, $\hbar\omega$, there is a fluxon which is $\hat{C}$ negative, showing clearly that the photon cannot be its own antiparticle, but carries a hidden quantum number which can be defined as the eigenvalue of $\hat{C}$ operating on $\Phi^{(0)}$,

$$\hat{C}\left(\Phi^{(0)}\right) = -\Phi^{(0)}. \tag{3.38}$$

In earlier work [9] this quantum number was labeled the F number, in analogy to charge, charm, baryon number and so forth. Equation (3.37) also signifies that without the presence of the elementary quantum e in the phenomenon known as the electromagnetic field, there can be no magnetic effects as observed empirically when circularly polarized light meets matter, such as solids or liquids in the inverse Faraday effect. In the last analysis the quantum e must be present in the electromagnetic field.

In the development so far there has been no mention of space-time, we have dealt completely in relativistic invariants. The fine structure constant α is the result of empirical measurement in phenomena such as the anomalous magnetic dipole moment of the electron. Of course it is also the product of intellect, namely quantum electrodynamics [27]. The equation $\Phi^{(0)} = \hbar/e$ is therefore a quantum hypothesis between fundamental

relativistic invariants, i.e., the fluxon is $\hbar$ multiplied by $1/e$. The quantum of energy is $\hbar$ multiplied by angular frequency, ω, but the latter is a space-time quantity which is not relativistically invariant, because En is the time-like part of a four-vector $p^\mu = (En/c, \boldsymbol{p})$. *Therefore, the fluxon is the relativistically invariant quantum hypothesis.* Analogously, $\boldsymbol{B}^{(3)}$ as we have argued, is the relativistically invariant electromagnetic spin, and phenomena of magnetization by $\boldsymbol{B}^{(3)}$ are in a sense indicative of one of the fundamentals of nature. It has taken a long time to realize this because experiments such as the inverse Faraday effect are difficult and the much easier resonance experiments described in Chaps. 1 and 2 of Vol. 3 [6] have yet to be carried out under optimized conditions.

The quantum of charge e is the quantity that gives the electromagnetic field its distinctive property of being the agent of transfer of influence from one electron to another. The quantum e is the ratio of the intrinsic angular momentum to the intrinsic fluxon, where, in each photon,

$$\hbar = 4110.8e^2, \qquad \Phi^{(0)} = 4110.8e. \tag{3.39}$$

It is known empirically that there exist in electromagnetic radiation space-time quantities, namely angular frequency and wavenumber, which are not invariants of special relativity. These space-time quantities must be related to each other in such a way as to keep intact the invariant constants. An example is the minimal prescription applied frequently in these volumes to the free photon,

$$eA^{(0)} = \hbar\kappa, \tag{3.40}$$

a prescription which keeps the ratio of $A^{(0)}$ to κ (i.e. $B^{(0)}/\kappa^2$), a constant at all electromagnetic frequencies and which introduces considerations of space-time. This is seen through the fact that $A^{(0)} = B^{(0)}/\kappa$ where $B^{(0)}$ is a flux density magnitude (in tesla = weber per unit area). The wavenumber κ is also an inverse length, so space-time has appeared in the analysis through a ratio of two quantities, $A^{(0)}$ and κ. The numerator and denominator are not relativistically invariant individually, but their ratio is a Lorentz invariant and a universal constant. Through seemingly mundane observations such as this we arrive at the fundamentally important conclusion that the electromagnetic field is a particular relation between space-time dependent quantities, a relation which ultimately leads to field equations exemplified by the linear Maxwell equations. We arrive at the conclusions of Chaps. 1 and 2, that the field is equivalent to space-time through the proportionality constant $\hbar/e$, which is the fluxon. The latter therefore has the role of the gravitational constant in the generally relativistic theory of gravitation.

The magnetic flux density magnitude $B^{(0)}$ can be developed into a magnetic dipole moment magnitude $m^{(0)}$, but only by introducing a volume V. Otherwise the development is dimensionally incorrect. Therefore the fundamental fluxon can be defined as

$$\Phi^{(0)} = \mu_0 \frac{Ar}{V} m^{(0)} = Ar B^{(0)} , \tag{3.41}$$

and the product $Arm^{(0)}/V$ must be a relativistic invariant. The ratio of area Ar to volume V is an inverse distance with the units of wavenumber, and so the wave in the electromagnetic field is a traveling wave. This can be inferred from the existence in the field both of $\hbar$, an angular momentum, and c, a forward signal velocity. This in turn gives the field a chiral (or handed) nature, and so gives it a sense of right or left handedness which manifests itself empirically as circular polarization. The ratio between the magnetic dipole moment $m^{(0)}$ and the intrinsic angular momentum $\hbar$ is a gyromagnetic ratio, as usually defined [31] for particles,

$$m^{(0)} = \frac{\hbar}{(\mu_0 \kappa e)} . \tag{3.42}$$

This gyromagnetic ratio is not however relativistically invariant because of the presence of κ in the denominator. In order to arrive at a relativistically invariant gyromagnetic ratio, a ratio between a magnetic property and an angular momentum, we use again the definition of the fluxon,

$$\Phi^{(0)} = \frac{\hbar}{e} , \tag{3.43}$$

and so $1/e$ becomes the invariant gyromagnetic ratio.

This line of argument shows again that the fundamental spin in electromagnetism is magnetic in nature, a conclusion which in one sense is almost tautological, because without spin and charge, there can be no magnetic phenomenon. The spin is governed at the most fundamental level by the intrinsic quantum $\hbar$ of angular momentum; and the charge by the intrinsic quantum e, whose inverse is an intrinsic gyromagnetic ratio. These arguments remain true for a (now practically accessible) beam made up of one photon, generated experimentally by a parametric downconverter [13]. The invariant longitudinal field $B^{(3)}$ is the fluxon $\Phi^{(0)}$ divided by an area and multiplied by the unit vector $e^{(3)}$.

Since $B^{(3)}$ is relativistically invariant, so must be the area Ar, which we have identified with $1/\kappa^2$. If this is done then the flux density magnitude $B^{(0)}$ must be relativistically invariant,

$$B^{(0)} = \frac{\Phi^{(0)}}{Ar} = \frac{\hbar}{ec^2}\omega^2 . \tag{3.44}$$

It is easily checked that the definition [4—6],

$$B^{(3)*} := -i\frac{e}{\hbar} A^{(1)} \times A^{(2)} , \tag{3.45}$$

produces Eq. (3.44) because $B^{(0)} = \kappa A^{(0)}$. The relativistically invariant area Ar is therefore directly proportional to the conjugate product $A^{(1)} \times A^{(2)}$. This is checked through the minimal prescription (3.40), which reduces $A^{(1)} \times A^{(2)}$ to a cross product of wavevectors, whose magnitude is Ar,

$$|A^{(1)} \times A^{(2)}| = \left(\frac{\hbar}{e}\right)^2 |\kappa^{(1)} \times \kappa^2| = \frac{\Phi^{(0)2}}{Ar} . \tag{3.46}$$

This is a precise, self-consistent definition of the area of the photon, an area which must be relativistically invariant, and which must be used to reduce $\Phi^{(0)}$ to $B^{(0)}$, the magnitude of $B^{(3)}$.

This procedure can be carried out only in a gauge whose group symmetry allows the self-consistent existence of $A^{(1)} \times A^{(2)}$. In space this group is O(3), which is a sub-group of the Poincaré group of space-time. It is not possible to define a photon area self-consistently in a gauge group such as U(1), in which $A^{(1)} \times A^{(2)}$ is undefined. The definition (3.46) also removes the paradox of the indefinite metric [27] in the quantized field. If we attempt to define Ar by the inverse of the magnitude of the Lorentz invariant square of light-like κ^μ, we find ,

$$\kappa^2 = \kappa^\mu \kappa_\mu = 0 , \tag{3.47}$$

for the massless photon, and the photon area $(1/\kappa^2)$ is indefinite. It may indeed become possible to remove all the difficulties associated with the indefinite metric [27] from electrodynamics.

The magnitude of the photon area can be expressed in terms of the magnitude of magnetic flux density, $B^{(0)2} = \mu_0 I/c$, where I is intensity in W m^2. The area, Ar, then becomes proportional to the square of angular frequency and inversely proportional to intensity I,

$$Ar = \frac{\hbar^2}{e^2 \mu_0 c}\left(\frac{\omega^2}{I}\right) . \tag{3.48}$$

In a phenomenon such as radiation induced resonance, as described in Chap. 1 and 2 of Vol. 3 [6], the proton resonance frequency, for example, becomes

$$\omega_{res} = \frac{5.5857}{2.002} \frac{\hbar}{m} \cdot \frac{1}{Ar}, \tag{3.49}$$

where m is the proton mass, and the ratio is that of Landé factors for proton and electron. The resonance frequency is inversely proportional to the area of the photon, i.e., proportional to the factor I/ω^2. It follows that a smaller photon is able to transfer its energy more efficiently to the proton in the radiation induced resonance process. It was also shown in Vol. 3 [6] that the Stephan-Boltzmann law for one photon can be expressed as

$$I = \frac{\hbar^2}{\mu_0 e^2 c^3} \omega^4, \tag{3.50}$$

in which there is no Planck distribution of energy levels. In other words there is only one energy level, $\hbar\omega$. In a beam made up of one photon, therefore, its area from Eqs. (3.48) and (3.50) is $1/\kappa^2$.

Precise empirical evidence for this result has been given by Hunter and Wadlinger [16] and by Hunter, Wadlinger and Engler [32]. It was found that within $\mp 0.5\%$, the diameter of a single photon is

$$\frac{\lambda}{\pi} = \frac{c}{\pi f} = 2\frac{c}{\omega} = \frac{2}{\kappa}, \tag{3.51}$$

which is also the result of the rigorously relativistic theory given by Hunter and Wadlinger [16] in which the photon volume was found to be an ellipsoid with circular cross section of circumference one wavelength. The magnitude of magnetic flux density for one photon is then given by the fluxon divided by the area of the photon. This is precisely what is indicated by Eq. (3.45) which defines $B^{(3)*}$ in terms of the cross section $A^{(1)} \times A^{(2)}$. *Therefore the experiment by Hunter et al.* [32] *is direct empirical evidence for the existence of the* $B^{(3)}$ *field in vacuo*. In view of its importance we provide some details as follows.

3.3.1 Empirical Evidence for the Photon Diameter and for $B^{(3)}$

The experiment by Hunter, Wadlinger and Engler [32] was motivated by the earlier theory by Hunter and Wadlinger [17] which developed the photon as an oscillating-rotating electromagnetic field contained within a circular ellipsoid, whose length and cross-sectional circumference are both one wavelength (λ); and whose long axis is the axis of propagation. The theory produces a photon that occupies a relativistically well-defined volume, which was identified as a *wavicle*. The theory is therefore three dimensional in nature, and violates U(1) symmetry. It predicts that a beam of monochromatic photons will pass through apertures whose smallest linear dimension is greater than the wavicle's diameter, λ/π. The wavicle cannot, a priori, pass through smaller apertures because of mechanical attenuation, and for larger, uniformly attenuated apertures it is expected a priori that the transmitted power be proportional to the difference between the area of the aperture and the wavicle's cross sectional area. Under ideal conditions one photon should be passed through a circular aperture of variable diameter in order to test the theory exactly. When there are many photons present in a beam, it is expected that only one wavicle at a time can pass through the aperture if the latter is adjusted to be the same as the photon's maximum area.

The wavicle theory of the photon corresponds exactly with the design of the helical microwave antenna, whose axial mode has a wavelength equal to the circumference of the cylindrical helix, with maximum radiation along the axis of the helix in a well-defined, circularly polarized, beam [32]. The axis of propagation and circumference of the antenna, and the circular polarization, correspond precisely with the axis, circumference and polarization of the photon-wavicle theory [16] of Hunter and Wadlinger.

The experiments by Hunter, Wadlinger and Engler [32] were carried out with circularly polarized microwaves which were passed through a metal screen containing a circular or rectangular aperture. The transmitted power was measured on the other side of the screen as a function of aperture diameter. The microwave generator, aperture and receiver were carefully aligned. Therefore if the photon has no measurable or meaningful area, as in U(1) theory, very different results are expected from those actually obtained experimentally [32] where a critical aperture was found below which no power was transmitted. In carrying out the experiment the following sources of artifact were carefully considered.

(1) The transmitted power decreases towards zero as the aperture size approaches the critical size λ/π; measuring very small radiative powers, and discriminating them from instrumental noise, is subject to large relative errors.

(2) The fraction of the transmitted radiation that is diffracted (i.e., bent through an angle as it passes through the aperture) increases as the aperture size is reduced towards the critical size λ/π. In terms of the wavicle model, diffraction occurs when a wavicle impinges upon the wall of the aperture, the angle of bending being a function of the impact parameter of the collision. The model predicts that the proportion of incident photons that collide with the aperture walls increases as the aperture size approaches the critical size λ/π. The experiment sets out to detect only the non-diffracted light; i.e., the wavicles that pass through the aperture without colliding with its walls. Separating diffracted from non-diffracted light requires an appropriate experimental arrangement.

(3) Currents induced in the nominally opaque screen may cause some radiation to appear on the far side of the screen and must be carefully removed.

(4) The extrapolation to zero power assumes monochromatic radiation. Harmonics in the incident beam will dominate the transmitted power for aperture sizes close to the critical size λ/π. This must be corrected for in analyzing the measurements.

The experiments were carried out in the research laboratory of Tribar Industries over a six month period from October 1985 to April 1986. The microwave generators were standard production models based upon a turnstile junction and a horn antenna; they produced a beam of circularly polarized microwaves. The transmitted power was measured initially with a receiver antenna coupled to an amplifier and millivolt meter, and later with a Hewlett-Packard power meter. The first set of measurements were made with circular apertures cut in a thin aluminum screen with compasses to within 0.1 mm. Measurements were made with X-band (10.525 GHz, λ = 28.48 mm; λ/π = 9.07 mm). *The results showed immediately that there was no transmission through holes smaller than λ/π. The experiment was repeated many times with apertures of different diameter, more accurately machined, with a calibrated Hewlett Packard power meter as detector. Results were also obtained with K band radiation* (24.15 GHz; λ = 12.41 mm; λ/π = 3.95 mm); and the complete experiment carefully repeated with slit apertures. The theoretically predicted photon diameter was verified empirically within 0.5 %. As reasoned already this is also evidence for the existence of the $B^{(3)}$ field, through Eq. (3.45). In other words the inverse Faraday effect and the photon diameter experiment confirms Eq. (3.45) empirically because the former shows the presence of magnetic flux, $\hbar/e$, per photon, in the beam; and the latter measures the area per photon, i.e., $1/\kappa^2$, occupied by $\hbar/e$. A combination of these two experiments leads immediately to Eq. (3.45).

The results also provide important confirmation of the three dimensional nature of the individual photon, and counter-indicate U(1) symmetry empirically. The wavicle can be observed whenever one looks through a metal screen into a microwave oven. The visible photons (λ/π = 0.0002 mm) readily pass through the 2 mm holes in the screen, while the microwave photons (λ/π = 40 mm) do not.

3.3.2 Other Expressions for the Photon's Gyromagnetic Ratio

There are several different ways in which the gyromagnetic ratio of the photon can be expressed; one of which is Eq. (3.42). In this section the ratio is derived self-consistently to be

$$g_p = \frac{ec}{\hbar\kappa}, \tag{3.52}$$

which is the ratio of the elementary charge quantum e to the mass $\hbar\kappa/c$. Experimental observation of the $B^{(3)}$ field therefore occurs through the observation of the photon's spin angular momentum $S^{(3)}$ in the Beth effect [31] or in atomic absorption. This is a consequence of the fact that the $B^{(3)}$ field is the fundamental spin field, and in the next section, atomic absorption is worked out entirely in terms of $B^{(3)}$.

The only observable spin angular momentum of the photon is that in the propagation axis, which we denote $S^{(3)}$. This is observed in the Beth effect [31] as a mechanical angular momentum induced in a crystal by circularly polarized radiation. It is observed spectroscopically in atomic and molecular absorption processes [31]. In this section it is argued that the $B^{(3)}$ field is directly proportional to $S^{(3)}$ through the photon's gyromagnetic ratio g_p,

$$B^{(3)} = \frac{\mu_0}{V} m^{(3)} = \frac{\mu_0}{V} g_p S^{(3)} . \tag{3.53}$$

The gyromagnetic ratio in Eq. (3.53) has the usual units of C kgm^{-1}, as for a fermion such as proton or electron. Accordingly, Eq. (3.53) is a self-consistent relation in theoretical physics between a magnetic moment and an angular momentum.

As argued throughout these volumes, the photon and concomitant field have present in their dual wave-particle structure the quantum of charge e, an inference that was developed in the opening essay of Chap. 1. The electromagnetic field in vacuo is negative under the charge conjugation operator $\hat{C}$, for example

$$\hat{C}(A_\mu) = -A_\mu, \quad \hat{C}(B) = -B, \quad \hat{C}(E) = -E, \tag{3.54}$$

where A_μ is the four-potential in vacuo and E and B the electric and magnetic component vectors in space. As argued in Vol. 3 [6] however, the photon as quantum of energy, is proportional to the square of e, so is its quantized linear momentum $\hbar\kappa$ and angular momentum $\hbar$. As seen in this section, the fine structure constant relates $\hbar$ to the square of e, while the fluxon is $\hbar$ multiplied by $1/e$. Therefore particulate properties of the photon can be regarded as $\hat{C}$ positive, but are always accompanied by $\hat{C}$ negative concomitant quantities, of which $\hbar/e$ is the relativistically invariant quantum hypothesis. The photon, by Eq. (3.54), is not therefore its anti-particle. The photon is emitted by the oscillating electron in matter, the anti-photon by the oscillating positron in anti-matter. In this argument, therefore, it is necessary only to demonstrate the existence of the gyromagnetic ratio to show the existence of $B^{(3)}$ in the vacuum. In other words there exists in the field-particle the quantum of charge e. Fields are linear in e; particulate photon properties are quadratic in the *same* e. Clearly, the field can never be separated philosophically from the particle, because both concepts can be traced ultimately to the existence of the elementary quantum of charge, e. If the latter were not present in the electromagnetic field itself, there would be no transmission of electromagnetic influence through the vacuum. One oscillating electron could not cause another to oscillate at the other end of the universe. Evidently, the charge quantum e can be localized (as on a fermion), or delocalized, as in the electromagnetic field $B^{(3)}$. In the case of instantaneous action at a distance it has infinite extent. The conceptual relation of electromagnetic field to *point* or localized charge is the same as that of the gravitational field to *point* or localized mass. Electromagnetism becomes in this view a theory of general relativity and an expression of the strong equivalence Principle of Mach. It is therefore no coincidence that $B^{(3)}$ allows the development of a novel unification theory of the electromagnetic and gravitational fields.

The gyromagnetic ratio can be derived in the received view through the conjugate product of electric field components

$$S^{(3)} = -\frac{i\epsilon_0}{2\omega}\int E^{(1)} \times E^{(2)} dV, \tag{3.55}$$

and in the $B^{(3)}$ theory this is equivalent to [6]

$$En = \omega |S^{(3)}| = \frac{1}{\mu_0} B^{(0)} |B^{(3)*}| V, \tag{3.56}$$

where $B^{(0)}$ is the magnitude of $B^{(3)}$. The magnetic dipole moment in vacuo, $m^{(3)}$, is deduced from Eq. (3.56) using

$$B^{(3)} = \left(\frac{\omega\mu_0}{VB^{(0)}}\right) S^{(3)}, \tag{3.57}$$

and is defined by

$$m^{(3)} = \frac{V}{\mu_0} B^{(3)} = \left(\frac{\omega}{B^{(0)}}\right) S^{(3)}. \tag{3.58}$$

Therefore one photon has a magnetic dipole moment $m^{(3)}$ proportional to the spin angular momentum $S^{(3)}$. In the received view, however, the photon is thought of as being *uncharged* and the magnetic dipole moment in consequence is thought to vanish.

In the new theory presented in these volumes and elsewhere [4—12] there occur relations between $B^{(0)}$ and the magnitude $A^{(0)}$ of the potential vector which recognize the existence of e in the vacuum. One of these relations is that underlined experimentally in the previous section ,

$$B^{(0)} = \frac{e}{\hbar} A^{(0)2}, \tag{3.59}$$

an equation which can be combined with [6]

$$B^{(0)} = \frac{\omega}{c} A^{(0)}, \qquad \hbar\omega = ecA^{(0)} = \frac{1}{\mu_0} B^{(0)2} \bar{V}, \tag{3.60}$$

to give three relations between $B^{(0)}$ and $A^{(0)}$. Using these, the volume occupied by the photon [17,32] can be deduced in the form,

$$\bar{V} = \frac{\mu_0 c \hbar^2}{eA^{(0)3}}, \tag{3.61}$$

and is inversely proportional to the cube of $A^{(0)}$. Using these relations gives a link between $m^{(3)}$ and $S^{(3)}$ in terms of the gyromagnetic ratio, as we set out to show,

$$m^{(3)} = \left(\frac{ec}{\hbar\kappa}\right) S^{(3)}, \tag{3.62}$$

We arrive at the following, self-consistent expressions for g_p,

$$g_p = \frac{ec^2}{\hbar\omega} = \frac{c}{A^{(0)}} = \frac{\omega}{B^{(0)}}. \tag{3.63}$$

It is notable that $\hbar\kappa/c$ has the units of mass, although it is not the photon mass [6], but simply the photon momentum divided by its velocity c. Equation (3.62) is self-consistent physically and mathematically, and shows that for one photon of energy $\hbar\omega$, g_p is a constant. Finally using [6],

$$B^{(0)} = \frac{e\mu_0 c}{\bar{V}\kappa}, \tag{3.64}$$

we obtain the result first proposed in 1992 [4—6] in the first paper on the $B^{(3)}$ field,

$$B^{(3)} = B^{(0)} \frac{S^{(3)}}{\hbar} \tag{3.65}$$

The one photon Beth effect therefore demonstrates the existence of $B^{(3)}$ through that of $S^{(3)}$ and $m^{(3)}$. In the received view [31] the Beth effect demonstrates only the existence of $S^{(3)}$, and the link between $S^{(3)}$ and $B^{(3)}$ is not established. Similarly, in the received view, the link between $B^{(1)} \times B^{(2)}$ and $iB^{(0)}B^{(3)*}$ is not made. At radio frequencies, the inverse Faraday effect leads to an induction profile proportional to the square root of intensity, I, and in the new theory this effect is due to $B^{(3)}$ acting at first order. The received view on the other hand regards it as an effect in the conjugate product $B^{(1)} \times B^{(2)}$ divided by $iB^{(0)}$ in the basis ((1), (2), (3)). This quantity, however, has the attributes of a magnetic field. At this point the received view becomes self-inconsistent.

As already seen in Sec. 3.2, a novel displacement current occurs in the new theory,

$$j^{(3)} = \frac{\kappa}{\mu_0} B^{(3)}, \tag{3.66}$$

and this is the space part of the four-current $j^{\mu} = (j^0, j)$ which defines the relativistic helicity. The four-current j^{μ} and Pauli-Lubanski four-vector B^{μ} are simply proportional through the scalar κ / μ_0. The four-current is also proportional to the relativistic, axial, spin four-vector S^{μ} through the equation

$$j^{\mu} = \frac{ec}{\bar{V}} \frac{S^{\mu}}{\hbar}. \tag{3.67}$$

The fundamental spin angular momentum four-vector of the field is therefore j^{μ} whose scalar part is the elementary charge e multiplied by the factor $c/\bar{V}$. This is an example of reducing quantities to multiples of e, which must clearly be present in the field. It follows that the quadratics $j_{\mu} j^{\mu}$ and $S_{\mu} S^{\mu}$ are Casimir spin invariants of the Poincaré group, and that the theory is relativistically covariant.

These results are also examples of the principle that e is the $\hat{C}$ negative scalar that links space-time to the electromagnetic field. In other words $\hbar / e$ multiplied by $R_{\mu\nu}$ gives $G_{\mu\nu}$. In this view the quantum of charge e is simply the universal constant that makes curved space-time $\hat{C}$ negative, thus producing the electromagnetic field. Point charge, on the other hand, is to be found only when there is point mass, as in an electron or positron, and point charge to the electromagnetic field is as point mass to the gravitational field. In this way, both the electromagnetic and gravitational fields become expressions of curved space-time, the former is curved antisymmetrically, the latter is curved symmetrically through the Einstein tensor. It is concluded that the concept of the $B^{(3)}$ field can be used quite readily to unify the gravitational and electromagnetic fields.

3.3.3 Tangential Velocity Versus Signal Velocity

From elementary, non-relativistic, classical considerations [31] the magnitude of the magnetic dipole moment generated by a charge moving with tangential velocity v in a circular orbit of radius r is

$$m^{(0)} = \frac{e}{2} r \mathrm{v}. \tag{3.68}$$

The relativistic expression for the magnetic dipole moment of a photon is on the other hand equation (3.42), in which the radius of the photon is $1/\kappa$. The tangential velocity given by a comparison of Eq. (3.42) with (3.68) is greater than c,

$$\mathrm{v} = \frac{c}{2\pi\alpha}. \tag{3.69}$$

This is the result of comparing a non-relativistic classical expression (3.68) with a quantum relativistic expression, Eq. (3.42), which shows that if the radius of the photon is $1/\kappa$, the effective tangential velocity is c divided by the factor $2\pi\alpha$, where α is the fine structure constant. In order to reduce the tangential velocity to c, then the photon radius must be increased to $(1/\kappa)/(2\pi\alpha)$. In deriving Eq. (3.42) it was assumed that $1/\kappa$ is the ratio of volume to area, and so if the area of the photon is fixed at π/κ^2, its volume must be greater than $4\pi(1/\kappa)^3/3$, the volume of a sphere. The volume of the photon was found by Hunter and Wadlinger [16,32] to be an ellipsoid with the maximum cross section and this ellipsoidal volume must be greater than the spherical volume, in our theory, by a factor $2\pi\alpha$. In the correctly relativistic theory of the photon's magnetic dipole moment, we arrive at qualitatively the same result as Hunter and Wadlinger [16,32].

We also arrive at the key result that a relativistically invariant photon area can be derived only from the conjugate product $A^{(1)} \times A^{(2)}$ through Eq. (3.46), because the light-like invariant $\kappa_{\mu}\kappa^{\mu}$ is zero by definition, giving an *infinite* area for the photon. This is the result of using plane wave analysis, in which the X and Y components are unattenuated, i.e., extend, theoretically, to infinity. The product $A^{(1)} \times A^{(2)}$ is relativistically invariant in the proper gauge group, which is O(3) in space; and the area generated by this cross product is finite because the dependence on the unattenuated $\boldsymbol{i}$ and $\boldsymbol{j}$ components is removed. (The cross product is a function only of $\boldsymbol{k}$ and of the finite amplitude $A^{(0)2}$.) The experimental results now available [17,32] counter-indicate a photon with an infinite area, because such an object would not be 100% attenuated at a critical aperture diameter — the intensity (watts per square meter) of radiation passing through the aperture would be a linear function of the area for all diameters, assuming the absence of artifacts.

There exists therefore a covariant magnetic dipole moment per photon of magnitude,

$$m^{(0)} = \frac{1}{4\pi\alpha}\frac{c}{\kappa}e = 10.911\frac{c}{\kappa}e, \tag{3.70}$$

that is the photon radius itself within a proportionality factor 10.911 ce, which is a fundamental constant. This result uses the definition of the fine structure constant. If this definition is expressed as

$$\hbar\kappa = e\left(\frac{e\mu_0\kappa c}{4\pi\alpha}\right) := eA^{(0)}, \tag{3.71}$$

it becomes the minimal prescription for the free photon,

$$eA^{(0)} = \hbar\kappa, \tag{3.72}$$

as used throughout these volumes, provided that the amplitude of the vector potential

$$A^{(0)} = \left(\frac{e\mu_0 c}{4\pi\alpha}\right)\kappa = \left(\frac{e\mu_0}{4\pi\alpha}\right)\omega. \tag{3.73}$$

The amplitude $A^{(0)}$ is therefore the angular frequency $\omega = \kappa c$ within a universal proportionality constant $e\mu_0/4\pi\alpha$. It is concluded that the key relation (3.72) is the definition of the fine structure constant, an empirical observable of high resolution spectroscopy [31]. This conclusion demonstrates vividly the role of e in vacuum electrodynamics, because $A^{(0)}$ multiplied by e is the quantized photon momentum in vacuo, $\hbar\kappa$. This equation defines and is inter alia defined by, the fine structure constant of quantum electrodynamics.

3.4 The Fundamental Quanta in Electromagnetic Radiation

Starting from the definition of the fine structure constant of quantum electrodynamics, it can be shown the basic quantities $\Phi^{(0)}$, $A^{(0)}$, and $B^{(0)}$ are interrelated,

$$A^{(0)} = \frac{\omega}{c}\Phi^{(0)}, \qquad B^{(0)} = \left(\frac{\omega}{c}\right)^2\Phi^{(0)}, \tag{3.74}$$

i.e., there is a *non-linear* relation between $A^{(0)}$ and $B^{(0)}$ in the vacuum. Such a result is obtained simply by multiplying both sides in the definition (3.36) by the wavenumber; once to obtain $A^{(0)}$; twice to obtain $B^{(0)}$. Not only does this simplest of procedures produce the vacuum minimal prescription, but also the non-linear equation (3.59). These two fundamental equations occur throughout these volumes and have been derived independently in several different ways. Furthermore, the relation between the flux density, $B^{(0)}$, of one photon, and the flux quantum per photon, the fluxon, $\hbar/e$, in Eq. (3.74), immediately gives a self-consistent definition of photon area, $1/\kappa^2$ [16,32]. These basic results emerge from the fine structure constant, which is a fundamental constant which has been measured with great precision in experimental tests for quantum electrodynamics. The area $1/\kappa^2$ is, more precisely, the mean area of the photon, whose maximum area according to the Hunter-Wadlinger theory is π/κ^2, the maximum cross section of the ellipsoidal volume occupied by the photon. To estimate the mean volume of the ellipsoid it is possible to use the definition of the quantum of energy, $\hbar\omega$, as an integral over the quantization volume [4—12],

$$\hbar\omega = \frac{1}{\mu_0}\int B^{(0)2}dV := \frac{1}{\mu_0}B^{(0)2}\bar{V}, \tag{3.75}$$

where $\bar{V}$ is a mean quantization volume, a first approximation to the integral. Use of Eq. (3.74) gives the ratio of mean volume to mean area,

$$\frac{\bar{V}}{\bar{Ar}} = \frac{4\pi\alpha}{\kappa} = \frac{0.0917}{\kappa}. \tag{3.76}$$

The corresponding ratio for a sphere of radius $1/\kappa$ would be $4/(3\kappa)$. Therefore Eq. (3.76) suggests that the volume to area ratio is more like that for a flattened ellipsoid of mean volume,

$$\bar{V} = 4\pi\alpha\left(\frac{1}{\kappa}\right)^3, \tag{3.77}$$

a result which depends, however, on the first approximation of the integral in Eq. (3.32). Equations (3.76) shows that of the mean area of the photon can be defined, then the mean volume can be defined with the use of Eq. (3.74). From this analysis emerges the idea of a photon as a well defined geometrical entity that can be described merely by inspection of the fine structure constant — an empirical observable.

Of particular significance in this development is the non-linear relation between $A^{(0)}$ and $B^{(0)}$ which defines the elementary fluxon $\hbar/e$; the ratio of action and charge quanta,

$$B^{(0)} = \frac{e}{\hbar}A^{(0)2} , \tag{3.78}$$

a relation which is consistent with, and derivable from, the fine structure constant. It is therefore *empirically supported*. Furthermore, both $B^{(0)}$ and $A^{(0)2}$ in this equation are relativistically covariant; and the structure of space-time is properly represented by it. For example, the area of the photon is defined covariantly and self-consistently. Equation (3.78) leads to the correct definition of $\boldsymbol{B}^{(3)}$ through the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, which also emerges self-consistently from the Dirac equation [6] describing the interaction of a fermion with the classical electromagnetic field. The conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ also allows a relativistically covariant definition of intensity (using $B^{(0)} = \omega A^{(0)}/c$),

$$I := \frac{c}{\mu_0}B^{(0)2} = \frac{\omega^2}{\mu_0 c}|\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}| , \tag{3.79}$$

and is gauge invariant in O(3). This result in turn is consistent with the fact that intensity is always defined with respect to an area, the minimum of which is the area of one photon.

In the received view [31] on the other hand, the gauge group is U(1), and this automatically prohibits the existence of Eq. (3.79), or any non-Abelian construct. Equation (3.79), however, is a direct result of the *empirical* fine structure constant, and so counter-indicates the use of U(1). Replacement of U(1) by O(3) defines $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ self-consistently [27] and leads to Eq. (3.79) through gauge theory [5,6]. The conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is relativistically covariant because $\boldsymbol{B}^{(3)}$, (to which it is proportional) is invariant under Lorentz transformation. The conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is gauge invariant in O(3) [27] using fundamental gauge theory. In U(1) gauge theory two components of the four-vector A_μ are discarded as unphysical, leaving only the transverse components. Such a procedure is used in the Coulomb gauge or transverse gauge [31] and is automatically incompatible with the existence of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. The U(1) gauge does not allow a field such as $\boldsymbol{B}^{(3)}$ to exist. By deriving Eq. (3.79) directly from the fine structure constant, however, it has been shown that the U(1) gauge is incompatible with the precise empirical evidence provided by quantum electrodynamics. We conclude that the existence of the fundamental spin field $\boldsymbol{B}^{(3)}$ is directly indicated by the most precise theory in physics.

Define now the quantum as the indivisible, intrinsic and irremovable unit of a physical and measurable quantity occurring both in radiation and in matter. Alternatively, the quantum can be thought of as a particle mediating a specific fundamental interaction. There are at least ten basic quanta in the electromagnetic field, five $\hat{C}$ positive and five $\hat{C}$ negative; and these are based on the existence of $\hbar$, which is known empirically to about

five decimal places [31]. The original definition of *the photon* is the bare minimum needed for description, being the quantum of energy.

$\hat{C}$ *Positive Quanta*:

1) The quantum of action or angular momentum, $\hbar$.
2) The quantum of energy, the original *photon*, $\hbar\omega$.
3) The quantum of power, $\hbar\omega^2$.
4) The quantum of energy per photon area, $\hbar\omega^3/c^2$.
5) The quantum of intensity, or power per photon area, $\hbar\omega^4/c^2$.

$\hat{C}$ *Negative Quanta*:

6) The quantum of magnetic flux (weber), the fluxon $\Phi^{(0)} = \hbar/e$.
7) The quantum of flux per photon radius (weber m^{-1}), $A^{(0)} = \kappa\Phi^{(0)}$.
8) The quantum of flux per photon area, or flux density (tesla), $B^{(0)} = \kappa^2\Phi^{(0)}$.
9) The quantum of flux per photon volume (weber m^{-3}), $\kappa^3\Phi^{(0)}$.
10) The quantum of flux density per photon area (tesla m^{-2}), $\kappa^4\Phi^{(0)}$.

The Planck hypothesis applies only to the quantum of energy, and the procedure adopted by Planck is a statistical calculation of the mean energy of an oscillator of frequency ν on the assumption that its energy levels are discrete, $0,\ \hbar\nu,\ \ldots,\ n\hbar\nu$, where h is the Planck constant. The result is that the mean energy per oscillator is

$$\overline{h\nu} = h\nu\frac{e^{-h\nu/kT}}{1 - e^{-h\nu/kT}} , \tag{3.80}$$

and depends on the frequency. Here k is Boltzmann's constant and T the temperature. The classical theory produces the result [31] that the mean energy per oscillator is kT, and this is not frequency dependent. In Planck's theory, oscillators can be excited only by absorbing one quantum of energy; whereas classical oscillators can be excited with an infinitesimal amount of energy. Clearly, such considerations, taken from the first quantum theory, can be adopted for every type of quantum listed above, provided that the parameters of the particular problem being considered are well defined.

In order to complete the construction of the Planck law for radiation the density of states is required, resulting in a premultiplication of the average oscillator energy in Eq. (3.80) by a factor proportional to the square of frequency. In the following, it is shown that the Planck law for one oscillator can be derived directly from the quantum of energy density, $En/\bar{A}r = \hbar\omega^3/c^2$ by averaging over the energy states 0, $h\nu$, ..., $nh\nu$ of the oscillator. The quantum of energy density is first re-written as the quantum of energy, $h\nu$, premultiplied by a factor $4\pi^2\nu^2/c^2$. Secondly, the mean oscillator energy at frequency ν is worked out using Planck's hypothesis, to give

$$\frac{\bar{En}}{\bar{Ar}} = 4\pi^2\frac{\nu^2}{c^2}h\nu\left(\frac{e^{-h\nu/kT}}{1-e^{-h\nu/kT}}\right). \tag{3.81}$$

The mean energy density per oscillator at frequency ν is therefore

$$\rho_0(\nu) := \frac{\bar{En}}{\bar{Ar}}, \tag{3.82}$$

and this has the form of the Planck law [31], or Rayleigh-Jeans law for one oscillator at low frequencies,

$$\rho_0(\nu\to 0) \to \frac{4\pi^2}{c^2}kT\nu^2 . \tag{3.83}$$

We have calculated the mean electromagnetic energy per area of one oscillator, applying Maxwell-Boltzmann averaging to the quantum of energy. It has been assumed inherently in the above that the area and volume occupied by one oscillator are the area and volume occupied by the quantum of intensity. These fundamental assumptions lead to the correct Planck law for one oscillator, except for a factor introduced by Rayleigh and corrected by Jeans in the calculation of the density of states in an octant of a sphere. Essentially [31] the number of ways of choosing the integers to fill this octant is equal to its volume. This calculation gives

$$\rho(\nu)(\text{Rayleigh-Jeans}) = \frac{1}{\pi}\rho_0 . \tag{3.84}$$

The new method of calculating the Planck (or Rayleigh-Jeans) law given here is much simpler, and gives the mean energy density per oscillator. The only volume considered is the volume occupied by the oscillator itself and there is only one frequency being considered. If there are many oscillators covering all frequencies then the integral with respect to frequency over equation (3.81) gives a black body law — the total energy density of black body radiation. Finally, this result becomes exactly the conventional one [31] if we reinstate the factor $1/\pi$, which, however, comes from Rayleigh's consideration of the density of states in a particular geometry.

We have therefore derived the black body law from the quantum of energy density by averaging over the quantum of energy. The important point is that this very simple calculation can be repeated for all the other quanta listed above with the obvious

exception of $\hbar$, which is not frequency dependent, and is a topological and fundamental angular momentum intrinsic to space-time.

The calculation can be repeated from a different starting point,

$$I := \frac{c}{\mu_0}B^{(0)2}, \tag{3.85}$$

by using the $\bar{Ar} = 1/\kappa^2$ rule verified empirically [16,32]. Equation (3.85) then becomes

$$I = \frac{c}{\mu_0}\left(\frac{\kappa\hbar}{e}\right)^2\cdot\frac{1}{\bar{Ar}}, \tag{3.86}$$

which reduces to

$$I = \frac{1}{4\pi\alpha}\frac{\hbar\omega^2}{\bar{Ar}}, \tag{3.87}$$

using the minimal prescription for the free photon, Eq. (3.72). This expression for the intensity per photon of energy, I, differs from our previously defined quantum of intensity by a factor $1/(4\pi\alpha)$, which has been introduced through Eqs. (3.85) and (3.72). Therefore the precise way in which intensity per photon, or oscillator, is defined is not standardized, but each expression has the units of power per photon area.

The foregoing development reveals that *the photon* is a term that has several connotations not usually given in the received literature. The usual assertion that the photon is its own anti-particle applies only to the $\hat{C}$ positive quanta. Each of the $\hat{C}$ negative quanta changes sign on application of $\hat{C}(e) = -e$ by definition. Taking the existence of such quanta into account, the photon is distinct from the anti-photon, and is not its own anti-particle. The existence of e in electromagnetic radiation has been discussed already in Chap. 1 of this volume. In the end analysis, all $\hat{C}$ negative quantities in the electromagnetic field, depend on e, including $A^{(0)}$ and $B^{(0)}$. The fundamental, frequency independent, $\hat{C}$ positive quantum is that of angular momentum, $\hbar$, and the $\hat{C}$ negative counterpart is the fluxon, which is angular momentum $\hbar$ divided by the charge quantum e. Significantly, both $\hbar$ and $\hbar/e$ are independent of any consideration of geometry; whereas all the other quanta are ω or κ dependent. If the area of the photon is $1/\kappa^2$, then all quanta apart from $\hbar$ and $\hbar/e$ depend on radius, area or volume. Accordingly, $\hbar$ and $\hbar/e$ are the only relativistically and gauge invariant, universal, scaling factors.

3.5 Atomic Absorption

In this section it is shown that ordinary stimulated atomic absorption and emission can be understood in terms of the fundamental spin field $B^{(3)}$. Simple expressions are given for the Einstein coefficients and for the ordinary dipole interaction terms. Selection rules are discussed in linear and circular polarizations, and it is shown that absorption and emission theory can be understood without reference to the usual transverse fields, showing that $B^{(3)}$ is the fundamental field of vacuum electromagnetism, and is an expression of its roto-translational nature. Recent research [33] has indicated that the existence of the $B^{(3)}$ field was inferred in the thirties by leading physicists of that era, names such as those of Dirac, Fock, Podolsky, Oppenheimer, Majorana and Wigner. The acausal solution of Maxwell's field equations has been discovered independently several times [33]. From the definition (3.65), and from the Dirac equation [6], the mode of interaction of the spin field with a fermion is understood to be a spin-spin interaction which is theoretically responsible for radiation induced fermion resonance as described in Chap. 1 and 2 of Vol. 3 [6]. Of particular interest is that the existence of $B^{(3)}$ implies that of a hidden quantum number, identified as the F number in Ref. 6. The photon is not its own anti-particle as discussed above in this chapter. In general, the U(1) sector of electromagnetism is enlarged to the Poincaré group by the existence of $B^{(3)}$, and this has theoretical ramifications throughout field theory. The existence of longitudinal solutions of the electromagnetic field equations implies the possible existence of photon mass, and allows a simple mechanism to be developed for field unification as discussed in Chaps. 1 and 2 of this volume.

The received view on the other hand disallows the existence of longitudinal solutions in vacuo of the electromagnetic field equations, which are taken to be Maxwell's equations in vacuo. This view is based, somewhat superficially, on Wigner's paper of 1939 [34]. The latter develops the spin and mass invariants of particles within the Poincaré group. A massless particle in this theory propagates at the speed of light in vacuo, and has two and only two states of polarization in four dimensional space-time. These are taken to be the transverse states of polarization, leading to severe problems in quantizing the potential four-vector A_μ [35]. This view is counter-indicated by the existence of the B cyclics as we have frequently argued in these volumes and elsewhere. The terms in the B cyclics depend only on the empirically established existence of conjugate products in radiation and on the geometry of space-time. The received view asserts ultimately that the right hand sides in the cyclics are not equal to the left hand sides, a reduction to absurdity equivalent to asserting that the magnetic field $B^{(3)}$ is not a magnetic field despite having the observable properties of such a field. The B cyclics underpin a $\hat{C}\hat{P}\hat{T}$ conserving field theory and are inter alia Lorentz covariant. This means again that $B^{(3)}$ is a field component. The alternative, the assertion that $B^{(3)}$ is not a field component, violates the $\hat{C}\hat{P}\hat{T}$ theorem, and contradicts the fact that the $B^{(3)}$ cyclics are automatically Lorentz covariant, being angular momentum relations. The B cyclics expose several difficulties of the received view, because the basic field equations become non-linear within a non-Abelian group structure; transverse components imply longitudinal components and vice versa; A_μ must be fully covariant; the existence of photon mass becomes possible; the Coulomb gauge must be abandoned; and the Maxwell equations become linear approximations to more general field equations as described in Chaps. 1 and 2 of this volume. The theory of electromagnetism is changed fundamentally simply by writing $B^{(1)} \times B^{(2)}$ as $iB^{(0)}B^{(3)*}$ in the vacuum.

Assertions [36,37] that the B cyclics violate $\hat{C}$ and $\hat{C}\hat{P}\hat{T}$ are counter-indicated by the fact that Eqs. (3.17) are field equations of special relativity, being angular momentum commutator relations, i.e., relations between infinitesimal rotation generators of the O(3) sub-group. This is alone sufficient to show that they are $\hat{C}\hat{P}\hat{T}$ conserving equations as discussed already and removes criticisms by Barron [36], Buckingham [37] and Buckingham and Parlett [38]. Specific replies to these criticisms are given elsewhere [39—42]. Early criticisms by Lakhtakia [43,44] and by Grimes [45] are insubstantial [39—42], being subjective attempts to assert that $B^{(3)}$ is either zero or *not fundamental*. However, the B cyclics are expressions of space-time geometry, and if $B^{(1)} = B^{(2)*}$ is fundamental so is $B^{(3)}$. Any other conclusion is incompatible with the fundamentals of space-time itself. The experimental paper by Rikken [46], reporting an unsuccessful attempt to observe laser induced Faraday rotation in liquid benzene, is answered elsewhere [40]. This experiment cannot have shown that $B^{(3)}$ is zero because the way $B^{(3)}$ interacts with matter is given by Eq. (3.45) as used in the Dirac equation. The latter correctly establishes the experimental conditions under which Faraday rotation may be observed due to intense lasers. The work by Rikken has served a good purpose however in establishing and emphasizing the Dirac equation as the correct description of the way in which $B^{(3)}$ interacts with one fermion, and magnetizes fermionic matter. In liquid benzene, the sample used by Rikken, there are no free electrons, and it is difficult to see how the experiment can test the rigorous one electron theory [6]. The most that can be said is that the phenomenological theory of $B^{(3)}$, the first to be developed [4] is applicable only in the high field limit, and so if an experiment is not carried out under the appropriate conditions, there will be a negative result. If the B cyclics are accepted as equations of field theory, there is no experiment that will establish the non-existence of $B^{(3)}$ without violating the $\hat{C}\hat{P}\hat{T}$ theorem. If the existence is accepted of the fluxon, $\hbar/e$, in vacuo the B cyclic relations will determine the fundamental frame itself within the O(3) group of space. Abandoning $B^{(3)}$ means abandoning the Poincaré group in view of the fact that $B^{(3)}$ is indicated empirically by the inverse Faraday effect [47] and related phenomena of inverse induction [48]. There has recently appeared a further debate [49] between Evans and van Enk, but this does not substantially add to the above conclusions. It appears that the main point of criticism by van Enk is that the $B^{(3)}$ field violates linear superposition in quantum mechanics. However, the complete field vector,

$$B = B^{(1)} + B^{(2)} + B^{(3)}, \tag{3.88}$$

is always a linear superposition in three space dimensions of three space components in any correct O(3) basis such as ((1), (2), (3)) or (X, Y, Z).

There are to date several ways in which the longitudinal nature of electromagnetism in vacuo can be demonstrated theoretically. For example, Ahluwalia and Ernst [33] have shown the presence of acausal longitudinal vacuum fields which have infinite range. Dvoeglazov [51] has pointed out that the acausal solution can be interpreted as $B^{(3)}$ in vacuo and has developed the view that $B^{(3)}$ is a Pauli Lubanski axial four-vector. These theories recognize $B^{(3)}$ as the simplest representation of electromagnetic spin in the vacuum. Chubykalo and Smirnov-Rueda [1] have demonstrated that Maxwell's linear approximation to the field equations of electromagnetism in vacuo produces longitudinal and transverse components in general. Their paper suggests ways in which the anomalies of classical field theory, such as infinite electron self energy, can be removed entirely using simultaneous concepts of action at a distance and Faraday Maxwell propagation at a finite light velocity, the signal velocity. Longitudinal solutions imply action at a distance, transverse solutions describe the Faraday Maxwell propagation of fields in vacuo at a finite signal velocity c. Prior to the appearance of this important paper [1], these concepts had appeared to be entirely incompatible. Work is in progress to define the role of $B^{(3)}$ in this theory [33]. Dvoeglazov, in a recent series of papers [51], has discussed non-Maxwellian theories of vacuum electromagnetism in terms of Majorana and Joos-Weinberg equations [52] for any spin, and has derived the form of the B cyclics within these representations of field theory. The overall trend is towards the conclusion that Maxwell's equations may not be the ultimate description of electromagnetism in vacuo and matter. Solutions to Maxwell's linear equations are Abelian in the received view, whereas the fundamental B Cyclic theorem [4—6] is non-Abelian by definition. If we accept that the structure of space is three dimensional and non-Abelian, something must be done to generalize the structure of Maxwell's equations themselves. Suggestions as to how to go about this in general relativity are given in Chaps. 1 and 2. Otherwise there will develop a philosophical difference within field theory itself. Three dimensional space remains so in any theory of electromagnetism, classical or quantized, and so the B cyclics remain valid for N photons in an ensemble, for one photon, or for the classical field theory. The B Cyclic theorem is therefore topological in its generality, being valid in all field theory. Mészáros [53], and Mészáros and Molnár [54], have recently described paradoxes in cosmology and statistical electrodynamics in which the existence of a third spatial degree of freedom is indicated in general thermodynamical terms. This type of theory provides a very general indication of the need for a third degree of freedom in the radiation laws. Mészáros *et al.* [55] have also developed a theory of excess, measurable, pressure using the $B^{(3)}$ concept. Okhlovsky and Recami [3] have developed the theory of action at a distance and tachyonic properties of electromagnetism in terms of longitudinal fields [50]. It is significant that the work of Chubykalo and Smirnov-Rueda [1] indicates that action at a distance is mediated in electromagnetic field theory by longitudinal solutions, something that was indicated by Dirac in the early thirties. These developments had been anticipated in the careful work of Hunter and Wadlinger [16], who have shown that relativistic principles lead to the photon as wavicle. The latter occupies a well-defined three dimensional (ellipsoidal) volume in the vacuum, indicating the presence of a third axis, the major axis of the ellipsoid, and this conclusion is automatically incompatible with U(1). Experimental evidence for the photon radius given by this theory has been obtained as described already in this chapter.

In this section, the Einstein coefficients are expressed in terms of $B^{(3)}$, which is thereby the only field term in the fundamental radiation-matter equations. The ordinary electric and magnetic dipole terms are worked out in terms of $B^{(3)}$ and selection rules are discussed in linear and circular polarization for a boson with **eigenvalues -1, 0, +1** as indicated by longitudinal components in vacuo.

Equations (3.17) conserve $\hat{C}$ and $\hat{C}\hat{P}\hat{T}$, and are locally Poincaré invariant, so $B^{(3)}$ is in the standard model a magnetic flux density of magnitude $B^{(0)}$. In this section it is shown that the ordinary Einstein coefficients in atomic absorption can be expressed in terms of $B^{(3)}$ without direct reference to transverse components. In radiation-matter equilibrium processes such as atomic absorption and emission therefore there is no sign of the existence of transverse modes in the Einstein coefficients. Transverse modes are indicated by a quite independent consideration of the Maxwell equations, and atomic absorption processes are described through the fundamental *spin field* $B^{(3)}$. This is shown straightforwardly as follows.

From the definition (3.59) we obtain

$$B^{(0)} = |B^{(3)}| = \frac{e}{\hbar\omega^2}E^{(0)2}, \tag{3.89}$$

indicating that the modulus of $B^{(3)}$ is proportional to $E^{(0)2}$. The energy per volume $\bar{V}$ for one photon is [4—6]

$$\frac{En}{\bar{V}} = \frac{3E^{(0)2}}{\mu_0 c^2}, \tag{3.90}$$

where μ_0 is the permeability in vacuo and where $B \cdot B^*$ from Eq. (3.88) is simply $3B^{(0)2}$. Therefore from Eqs. (3.89) and (3.90) the energy per unit volume for one photon is proportional to $B^{(0)}$, the magnitude of $B^{(3)}$,

$$\frac{En}{\bar{V}} = \left(\frac{3\hbar\omega^2}{e\mu_0 c^2}\right)B^{(0)}. \tag{3.91}$$

It is now possible to incorporate this result in a textbook theory of atomic absorption such as that given by Atkins [31], where the energy density of states at frequency ν is given by

$$\rho(\nu) = \frac{En}{\bar{V}} \hat{\rho}(\nu_{fi}), \tag{3.92}$$

which is the product of the energy of an electromagnetic mode with the number of modes per unit volume in the frequency range ν to $\nu + d\nu$. Therefore,

$$\rho(\nu) = \frac{3\hbar\omega^2}{e\mu_0 c^2} |B^{(3)}| \hat{\rho}(\nu_{fi}), \tag{3.93}$$

in which there is no indication that the mode must be made up of transverse components; the energy density of states at frequency ν is proportional directly to the modulus of $B^{(3)}$. The transition rate in atomic absorption [31] is therefore also directly proportional to $B^{(3)}$ in *any* absorption from i to f inside an atom or molecule.

The total rate of absorption is $B_{if}\rho$ and the total rate of emission is $A_{fi} + B_{fi}\rho$ where A_{fi} and $B_{fi} = B_{if}$ are the Einstein coefficients. Therefore we reach the simple result that the rate of absorption of radiation by one atom from energy level i to energy level f within the atom is

$$W_{f-i} = B'_{if} |B^{(3)}|, \tag{3.94}$$

where

$$B'_{if} = \frac{3\hbar\omega^2}{e\mu_0 c^2} \hat{\rho}(\nu_{fi}) B_{if}, \tag{3.95}$$

and where $\hat{\rho}(\nu_{fi})$ is the frequency density of states at ν_{fi}. The rate of stimulated emission from level f to level i within the atom is conversely

$$W_{f-i} = B'_{fi} |B^{(3)}|, \tag{3.96}$$

and is also directly proportional to the magnitude of $B^{(3)}$.

Therefore the process of absorption and emission encompasses energy changes accompanied by spin changes, i.e., transfer of angular momentum. The simplest way to describe this process is through the $B^{(3)}$ field in vacuo, and not through the transverse fields. In the following, the ordinary dipole terms of radiation-matter theory are developed using $B^{(3)}$ rather than the usual phase dependent transverse components.

The interaction Hamiltonian between radiation and one electron contains the standard term [31],

$$H_1 = \frac{e}{2m} A^{(1)} \cdot p, \tag{3.97}$$

where $A^{(1)}$ is a transverse plane wave representation of the vector potential of the vacuum electromagnetic field, p the electron's real linear momentum, and e/m its charge to mass ratio. The factor two in the denominator of Eq. (3.97) comes from the Dirac equation in a non-relativistic limit [6]. Equation (3.97) can be worked out in terms of $B^{(3)}$, both for circular and linear polarization, showing that ordinary dipole absorption can be expressed in terms of the longitudinal field of vacuum electromagnetic radiation. Using the vacuum relations between the vector potential and magnetic field,

$$A^{(0)} = \frac{B^{(0)}}{\kappa} = \frac{c}{\omega} B^{(0)}, \tag{3.98}$$

where the wavenumber is expressed as $\kappa = \omega/c$, and using the vacuum minimal prescription (or quantum hypothesis), Eq. (3.72), Eq. (3.45), through which $B^{(3)}$ is defined in vacuo, becomes Eq. (3.17), which is,

$$B^{(3)*} = -\frac{i\omega}{cA^{(0)}} A^{(1)} \times A^{(2)}. \tag{3.99}$$

Using the vector identity (3.19) it is deduced that

$$A^{(1)} = i\frac{c}{\omega} \cdot \frac{1}{A^{(0)}} A^{(1)} \times B^{(3)*}, \tag{3.100}$$

an equation which expresses $A^{(1)}$ in terms of a cross product of itself with $B^{(3)*}$. Expressing the transverse plane wave $A^{(1)}$ in circular polarization as [6]

$$A^{(1)} = \frac{A^{(0)}}{\sqrt{2}} (ii + j) e^{i\phi}, \tag{3.101}$$

and using the vector identity (3.20) the dipole term (3.97) becomes

$$H_1 = i^2 \frac{c}{\omega} e^{(1)} \cdot (B^{(3)*} \times p) e^{i\phi}, \tag{3.102}$$

in which ϕ is the electromagnetic phase in vacuo. Here ω is the angular frequency at instant t, κ the wavenumber at point r in space. Defining the magnetic dipole moment,

$$m = \frac{1}{\sqrt{2}}\frac{e}{2m}\frac{c}{\omega}p, \tag{3.103}$$

and working out the real part of Eq. (3.102) gives

$$H_1 = (i\cos\phi + j\sin\phi)\cdot m \times B^{(3)*}. \tag{3.104}$$

Using again the vector identity (3.20) gives our final result,

$$H_1 = ((i\cos\phi + j\sin\phi)\times m)\cdot B^{(3)*} := m(\phi)\cdot B^{(3)*}, \tag{3.105}$$

in which the dipole moment is a function of the phase and the electromagnetic field is phase free and longitudinal. This is the opposite to, but the precise equivalent of, the usual equation for the interaction of a phase dependent, transverse magnetic plane wave with a static magnetic dipole moment. Therefore we have shown that

$$H_1 = m(\phi)\cdot B^{(3)*} = \frac{e}{2m}A^{(1)}(\phi)\cdot p, \tag{3.106}$$

in circular polarization. The importance of this result is that everything that is usually expressed in terms of the transverse component $B^{(1)} = B^{(2)*} = \nabla \times A^{(1)}$ can be expressed equivalently in terms of the longitudinal component $B^{(3)}$. This result is in turn established by the empirical evidence [4] for the existence of $B^{(1)} \times B^{(2)}$, and by geometry, which shows that this conjugate product is equal to $iB^{(0)}B^{(3)*}$, where $B^{(3)}$ must be longitudinal. The components $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ evidently do not exist independently of each other in the vacuum, a result which is expressed in the B cyclic theorem (3.17), which is fundamental in field theory, but which is novel and changes the received view of electromagnetism in vacuo. If $B^{(3)}$ existed, hypothetically, independently of $B^{(1)} = B^{(2)*}$ there would be no reason for the appearance of the phase in Eq. (3.105). The phase is derived, however, from the cyclic structure (3.17) and the plane wave $B^{(1)} = B^{(2)*}$ is one possible solution which fits in with this structure. More generally these solutions are spherical harmonics as in spin angular momentum theory. Therefore the structure of three dimensional space itself has been shown to dictate the nature of electromagnetic fields in vacuo, through the fundamental equivalence (3.72). The older approach due to Maxwell "decouples" the field from space-time and asserts that the field is transverse as the result of equations between field components — the Maxwell equations — in which the key term is the displacement current in vacuo [4]. This is inconsistent with the O(3) rotation group symmetry of Eqs. (3.17). If one accepts three dimensional space, one is led to theorem (2) for fields. This theorem is for magnetic fields because it is a theorem for axial vectors — the cross product in space of two axial vectors is a third, axial, vector. The cross product of two polar vectors (electric fields) is also an axial vector, and for this reason there exists no precise analogy of Eq. (3.17) for three electric fields unless the longitudinal component is pure imaginary as developed extensively elsewhere [4—6]. This is why the fundamental dipole term (3.105) has been expressed through the pure real and physical $B^{(3)}$.

If linear polarization is defined as usual as a *50-50* superposition of right and left circular polarization then the sign of $B^{(3)}$ is reversed with the handedness of the radiation, because the sign of $B^{(1)} \times B^{(2)}$ is reversed [4]. However, the left and right interaction Hamiltonians are

$$\begin{aligned} H_{1L} &= (j\sin\phi + i\cos\phi)\cdot B^{(3)*} \times m, \\ H_{1R} &= (j\sin\phi - i\cos\phi)\cdot(-B^{(3)*})\times m, \end{aligned} \tag{3.107}$$

so the half sum $1/2\,(H_{1L} + H_{1R})$, the mean interaction Hamiltonian in linear polarization, is non-zero and given by

$$H_{1\,\mathrm{linear}} = -(m \times i\cos\phi)\cdot B^{(3)*}, \tag{3.108}$$

i.e., is again the dot product of a phase dependent magnetic dipole moment and a phase free and longitudinal magnetic field, $B^{(3)}$, defined by Eq. (3.17) in vacuo.

3.5.1 Selection Rules in Terms of $B^{(3)}$

Spin angular momentum selection rules for atomic absorption can be understood simply from the relation between the photomagneton, $\hat{B}^{(3)}$ [6], and the longitudinal angular momentum operator $\hat{J}^{(3)}$, for one photon interacting with an atom,

$$\hat{B}^{(3)} = B^{(0)}\frac{\hat{J}}{\hbar}. \tag{3.109}$$

The eigenvalues of $\hat{J}$ are $\pm\hbar$ for the usual picture of a two dimensional photon, which is augmented to $0, \pm\hbar$ because of the presence of the longitudinal $\hat{B}^{(3)}$ [6]. The selection rules can therefore be worked out as usual [31] from the assumption that all the photon angular momentum is transferred to an electron within the atom during an absorption process, causing a transition from energy level i to energy level f within the atom. When a photon is emitted it carries away angular momentum from the electron, which drops from energy level f to energy level i within the atom. The electric dipole transition rule [31], i.e., $\Delta l = \pm 1$, remains the same as in ordinary absorption theory, but changes have to be made to the usual point of view when dealing with the selection rule for spin angular momentum. The spin angular momentum quantum number for the photon in the presence of non-zero concomitant $\hat{B}^{(3)}$ is -1, 0, +1, in one sense of circular polarization, and +1, 0, -1 in the other. In linear polarization therefore we retain $m_s = 0$; which is transferred to $m_l = 0$ for an electron within the absorbing atom. The presence of $m_s = 0$ in the linearly polarized radiation is an expression of the dipole coupling term [31]; and neatly leads to the result expected in the conventional picture, $m_l = 0$, but in a different but equivalent way. In the conventional picture [31] plane polarized photons are linear superpositions of circularly polarized photons. For example, x polarized light is considered to be a superposition of $m_s = +1$ and -1, generating $p_{+1} + p_{-1}$; the p_x orbital [31]. Because of this superposition, the net angular momentum transferred to the atom is represented by $m_l = 0$ in linear polarization. In the new picture, we have $m_s = 0$ of the photon becoming $m_l = 0$ of the electron by conservation of angular momentum. The presence of $m_s = 0$ for the photon is possible because $\hat{B}^{(3)}$ is not zero in the new picture. In the old picture $\hat{B}^{(3)}$ is zero and $m_s = 0$ does not exist, because the photon's world is flat.

3.6 Lorentz Transformation of the B Cyclics and the Fundamental Displacement Current $J^{(3)}$

Consider an electromagnetic field in a frame K', propagating at c in vacuo. In any other frame K it must also propagate at c, and so in the quantum theory, the photon has no rest frame. This is a counter-intuitive feature of special relativity, but follows from the first principle, which asserts that c is a universal, invariant, constant. This feature must also be compatible with the result of the Lorentz transform applied to the field strength tensor $F_{\mu\nu}$, a process which produces

$$B^{(1)\prime} = \gamma\left(B^{(1)} - \frac{\mathbf{v}\times E^{(2)}}{c^2}\right),$$

$$B^{(2)\prime} = \gamma\left(B^{(2)} - \frac{\mathbf{v}\times E^{(2)}}{c^2}\right), \tag{3.110}$$

$$B^{(3)\prime} = B^{(3)}, \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$

with v as the constant speed of K' in the Z = (3) axis with respect to K. For a plane wave, Eqs. (3.110) produce the result

$$B^{(0)\prime} = \left(\frac{1-\frac{v}{c}}{1+\frac{v}{c}}\right)^{1/2} B^{(0)}, \tag{3.111}$$

where we have used $E^{(0)} = cB^{(0)}$, a relation which holds only at the speed of light in vacuo. The phase of the plane wave is a Lorentz invariant [31], and so the amplitude of the plane wave would gradually diminish to zero in frame K' if v were allowed to approach c. If the hypothetical rest frame K' could be defined in this way and if the plane wave continued to propagate at c with respect to K', there would be no radiation, a reduction to absurdity. The root of this paradox is found in the fact that the plane wave is already propagating at c in frame K', and in consequence frame K' cannot move with respect to K at any velocity, (other than 0 or c), without changing the value of c measured with respect to frame K, and thus violating the first principle of relativity, that c is a *Lorentz-frame invariant* universal constant. The only possible solution of the paradox is v = 0 in Eq. (3.111), leading to $B^{(0)\prime} = B^{(0)}$. This means that the plane wave propagates at c in any frame, and has no rest frame. It is frequently stated that the massless photon has no rest frame.

There follows the important result that the B Cyclic theorem, when applied to a plane wave propagating at c in vacuo, is also a Lorentz invariant construct. It remains the same in any Lorentz frame of reference, and is therefore an invariant feature of the Poincaré group of special relativity, forming an invariant Lie algebra. It is therefore automatically Lorentz covariant in vacuo, and a $\hat{C}\hat{P}\hat{T}$ conserving field equation. It is concluded that in standard special relativity, $B^{(3)}$ is a bona fide magnetic flux density. If the converse were true the B Cyclic theorem would violate the $\hat{C}\hat{P}\hat{T}$ theorem, probably the most fundamental theorem in physics. Note that this conclusion follows directly from application of the Lorentz transform to the field strength tensor of electromagnetic radiation propagating at c in vacuo. Similarly [4—6], the B Cyclic theorem conserves the other six symmetry combinations, i.e., $\hat{C}$, $\hat{P}$, $\hat{T}$, $\hat{C}\hat{P}$, $\hat{C}\hat{T}$, $\hat{P}\hat{T}$, and at the classical level violates no discrete symmetry in physics. The $\boldsymbol{B}^{(3)}$ field in vacuo is phase free and Lorentz invariant, as indicated directly by Eq. (3.110).

In the received view the field $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ conserves the seven symmetry combinations of physics, but it is asserted that the cross product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ does not produce another field. At this point, the received view violates the $\hat{C}\hat{P}\hat{T}$ theorem, which, as we have just seen, asserts that $\boldsymbol{B}^{(3)}$ must be a field.

If for some reason the electromagnetic radiation does not propagate at c in frame K, e.g. if the medium of propagation in frame K is in general magnetizable and polarizable, then $\boldsymbol{B}$ must be replaced by the magnetic field strength $\boldsymbol{H}$, and $\boldsymbol{E}$ by the displacement $\boldsymbol{D}$. The Lorentz transform (3.110) retains its form only if there is no magnetization ($\boldsymbol{M}$) or polarization ($\boldsymbol{P}$), so that

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0}, \quad \boldsymbol{D} = \epsilon_0 \boldsymbol{E}, \tag{3.112}$$

and in frame K' there exists a relation,

$$\boldsymbol{B}^{(1)\prime} \times \boldsymbol{B}^{(2)\prime} = i\zeta \boldsymbol{B}^{(3)*}, \tag{3.113}$$

where ζ is a factor due to Lorentz transformation. Using $\boldsymbol{B}^{(3)} = \boldsymbol{B}^{(3)\prime}$ we obtain

$$\boldsymbol{B}^{(3)*} = -i\frac{\gamma^2}{\zeta}\Big[\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} - \frac{1}{c^2}\left(\mathbf{v} \times \boldsymbol{E}^{(1)}\right) \times \boldsymbol{B}^{(2)} - \frac{1}{c^2}\boldsymbol{B}^{(1)} \times \left(\mathbf{v} \times \boldsymbol{E}^{(2)}\right) + \frac{1}{c^4}\left(\mathbf{v} \times \boldsymbol{E}^{(1)}\right) \times \left(\mathbf{v} \times \boldsymbol{E}^{(2)}\right)\tag{3.114}$$

If $\mathrm{v} \rightarrow c$ in this equation, then

$$\boldsymbol{B}^{(1)} \rightarrow \frac{1}{c^2}\boldsymbol{c} \times \boldsymbol{E}^{(1)} \text{ etc.}, \tag{3.115}$$

and this is a relation for the vacuum, because otherwise the propagation of electromagnetic radiation could not take place at c. In the vacuum there is no magnetization, and so

$$\boldsymbol{B}^{(3)*} \rightarrow -i\frac{\gamma^2}{\zeta}\left[\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} - \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} - \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} + \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}\right] = -\frac{i}{\zeta}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}\left(\frac{0}{0}\right). \tag{3.116}$$

Equation (3.116) becomes the B Cyclic theorem in the vacuum if v approaches c. Equation (3.114) applies in general when a magnetizable and polarizable medium in frame K moves at $-\mathrm{v}$ with respect to frame K'.

Equation (3.116) shows once more that for electromagnetic radiation propagating at c in the vacuum, there is no rest frame, and the Lorentz transform in this condition is compatible with the first principle of special relativity if and only if $\mathrm{v} = 0$. This is also true for Lorentz transform in axes other than Z. These conclusions can be arrived at in several ways, one of which is as follows. Consider the Lorentz transformation matrices of a four-vector in the X, Y, and Z axes. For a single Lorentz transform from frame K to K' these are, in the conventional notation $\beta = \mathrm{v}/c$; $\gamma = \left(1 - \mathrm{v}^2/c^2\right)^{-1/2}$,

$$L_X := \begin{bmatrix} \gamma & 0 & 0 & i\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma\beta & 0 & 0 & \gamma \end{bmatrix}, \quad L_Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma & 0 & i\gamma\beta \\ 0 & 0 & 1 & 0 \\ 0 & -i\gamma\beta & 0 & \gamma \end{bmatrix}, \quad L_Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\gamma\beta \\ 0 & 0 & -i\gamma\beta & \gamma \end{bmatrix}. \tag{3.117}$$

If we now consider a double boost, one from X to X', followed by one from X' to X'', the resultant matrix is

$$X_1 X_2 = \begin{bmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & 0 & 0 & i\gamma_1\gamma_2(\beta_1+\beta_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma_1\gamma_2(\beta_1+\beta_2) & 0 & 0 & \gamma_1\gamma_2(1+\beta_1\beta_2) \end{bmatrix}, \tag{3.118}$$

and this becomes the matrix L_x for a single boost from K to K'' if

$$\gamma := \gamma_1\gamma_2(1+\beta_1\beta_2), \quad \beta\gamma := \gamma_1\gamma_2(\beta_1+\beta_2). \tag{3.119}$$

These equations give the relativistic velocity addition rule [59],

$$v = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}, \tag{3.120}$$

where v_1 is the speed of K' relative to K; and v_2 is the speed of K'' relative to K''. Equation (3.120) gives the result that if $v_1 = v_2 = c$, then $v = c$; and if $v_1 = v_2 = c$, then $v = c$. So if K' moves at c relative to K, K' also moves at c relative to both K and K'. If however, K' moves at c relative to K, and K'' moves at a velocity $v_2 \ll c$ relative to K', then K'' from Eq. (3.120) must move at $\sim c + v_2$ relative to K. This violates the principle that c is the same in every frame of reference. A field $B = B^{(1)} + B^{(2)} + B^{(3)}$ which propagates at c in one frame propagates at c in every other frame and cyclic relations between these field are Lorentz invariant in the vacuum.

If we consider consecutive boosts in orthogonal directions (e.g. a boost in Z followed by one in X), the relevant matrix product is

$$X_1 Z_2 := \begin{bmatrix} \gamma_1 & 0 & 0 & i\gamma_1\beta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma_1\beta_1 & 0 & 0 & \gamma_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_2 & i\gamma_2\beta_2 \\ 0 & 0 & -i\gamma_2\beta_2 & \gamma_2 \end{bmatrix} = \begin{bmatrix} \gamma_1 & 0 & \gamma_1\gamma_2\beta_1\beta_2 & i\gamma_1\gamma_2\beta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_2 & i\gamma_2\beta_2 \\ -i\gamma_1\beta_1 & 0 & -i\gamma_1\gamma_2\beta_2 & \gamma_1\gamma_2 \end{bmatrix}, \tag{3.121}$$

and this does not have the structure of an individual boost in X or in Z, the reason being that a commutator of boost matrices is a rotation generator in space-time [4]. For example, the commutator of boosts represented by $XZ - ZX$ is an off diagonal 4 x 4 matrix representing a rotation about Y

$$XZ - ZX = \begin{bmatrix} 0 & 0 & \gamma^2\beta^2 & 0 \\ 0 & 0 & 0 & 0 \\ -\gamma^2\beta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.122}$$

Comparing the (0,3) elements of matrices (3.121) and (3.122) we see that the extra (0,3) element in Eq. (3.121) is caused by rotation, and

$$\beta = (\beta_1\beta_2)^{1/2}, \quad v = (v_1 v_2)^{1/2}, \tag{3.123}$$

is a relativistic velocity multiplication rule equivalent to the addition rule given by Eq. (3.120). If $v_1 = v_2 = c$ then again, $v = c$ from Eq. (3.123), and if $v_1 = v_2 = 0$, $v = 0$. If however, we try $v_1 = c$ and $v_2 < c$, the resultant v becomes again different from c. This violates the principle of special relativity that c is the same in every Lorentz frame of reference. Again, we arrive at the conclusion that a field moving at c in one Lorentz frame moves at c in all Lorentz frames. Therefore the B cyclics in vacuo form a Lorentz invariant field theory. This is an important result because it follows that the B cyclics must form a $\hat{C}\hat{P}\hat{T}$ conserving theory of fields. Therefore $B^{(3)}$ is a physical magnetic flux density in the vacuum. In matter, the H cyclics are Lorentz covariant, where H is magnetic field strength, but no longer invariant.

Covariance is sufficient to show that the H cyclics also form a $\hat{C}\hat{P}\hat{T}$ conserving field theory or structure, and a theory which is invariant is also covariant.

Equation (3.66) shows that there exists a longitudinal current density $j^{(3)}$ in the vacuum, related to $B^{(3)}$ through

$$j^{(3)} = \frac{1}{\mu_0}\frac{\omega}{c}B^{(3)}, \tag{3.124}$$

and this current density can be expressed as part of the j cyclics in the vacuum.

$$j^{(1)} \times j^{(2)} = ij^{(0)}j^{(3)*}, \text{ et cyclicum}, \tag{3.125}$$

where $j^{(1)}$ and $j^{(2)}$ are transverse current densities,

$$j^{(1)} = j^{(2)*} = \frac{j^{(0)}}{\sqrt{2}}(ii + j)\, e^{i\phi}. \tag{3.126}$$

Therefore the magnitude $B^{(0)}$ is proportional to the magnitude of the current density,

$$B^{(0)} = \mu_0\frac{c}{\omega}j^{(0)} = \mu_0 c\frac{e}{Ar}, \tag{3.127}$$

and if the photon area is $1/\kappa^2$, we recover, for every photon,

$$j^{(0)} = \omega\frac{e}{Ar} = \omega e\kappa^2 = c\rho^{(0)}, \tag{3.128}$$

where $\rho^{(0)}$ is the charge density of the photon. This type of radiated vacuum current needs for its existence a finite ω, and a finite, constant tangential velocity, defined by $v = \omega r$ where r is a radius. In the vacuum, the radius is $1/\kappa$ and the constant tangential velocity is c. The forward velocity is also c, and the resultant velocity is c as we have shown already using the relativistic multiplicative and addition rules.

References

[1] A. Chubykalo and R. Smirnov-Rueda, *Phys. Rev. E*, **53**, 5373 (1996); also V. Dvoeglazov, *Phys. Rev. D*, submitted for publication (Feb., 1996).
[2] H. Yilmaz, *Annals N. Y. Acad. Sci.* 755, 476 (1995), Wheeler Honorarium.
[3] See Chap. 6, this volume.

[4] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $B^{(3)}$ (Kluwer Academic, Dordrecht, 1994).
[5] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).
[6] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $B^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).
[7] M. W. Evans, *Physica B* **182**, 227, 137 (1992); **183**, 103 (1993); *Physica B* **190**, 310 (1993); *Physica A* **214**, 605 (1995).
[8] M. W. Evans and S. Kielich, eds., *Modern Nonlinear Optics,* Vol. 85(2) of *Advances in Chemical Physics,* I. Prigogine and S. A. Rice, eds. (Wiley Interscience, New York, 1993).
[9] M. W. Evans, *The Photon's Magnetic Field* (World Scientific, Singapore, 1992); A. A. Hasanein and M. W. Evans, *The Photomagneton in Quantum Field Theory* (World Scientific, Singapore, 1994).
[10] M. W. Evans, *Found. Phys.* **24**, 892, 1519, 1671 (1994); **25**, 175, 383 (1995); M. W. Evans and S. Roy, *Found. Phys.*, in press.
[11] M. W. Evans, *Found. Phys. Lett.* 7, 67, 209, 379, 437, 577, 591 (1994); ibid. **8**, 83, 187, 253, 279 (1995).
[12] F. Farahi, Y. Aktas, and M. W. Evans, *J. Mol. Structure* **285**, 47 (1993); M. W. Evans and F. Farahi, ibid., **300**, 435 (1993); M. W. Evans, *Mod. Phys. Lett.* **B**7, 1247 (1993); M. W. Evans, S. Roy and S. Jeffers, *Il Nuovo Cim. D* (1995).
[13] M. W. Evans in A. van der Merwe and A. Garuccio, eds., *Waves and Particles in Light and Matter* (Plenum, New York, 1994).
[14] M. W. Evans, *Found. Phys. Lett.* and *Found. Phys.*, in press (1996 to 1997); M. W. Evans and S. Roy, *Found. Phys.*, submitted.
[15] H. Múnera and O. Guzmán, *Found. Phys. Lett.*, submitted (Feb. 1996).
[16] G. Hunter and R. P. L. Wadlinger, *Phys. Essays* **2(2)**, 158 (1989).
[17] D. V. Ahluwalia and D. J. Ernst, *Mod. Phys. Lett.* **A7**, 1967 (1992); also D. V. Ahluwalia, M. B. Johnson, and T. Goldman, *Phys. Lett.* **B316**, 102 (1993); D. V. Ahluwalia and T. Goldman, *Mod. Phys. Lett.* **A8**, 2623 (1993).
[18] A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Dover, New York, 1980).
[19] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill Book Co., New York, 1980).
[20] K. Huang, *Quarks, Leptons and Gauge Fields* (World Scientific, Singapore, 1992).
[21] N. Calder, *Einstein's Universe* (Viking, New York, 1979).
[22] G. H. Wagnière, *Linear and Non-Linear Optical Properties of Molecules* (VCH, Basel, 1993).
[23] B. Lehnert, *Optik* **99**, 113 (1995).
[24] B. Lehnert, Physica Scripta **53**, 204 (1996).
[25] B. Lehnert, *Spec. Sci. Tech. 17, 259 (1994).*
[26] Y. Aharonov and D. Bohm, Phys. Rev. **115** 485 (1959).

[27] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).
[28] D. S. Jones, *The Theory of Electromagnetism* (Pergamon, Oxford, 1964).
[29] R. B. Leighton, *Principles of Modern Physics* (McGraw Hill, New York, 1959)
[30] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[31] P. W. Atkins, *Molecular Quantum Mechanics* (Oxford University Press, Oxford, 1983). For a relativistic treatment see L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th edn. (Pergamon, Oxford, 1975).
[32] Personal communication.
[33] D. Ahluwalia and D. J. Ernst, *Mod. Phys. Lett* **A7**, 1967 (1992).
[34] E. P. Wigner, *Ann. Math* **140**, 149 (1939).
[35] J. A. Stratton, *Electromagnetic Theory* (McGraw Hill, New York, 1941).
[36] L. D. Barron, *Physica B* **190**, 307 (1993)
[37] A. D. Buckingham, *Science* **266**, 665 (1994).
[38] A. D. Buckingham and L. C. Parlett, *Science* **264**, 1748 (1994).
[39] M. W. Evans *Physica B* **190**, *310 (1993).*
[40] M. W. Evans Found. Phys. Lett. **8**, 563, (1995).
[41] M. W. Evans and A. A. Hasanein, *The Photomagneton in Quantum Field Theory* (World Scientific, Singapore, 1994).
[42] M. W. Evans, *Found Phys. Lett.* **8**, 187 (1995).
[43] A. Lakhtakia, *Physica B* **191**, 362, (1993).
[44] A. Lakhtakia, *Found. Phys. Lett.* **8**, 183 (1995).
[45] D. M. Grimes, *Physica B* **191**, 367 (1993).
[46] G. L. J. A. Rikken, *Opt. Lett.* **20**, 846 (1995).
[47] M. W. Evans, *The Photon's Magnetic Field* (World Scientific, Singapore, 1992).
[48] Ref. 6, Chap. 9.
[49] M. W. Evans, *Found. Phys. Lett.* **8**, 187 (1995); S. J. van Enk, *Found. Phys. Lett.* **8**, 183 (1995).
[50] E. Recami, *Scientia*, **110**, 589 (1975).
[51] V. V. Dvoeglazov, *Int. J. Mod. Phys.* **34**, 2467 (1995).
[52] S. Weinberg, *Phys. Rev.* **B134**, 882 (1964).
[53] M. Mészáros, *Found. Phys. Lett.*, in press.
[54] M. Mészáros and P. Molnar, *Ann. Phys.* **45** 155 (1988); **46**, 153 (1989); *Phys. Essays* **3**, 284 (1990).
[55] M. Mészáros, this volume.

Chapter 4. The A Cyclics Derivation and Reduction

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We have argued that the B cyclics are $\hat{C}\hat{P}\hat{T}$ invariant, which means that they are Lorentz covariant. They indicate that the gauge transformation that should be applied in the vacuum to the electromagnetic field must be extended to take account of the existence of the longitudinal $\boldsymbol{B}^{(3)}$. In this chapter we adapt the well known Yang Mills theory [1] to suggest a way of doing this by using the notion of an axial vector potential with three components, $\boldsymbol{A}^{(1)}$, $\boldsymbol{A}^{(2)}$, and $\boldsymbol{A}^{(3)}$. These components make up a complex vector field in three dimensional space to which a gauge transformation can be applied. Following the rules of gauge transformation in the Yang Mills theory we arrive at cyclical relations between the A components in the basis ((1), (2), (3)). These relations, the A cyclics, form part of an electromagnetic field theory in which the ordinary derivatives of Maxwell's theory are replaced by covariant derivatives similar to those appearing in the opening chapters of this book. From previous chapters it is seen that each component of the A cyclics must be an axial vector, whereas the potential vector is usually regarded as polar [1] in the various gauge theories. It is shown in this chapter that $\boldsymbol{A}$ can indeed be axial, and directly proportional to the magnetic flux density: the choice of polar $\boldsymbol{A}$ is arbitrary, and originates in $\boldsymbol{B}$ being the curl of a polar $i\boldsymbol{A}$. It is possible for the electric field to be the curl of an axial $\boldsymbol{A}$. The complex $i\boldsymbol{A}$ is transformed by duality to the complex $\boldsymbol{A}$ in vacuo. Both $i\boldsymbol{A}$ and $\boldsymbol{A}$ have real components. In this view there can be a real, axial and longitudinal $\boldsymbol{A}^{(3)}$ and an imaginary, polar and longitudinal $i\boldsymbol{A}^{(3)}$.

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4.1 Definition of the Complex Vector Field, and Gauging

The complex vector field prior to gauging in the Yang Mills theory is defined by the standard formulae [1] usually used for the ϕ field in Yang Mills theory. The ϕ field is not regarded in standard theory as the electromagnetic field itself. However, any field can be subjected to a gauge transform, which is a rotation within a well defined group structure. In standard gauge theory this structure is U(1) as described in previous volumes and in the literature [4—10,1]. In the Yang Mills theory, however, this space becomes O(3). So we start by defining the electromagnetic field to be gauged in O(3). This has three axial components defined by

$$A^{(1)} := \frac{1}{\sqrt{2}}\left(A_X - iA_Y\right)e^{i\phi}, \tag{4.1a}$$

$$A^{(2)} := \frac{1}{\sqrt{2}}\left(A_X + iA_Y\right)e^{-i\phi}, \tag{4.1b}$$

$$A^{(3)} := A_Z, \tag{4.1c}$$

where ϕ is the electromagnetic phase. After gauging this field in O(3) it will be shown that we recover, self-consistently, the cyclic relations,

$$A^{(1)} \times A^{(2)} = iA^{(0)}A^{(3)*}, \tag{4.2a}$$

$$A^{(2)} \times A^{(3)} = iA^{(0)}A^{(1)*}, \tag{4.2b}$$

$$A^{(3)} \times A^{(1)} = iA^{(0)}A^{(2)*}, \tag{4.2c}$$

This is not a trivial exercise, however, because the same method leads to a generalization of the Maxwell equations to field equations of the same type as in previous chapters, with covariant derivatives replacing ordinary four-derivatives. In order to arrive at this self-consistency within Yang Mills theory the isospace has to be identified with the three dimensional space ((1), (2), (3)) or (X, Y, Z). From the same method emerges the B cyclics, which inter-alia imply that the Maxwell equations are incomplete. In further sections of this chapter we will develop a method for reducing the A cyclics to fully quantized Maxwell equations using only the correspondence principle of quantum mechanics.

Another important consequence of the method in this chapter is that the gauge theory of the B cyclics is self-consistent only if it is a gauge theory of type two, and one developed in the O(3) group. If $A^{(1)}$ and $A^{(2)}$ in Eqs. (4.1) are defined as axial, then $A^{(3)}$ is real and physical. If $A^{(1)}$ and $A^{(2)}$ in Eqs. (4.1) are defined as polar,

then $iA^{(3)}$ occurs within the A cyclics and we obtain the Gupta-Bleuler result [4—10,1], i.e., longitudinal and time-like modes of the polar A field are pure imaginary and cancel each other, leaving no physical, longitudinal component. The physical longitudinal component of the A field is therefore an axial vector directly proportional to the physical and real $B^{(3)}$ field observable empirically [4—6].

In the Cartesian basis, Eqs. (4.1) are gauged by rotation about the Z axis through an angle Λ_3, which in type two gauge theory is a function as usual of x_μ,

$$A_X' = A_X \cos\Lambda_Z + A_Y \sin\Lambda_Z, \tag{4.3a}$$

$$A_Y' = -A_X \sin\Lambda_Z + A_Y \cos\Lambda_Z, \tag{4.3b}$$

$$A_Z' = A_Z. \tag{4.3c}$$

This is a clockwise rotation of the vector field in frame (X, Y, Z) through an angle Λ_Z; equivalent to keeping the field constant and rotating the frame anti-clockwise through the same angle. There is no a priori objection therefore to carrying out this procedure for the electromagnetic field itself, because it is simply a frame rotation. In the circular basis it is a frame rotation around the vector field,

$$A := A^{(1)} + A^{(2)} + A^{(3)}, \tag{4.4}$$

whose components are defined as being axial. If this process were not possible, the electromagnetic field could not be defined with respect to a frame of reference, a reduction to absurdity.

The usual methods of Yang Mills theory can now be applied. If Λ_Z is infinitesimally small, then Eqs. (4.3) become

$$A_X' = A_X + \Lambda_Z A_Y, \quad A_Y' = -\Lambda_Z A_X + A_Y, \quad A_Z' = A_Z, \tag{4.5}$$

which is the Z component of the vector equation,

$$\delta A := A' - A = -\Lambda \times A. \tag{4.6}$$

In the type two gauge transformation the angle becomes a function of x_μ as a consequence of locality in special relativity [1]. The vector field A is therefore changed (or *gauged*) to

$$\delta A = A' - A = -\Lambda(x_\mu) \times A, \tag{4.7}$$

but its four-derivative is changed to

$$\delta(\partial_\mu A) = -\Lambda \times (\partial_\mu A) - (\partial_\mu \Lambda) \times A . \tag{4.8}$$

So A and its four-derivative $\partial_\mu A$ do not respond in the same way to gauge transformation, i.e., are not covariant. In order to make the theory covariant, a new derivative is introduced,

$$D_\mu A = (\partial_\mu + g A_\mu \times) A , \tag{4.9}$$

where A_μ is a four-vector and g a universal constant. The four-vector A_μ is identified with the electromagnetic field, which is said to be a universal influence needed to keep the original field Lorentz covariant. The correct covariant result,

$$\delta(D_\mu A) = -\Lambda \times (D_\mu A), \tag{4.10}$$

is obtained from this procedure if and only if

$$\delta(A_\mu) = -\Lambda \times A_\mu + \frac{1}{g}\partial_\mu \Lambda , \tag{4.11}$$

and if the field strength tensor is defined as

$$G_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + g A_\mu \times A_\nu . \tag{4.12}$$

Therefore $G_{\mu\nu}$ is a vector in ((1), (2), (3)) and transforms covariantly as

$$\delta(G_{\mu\nu}) = -\Lambda \times G_{\mu\nu}. \tag{4.13}$$

The corresponding Lagrangian is

$$\mathcal{L} = D_\mu A \cdot D^\mu A - m^2 A \cdot A - \frac{1}{4} G_{\mu\nu} \cdot G^{\mu\nu}, \tag{4.14}$$

(where m is a field mass) and the Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial (A_\mu^i)} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu^i)} \right) , \tag{4.15}$$

gives the result

$$D^\nu G_{\mu\nu} = g(D_\mu A) \times A := g J_\mu , \tag{4.16}$$

where J_μ is a current. This is Eq. (2.26) of Chap. 2, derived from unified field theory in a different way. The current vanishes, if and only if

$$D_\mu A = \partial_\mu A + g A_\mu \times A = 0 , \tag{4.17}$$

and this is the condition for the source free field equation with no vacuum current. The condition (4.17) also means that

$$\partial^\nu G_{\mu\nu} + g A^\nu \times G_{\mu\nu} = 0. \tag{4.18}$$

We shall see that Eq. (4.17) gives the A cyclics, and Eq. (4.18) gives the B cyclics.

If we let $\mu = Z$ in Eq. (4.17) then the Z component of A_μ is A_Z, which is $A^{(3)}$ in the circular basis. Now take the $A^{(1)}$ component of A in Eq. (4.17) and we obtain

$$\frac{\partial}{\partial Z} A^{(1)} = -g A^{(3)} \times A^{(1)}, \tag{4.19}$$

which can be re-arranged following differentiation to give

$$A^{(3)} \times A^{(1)} = i \frac{\kappa}{g} A^{(2)*} . \tag{4.20}$$

If the universal factor g is identified as

$$g = \frac{\kappa}{A^{(0)}} = \frac{e}{\hbar}, \tag{4.21}$$

Eq. (4.20) becomes the A cyclical structure,

$$A^{(3)} \times A^{(1)} = i A^{(0)} A^{(2)*} , \tag{4.22}$$

which is also inferred directly by the original definition (4.1), thus self-checking the calculation.

The A cyclics are a direct consequence of Yang Mills gauge theory if the original field being gauged is taken to be the electromagnetic field as represented in Eq. (4.1). This finding has a number of consequences, among them is charge quantization according to the condition (4.21), which was first derived in Vol. 2 of this series and referred to as the charge quantization condition. It is also the minimal prescription applied to the free field itself. The electromagnetic field in this view is consistently non-Abelian, and the gauge transformation induces a change in the electromagnetic phase, because the gauge condition (4.11) is the infinitesimal form of the complex exponential [1],

$$S := \exp\left(\frac{i}{2}\tau\cdot\Lambda\right), \tag{4.23}$$

where τ is a matrix of $SU(2)$. The gauge transformation is therefore summarized as

$$\delta A = SA, \tag{4.24}$$

and it is seen that the cross product $A^{(1)}\times A^{(2)}$ is gauge invariant. It is proportional to the gauge invariant $A^{(3)}$ and $B^{(3)}$. This result is self-consistent because the gauge transformation is a rotation about the Z = (3) axis. The phase shift introduced into the electromagnetic fields $A^{(1)}$ and $A^{(2)}$ is one which resembles the empirically observable [4] topological phase introduced independently by Berry [2] and by Hunter and Wadlinger [2]. This is again self consistent because $A^{(1)}$ and $A^{(2)}$ are affected by the rotation through which the gauge transformation is defined. The frame rotation leaves the complete vector A unchanged, as for any rotation, but changes the individual components. This is the origin of a physical and observable topological phase shift in the transverse components of the field.

If the mass term m is taken to be zero, then the Lagrangian in free space becomes

$$\mathscr{L}_{\text{free}} = -G_{\mu\nu}\cdot G^{\mu\nu}, \tag{4.25}$$

and in three space is made up of the three components of the complete vector $B = B^{(1)} + B^{(2)} + B^{(3)}$ precisely as argued elsewhere [4—6].

4.2 Gauge Transformation and Measurable Effects of Topology

The gauge transformation of Eq. (4.3) introduces a measurable topological phase so that the electromagnetic phase becomes $e^{i(\phi+\Lambda)}$ or $e^{-i(\phi+\Lambda)}$. This is shown through the fact that Eq. (4.3a) becomes

$$A_X' = A_X e^{i\Lambda}, \tag{4.26}$$

if $A_Y = iA_X$. Similarly,

$$A_Y' = A_Y e^{i\Lambda}. \tag{4.27}$$

It follows that

$$A^{(1)\prime} := \frac{1}{\sqrt{2}}\left(A_X' - iA_Y'\right) = \frac{1}{\sqrt{2}}\left(A_X - iA_Y\right)e^{i\Lambda}, \tag{4.28}$$

$$A^{(2)\prime} := \frac{1}{\sqrt{2}}\left(A_X' + iA_Y'\right) = \frac{1}{\sqrt{2}}\left(A_X + iA_Y\right)e^{-i\Lambda}, \tag{4.29}$$

$$A^{(3)\prime} := A_Z' = A_Z. \tag{4.30}$$

The gauge transformation of the vector potential is therefore

$$A^{(1)} \rightarrow e^{i\Lambda}A^{(1)} := SA^{(1)} := A^{(1)\prime}, \tag{4.31}$$

$$A^{(2)} \rightarrow e^{-i\Lambda}A^{(2)} := S^{*}A^{(2)} := A^{(2)\prime}, \tag{4.32}$$

$$A^{(3)} \rightarrow A^{(3)} := A^{(3)\prime}. \tag{4.33}$$

The magnetic flux density is normally defined by the curl of the vector potential, for example, $B^{(1)} = \nabla\times A^{(1)}$. We now find the conditions under which $\nabla\times A^{(1)}$ transforms covariantly with $A^{(1)}$ (*cf.* Eqs (4.10) and (4.13)) and find that these conditions imply the existence of

$$B^{(3)*} = -i\frac{e}{\hbar}A^{(1)}\times A^{(2)}, \tag{4.34}$$

from fundamental topology through the commutator of covariant derivatives. The latter also implies a non-local effect of $B^{(3)}$ — the optical equivalent of the Aharonov-Bohm effect [6].

The gauge transformation of $B^{(1)}$ is given by

$$B^{(1)} := \nabla\times A^{(1)} \rightarrow \nabla\times A^{(1)\prime} := \nabla\times\left(e^{i\Lambda}A^{(1)}\right). \tag{4.35}$$

Using the vector identity [1],

$$\nabla\times(uF) = u(\nabla\times F) + (\nabla u)\times F, \tag{4.36}$$

gives

$$\nabla \times A^{(1)} \rightarrow e^{i\Lambda}\nabla \times A^{(1)} + \left(\nabla e^{i\Lambda}\right) \times A^{(1)} . \tag{4.37}$$

If $\nabla e^{i\Lambda}$ were zero, Eq. (4.37) would be covariant with Eq. (4.31) under gauge transformation, and ∇ would be unchanged. In special relativity, however, $\Lambda = \Lambda(x_\mu)$, so

$$\nabla e^{i\Lambda} = i\nabla\Lambda e^{i\Lambda} , \tag{4.38}$$

and

$$D' \times A^{(1)\prime} := \left(\nabla + i\nabla\Lambda\right) \times \left(e^{i\Lambda}A^{(1)}\right), \tag{4.39}$$

where

$$D' := \nabla + i\nabla\Lambda := \nabla + igA', \qquad A' := \frac{1}{g}\nabla\Lambda , \qquad D := \nabla + igA , \qquad A := 0 . \tag{4.40}$$

Therefore the following covariant derivatives have been introduced by the assumption that Λ is a function of space-time,

$$D' \times A^{(1)\prime} := \left(\nabla + i\nabla\Lambda\right) \times \left(SA^{(1)}\right), \qquad D \times A^{(1)} := \left(\nabla + 0\right) \times A^{(1)} . \tag{4.41}$$

The gauge transform of type two is therefore

$$D \times A^{(1)} \rightarrow D' \times A^{(1)\prime} , \tag{4.42}$$

with the covariant derivative,

$$D' := \nabla + igA' . \tag{4.43}$$

This has the same form as in general gauge theory [1]. The curl of $A^{(1)}$ transforms covariantly with $A^{(1)}$ itself if the del operator transforms as

$$\nabla \rightarrow \nabla + i\frac{e}{\hbar}A . \tag{4.44}$$

Since A is complex,

$$A^{(1)} := \frac{\hbar}{e}\nabla\Lambda^{(1)}, \qquad A^{(2)} := \frac{\hbar}{e}\nabla\Lambda^{(2)} . \tag{4.45}$$

If $A' := A^{(1)}$ in Eq. (4.42) then

$$D' \times A^{(1)\prime} = e^{i\Lambda}\left(D' \times A^{(1)}\right) = e^{i\Lambda}\nabla \times A^{(1)} , \tag{4.46}$$

and

$$\nabla \times A^{(1)} \rightarrow e^{i\Lambda}\nabla \times A^{(1)} , \tag{4.47}$$

$$A(=0) \rightarrow A(=0) + \frac{\hbar}{e}\nabla\Lambda^{(1)} , \tag{4.48}$$

represents a covariant gauge transformation with $A = 0$. The quantity $\nabla\Lambda^{(1)}$ exists in regions where $B^{(1)}$ does not exist, and causes an optical Aharonov-Bohm effect through the non-local conjugate product $\nabla\Lambda^{(1)} \times \nabla\Lambda^{(2)}$. Equation (4.48) is an example of the general result [1],

$$A_\mu \rightarrow SA_\mu A^{-1} - \frac{i}{g}\left(\partial_\mu S\right)S^{-1} , \tag{4.49}$$

in gauge theory for any topological group. Equation (4.48) is the rule for gauge transform of the potential introduced in the covariant derivative and identified as the electromagnetic field.

We are now led to a self-consistent definition of the complete magnetic field as the commutator of covariant derivatives [1],

$$B := -i\frac{A^{(0)}}{\kappa}D^{(1)} \times D^{(2)} , \tag{4.50}$$

again a result of general gauge theory. From Eq. (4.50),

$$B = -i\frac{A^{(0)}}{\kappa}\left(\nabla\times\nabla + i\frac{\kappa}{A^{(0)}}\nabla\times A^{(2)} - i\frac{\kappa}{A^{(0)}}A^{(1)}\right.$$

$$\left.\times\nabla + \left(\frac{\kappa}{A^{(0)}}\right)^2 A^{(1)}\times A^{(2)}\right) = \nabla\times A^{(1)} + \nabla\times A^{(2)} \tag{4.51}$$

$$- i\frac{\kappa}{A^{(0)}}A^{(1)}\times A^{(2)} = B^{(1)} + B^{(2)} + B^{(3)}.$$

Equation (4.51) is the vector form of the general definition (4.12),

$$G_{\mu\nu} := \partial_\mu A_\nu^{(1)} - \partial_\nu A_\mu^{(1)} + \partial_\mu A_\nu^{(2)} - \partial_\nu A_\mu^{(2)}$$

$$- i\frac{\kappa}{A^{(0)}}A_\mu^{(1)}\times A_\nu^{(2)}. \tag{4.52}$$

Under gauge rotation,

$$B^{(1)} \rightarrow e^{i\Lambda}B^{(1)},\qquad B^{(2)} \rightarrow e^{-i\Lambda}B^{(2)},$$

$$B^{(3)} \rightarrow B^{(3)}, \tag{4.53}$$

and all three components transform covariantly with the three components of A. Gauge rotation has no direct effect on $B^{(3)}$ and $A^{(3)}$, but introduces the topological phases $e^{i\Lambda}$ and $e^{-i\Lambda}$ into the transverse components. This can be identified with the Berry phase, also introduced by Hunter and Wadlinger [2], and has been measured empirically. The optical Aharonov-Bohm effect is caused by the conjugate product $\nabla\Lambda^{(1)}\times\nabla\Lambda^{(2)}$ which appears as the result of gauge rotation and which represents the effect of $B^{(3)}$ in regions where $B^{(3)}$ itself is excluded experimentally. This parallels the original Aharonov-Bohm effect [6].

In summary, covariant gauge transformation of $\nabla\times A^{(1)}$ and $\nabla\times A^{(2)}$ is possible self-consistently only if $B^{(3)}$ is introduced through the covariant derivatives and their cross product (commutator). This is a fundamental topological argument for the existence of $B^{(3)}$ and the optical equivalent of the non-local Aharonov-Bohm effect. The gauge transform of type two on the electromagnetic field introduces the topological phase into the transverse components of the field, and this same process introduces the locally gauge invariant $B^{(3)}$. Without $B^{(3)}$ we cannot gauge the electromagnetic field self-consistently in spacetime. In simpler language, rotation of the $A^{(1)} = A^{(2)*}$ vector through an angle Λ_Z requires the existence of an axis about which this rotation takes place, and this is the axis of definition of the $B^{(3)}$ field.

4.3 The Cyclic Structure of Vacuum Electromagnetism: Quantization and Derivation of Maxwell's Equations

Starting from the A cyclics, Eq. (4.2), the Faraday and Ampère equations are derived in this section in quantized form, these being two of the Maxwell equations. The third A cyclic can be quantized self-consistently using the same operators and de Broglie wave-function. This method shows that if $B^{(3)} =? \mathbf{0}$ the Maxwell equations vanish. The second and third equations of the A cyclics can be quantized to give two of the vacuum Maxwell equations: the same method self-consistently quantizes the first equation of the A cyclics (4.2) and shows that there is no Faraday induction law for $B^{(3)}$. Consistently, no Faraday induction has been observed in a circularly polarized laser beam modulated inside an evacuated induction coil [4—6]. In this method, $A^{(3)}$ quantizes to the $\hbar\partial/\partial Z$ operator and is not zero. If set to zero, all three A cyclics vanish, and with them the Maxwell equations. The Maxwell equations for $B^{(1)} = B^{(2)*}$ imply the existence of $B^{(3)}$, and if the latter is set arbitrarily to zero, the Maxwell equations vanish. Finally the method allows direct quantization of the A cyclics to the Maxwell equations, which become equations of the quantum field theory. The method is therefore direct, simple, and easy to interpret.

Write Eq. (4.2b) as the classical eigenvalue equation,

$$-A^{(3)}\times A^{(2)} = iA^{(0)}A^{(2)}. \tag{4.54}$$

Use the minimal prescription in the form [4—6],

$$p^{(3)} = ieA^{(3)},\qquad p^{(0)} = i\,eA^{(0)}, \tag{4.55}$$

and identify $A^{(2)}$ with the classical eigenfunction $\Psi^{(2)}$. Here e is the elementary charge. This procedure results in the classical equation,

$$-p^{(3)}\times\Psi^{(2)} = ip^{(0)}\Psi^{(2)}, \tag{4.56}$$

and the vector potential has taken on a dual role of operator and function in a classical eigenequation. Its ability to do this springs from the duality transform $A \rightarrow iA$ [11—14] in the complex three space ((1), (2), (3)). Therefore if iA is a polar vector multiplied by i, then A is an axial vector. The same duality transform takes the axial vector $\boldsymbol{B}$ to $i\boldsymbol{E}/c$, a polar vector multiplied by i. The fact that A is both polar and axial signifies that electromagnetism is chiral, with two enantiomeric forms — right and left circularly polarized [1]. Chirality in Dirac algebra becomes the eigenvalues of the γ_5 operator, playing the role of i in Pauli algebra [3]. This dual polar-axial nature of A allows it to be both an operator (polar vector) and function (axial vector).

The classical eigenvalue Eq. (4.56) is now quantized with the correspondence principle, whose operators $p^{(3)} \rightarrow i\hbar\frac{\partial}{\partial Z}$ and $p^{(0)} \rightarrow -i\frac{\hbar}{c}\frac{\partial}{\partial t}$ act on a wavefunction in our complex three space. Let this wavefunction be [15]

$$\Psi^{(2)} = c\boldsymbol{B}^{(2)} - i\boldsymbol{E}^{(2)}, \tag{4.57}$$

as used by Majorana. Here c is the speed of light in vacuo, $\boldsymbol{B}$ is magnetic flux density and $\boldsymbol{E}$ is electric field strength. The function (4.57) includes the electromagnetic phase in the form of the scalar de Broglie wavefunction [15], and it is understood that the operators introduced by the correspondence principle operate on this. Therefore the operators $p^{(3)}$ and $p^{(0)}$ are phase free, the function $\Psi^{(2)}$ is phase dependent. The quantum field equation derived in this way from the classical equation (4.56) is

$$\nabla \times \left(c\boldsymbol{B}^{(2)} - i\boldsymbol{E}^{(2)}\right) = \frac{i}{c}\frac{\partial}{\partial t}\left(c\boldsymbol{B}^{(2)} - i\boldsymbol{E}^{(2)}\right). \tag{4.58}$$

Compare real parts to give an equation of *quantized* field theory in the form of Ampère's law modified by Maxwell's vacuum displacement current,

$$\nabla \times \boldsymbol{B}^{(2)} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(2)}}{\partial t}. \tag{4.59}$$

Compare imaginary parts to give an equation of quantized field theory in the form of Faraday's law of induction,

$$\nabla \times \boldsymbol{E}^{(2)} = -\frac{\partial \boldsymbol{B}^{(2)}}{\partial t}. \tag{4.60}$$

Equations (4.59) and (4.60) are two of the four-vacuum Maxwell equations, but have been derived through the correspondence principle and are therefore also equations of the quantum field theory. These take the same form as the classical Ampère-Maxwell and Faraday laws but are also equations of a novel, fully relativistic field theory.

Similarly, Eq. (4.2c) quantizes to

$$\nabla \times \boldsymbol{B}^{(1)} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(1)}}{\partial t}, \tag{4.61}$$

$$\nabla \times \boldsymbol{E}^{(1)} = -\frac{\partial \boldsymbol{B}^{(1)}}{\partial t}. \tag{4.62}$$

4.3.1 The d'Alembert Equation, Lorentz Condition and Acausal Energy Solution

The dual nature of the vector potential, once recognized, leads immediately to the d'Alembert equation, because A_μ is light-like. Therefore,

$$A_\mu A^\mu = 0, \tag{4.63}$$

and taking the operator definition this becomes the d'Alembertian operating on a wavefunction in spacetime, i.e.,

$$\partial_\mu \partial^\mu \psi_\nu = \Box \psi_\nu = 0. \tag{4.64}$$

This is the quantized d'Alembert equation written for the four-vector ψ_ν. The latter in general has a space-like and time-like component. In this view A_μ must be a polar four-vector proportional to the generator of spacetime translations, and so the d'Alembert equation (4.64) is the first (mass) Casimir invariant of the Poincaré group [1]. The invariant is zero because we have assumed that c is the speed of light and have taken photon mass to be zero.

If, in the condition $A_\mu A^\mu = 0$, we take the first A_μ as an operator through the correspondence principle, and interpret the second A^μ as a wavefunction ψ^μ, we obtain the quantized Lorentz condition for a massless particle,

$$\partial_\mu \psi^\mu = 0. \tag{4.65}$$

This is the orthogonality condition of the Poincaré group, which states that A_μ in operator form is orthogonal to A^μ in function form. The latter becomes the Pauli- Lubanski axial four-vector of the Poincaré group [4,1]. (See also Chap. 8).

The condition $A_\mu A^\mu = 0$ interpreted as a condition on the wavefunction, gives the acausal energy condition,

$$\psi_\mu \psi^\mu = 0, \tag{4.66}$$

which is the second (spin) invariant of the Poincaré group. Therefore we are dealing with a quantized particle with spin described by the three A cyclics (4.2). Evidently, this is the photon of the new relativistic quantum field theory developed here. The empirical evidence for the existence of this photon can be traced to the magneto-optical evidence for $\boldsymbol{B}^{(3)}$ in the inverse Faraday effect [4—6] and other effects. Without $\boldsymbol{B}^{(3)}$, this photon is undefined.

Finally, the energy condition (4.66) is the acausal solution [16—21]. It is longitudinal because the Pauli-Lubanski four-vector ψ_μ can be expressed in terms of the purely longitudinal [4],

$$\psi^{(3)} = cB^{(3)} + iE^{(3)}, \tag{4.67}$$

in the vacuum.

4.3.2 Self-Consistent Quantization of Equation (4.2a)

The quantization of Eq. (4.2a) occurs in a self-consistent way using the same operator interpretation of $iA^{(0)}$ and $iA^{(3)} = -iA^{(3)*}$. This gives the relativistic Schrödinger equation,

$$\frac{i}{c}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial Z}\psi_0\right) = \left(\frac{e}{\hbar}A^{(0)}\right)^2\psi_0\,, \tag{4.68}$$

where ψ_0 is the scalar de Broglie wavefunction [4—6],

$$\psi_0 = \exp\left(i\phi\right), \tag{4.69}$$

where $\phi = \omega t - \kappa Z$ is the electromagnetic phase. Here ω is the angular frequency at an instant t and κ the wavevector at point Z as usual. Using the vacuum minimal prescription [4—6],

$$eA^{(0)} = \hbar\kappa\,, \tag{4.70}$$

it is seen that Eq. (4.68) is self-consistent and consistent with the correspondence principle in the form (4.55). The method used to transform the second and third A cyclics into the Maxwell equations gives a fully consistent Schrödinger equation for the third cyclic. In this method $iA^{(3)}$ is clearly not zero, and since $\boldsymbol{B}^{(3)} = \kappa A^{(3)}$, neither is $\boldsymbol{B}^{(3)}$. If we try to set $iA^{(3)}$ to zero the del operator vanishes along with all three A cyclic equations. The

Maxwell equations themselves vanish if we try $\boldsymbol{B}^{(3)} =?\ \boldsymbol{0}$. There is no vacuum Faraday induction law involving $\boldsymbol{B}^{(3)}$, because of the structure of Eq. (4.2), and this is again consistent with the experimental finding that there is no Faraday induction in a coil wound around a modulated monochromatic laser beam propagating in a vacuum [4—6]. The fundamental reason for this is that $\boldsymbol{B}^{(3)}$ is an unchanging property of one photon, i.e., $\hbar/e$ divided by the photon area.

The duality transform $\boldsymbol{A} \to i\boldsymbol{A}$ in the vacuum shows that $\boldsymbol{A}$ can act as an operator and as a function. This transforms two of the A cyclics into two of the Maxwell equations in fully quantized form, producing a new quantum field theory for the photon, which acquires in the process three degrees of polarization. The first equation (4.2a) of the A cyclics is quantized self-consistently. The structure of these equations shows that there is no Faraday induction law for $\boldsymbol{B}^{(3)}$, as observed experimentally. The explanation of magneto-optical phenomena [4—6] requires the use of the conjugate product; a product which demonstrates the existence of $iB^{(0)}\boldsymbol{B}^{(3)*}$ in the vacuum, and therefore of $\boldsymbol{B}^{(3)}$. Since $\boldsymbol{B}^{(3)}$ is $\kappa\boldsymbol{A}^{(3)}$, an attempt to set $\boldsymbol{A}^{(3)}$ to zero removes the three equations of the A cyclics, and so removes the Maxwell equations themselves. Therefore the A and B cyclics become fundamental classical structures from which the Maxwell equations can be *derived* in quantized form using the correspondence principle.

There are clear differences between this theory of electrodynamics and the received theory.

(1) The Maxwell equations are no longer the fundamental classical equations, they can be simultaneously derived and quantized from a more fundamental classical structure in which $\boldsymbol{B}$ and the axial $\boldsymbol{A}$ are infinitesimal rotation generators of O(3).

(2) The potential four-vector A_μ is fully covariant and has four non-zero components inter-related as in Eqs. (4.2). The older view allows a non-covariant A_μ such as the Coulomb gauge.

(3) The quantized d'Alembert equation becomes the first Casimir invariant of the Poincaré group; the quantized Lorentz condition becomes an orthogonality condition; and the quantized acausal energy condition becomes the second Casimir invariant. These results can be derived from the fact that A_μ plays the dual role of operator and function. Since $\boldsymbol{A}^{(3)}$ is directly proportional to $\boldsymbol{B}^{(3)}$ it is gauge invariant; a property which is consistent with the fact that the cross product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is gauge invariant [5] in the Poincaré group, but not in the U(1) group of the received view.

The Maxwell equations become derivative equations of a cyclical structure for electromagnetism in the vacuum. A similar result can be derived for the equations in the presence of sources (charges and currents).

4.4 The Cyclic Structure of Vacuum Electromagnetism: the Lorentz Equation

In order to complete the derivation of the equations of electromagnetism from the structure of space-time, consideration needs to be taken of the Lorentz equation, in which the Lorentz force was introduced as an empirical construct. In this section (where we use Minkowski notation) the Lorentz equation is shown to be a structural property of space-time, and can be derived within the Poincaré group, again using the concept of Pauli-Lubanski pseudo-vector. The Lorentz equation is shown to be identical in structure to the definition of relativistic helicity, proportional to a conserved current,

$$\zeta_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G_{\alpha\beta}A_\nu . \tag{4.71}$$

The similarity in structure to Eq. (4.71) of the standard Lorentz force equation is evident when we write the latter as [1]

$$\frac{dp_\mu}{d\tau} = \frac{e}{m}G_{\mu\nu}p_\nu , \tag{4.72}$$

where e/m is the charge to mass ratio of the electron, and p_μ the relativistic four-momentum. Here τ is the proper time, a Lorentz invariant scalar. Equation (4.72) is for the interaction of the electron with electromagnetic radiation, but is structurally the same as the definition (4.71) of the relativistic helicity. This suggests that a form of the Lorentz equation can be derived for the photon in vacuum, a form in which the gyromagnetic ratio becomes $c/A^{(0)}$ as argued in earlier chapters.

To make this surmise more precise, consider the electromagnetic torque density introduced in Vol. 3,

$$\begin{aligned} \boldsymbol{T}_V^{(3)*} &= -\frac{i}{\mu_0}\boldsymbol{B}^{(1)}\times\boldsymbol{B}^{(2)} = \frac{B^{(0)}}{\mu_0}\boldsymbol{B}^{(3)*} \\ &= \omega\boldsymbol{J}_V^{(3)*}, \end{aligned} \tag{4.73}$$

where $\boldsymbol{J}_V^{(3)*}$ is the angular momentum density of vacuum radiation. Introduce the four-current (C s^{-1})

$$j_\mu := \frac{4\pi}{\mu_0}A_\mu , \tag{4.74}$$

where there is a translational and rotational four-potential,

$$\begin{aligned} A_\mu^{(T)} &:= -iA^{(0)}(0,0,1,i) , \\ A_\mu^{(R)} &:= A^{(0)}(0,0,1,i) , \end{aligned} \tag{4.75}$$

as defined in the foregoing sections of this chapter. The Pauli-Lubanski pseudo four-vector V_μ is defined [4] by

$$A^{(0)}V_\mu := \tilde{G}_{\mu\nu}A_\nu^{(T)} , \tag{4.76}$$

in terms of the dual of the field strength tensor $\tilde{G}_{\mu\nu}$ and the potential $A^{\nu(T)}$. The dual tensor is defined as (Chap. 8),

$$\tilde{G}_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma} . \tag{4.77}$$

The electromagnetic torque density (4.73) is therefore defined by

$$T_\mu = \frac{\omega}{4\pi c}\tilde{G}_{\mu\nu}j_\nu , \tag{4.78}$$

which becomes

$$T_\mu = \omega J_\mu = \frac{\omega}{8\pi c}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}j_\nu , \tag{4.79}$$

in units of J m^{-3}. Using Eq. (4.74), the angular momentum density in this notation becomes

$$J_\mu = \frac{1}{2\mu_0 c}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}A_\nu , \tag{4.80}$$

and is closely related to the relativistic electromagnetic helicity,

$$\zeta_\mu := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}A_\nu . \tag{4.81}$$

It is seen that the electromagnetic torque and angular momentum densities and the relativistic helicity, ζ_μ, are each Pauli-Lubanski pseudo-vectors. Therefore,

$$T_\mu = \frac{eA^{(0)}}{4\pi\hbar}\tilde{G}_{\mu\nu}j_\nu , \tag{4.82}$$

is a Lorentz torque equation in radiation or matter. The gyromagnetic ratio in matter is given by the ratio e/m, where m is particle mass; but in electromagnetic radiation, this ratio becomes $c/A^{(0)}$, identified in earlier chapters as the gyromagnetic ratio of the photon.

It is now shown that cyclic relations among A components of the type (4.2) emerge self-consistently from the definition of the Pauli-Lubanski vectors used in this analysis. The overall conclusion, therefore, is that the underlying structure of the Lorentz force equation can be obtained from the structure of space-time. The required result is obtained from the following structure,

$$\begin{bmatrix} V_X \\ V_Y \\ V_Z \\ iV_0 \end{bmatrix} = -i \begin{bmatrix} 0 & -iE_Z & iE_Y & cB_x \\ iE_Z & 0 & -iE_X & cB_Y \\ -iE_Y & iE_X & 0 & cB_Z \\ -cB_X & -cB_Y & -cB_X & 0 \end{bmatrix} \begin{bmatrix} E_Y + cB_X \\ -E_X + cB_Y \\ cB_Z \\ icB_Z \end{bmatrix}, \tag{4.83}$$

which is the expanded form of Eq. (350) of Vol. 1. For a circularly polarized plane wave propagating in vacuo the first two entries of the right hand side column vector vanish, and the complete column vector is proportional to the translational potential four-vector defined in Eq. (4.75). The column vector on the left hand side of this equation is proportional to the rotational potential four-vector defined in Eq. (4.75). Therefore the first two entries of the column vector on the left hand side are also zero. Using the plane wave relations

$$E_Y = -cB_X, \qquad E_X = cB_Y, \tag{4.84}$$

it is found that the product on the right hand side of Eq. (4.83) is made up of cyclic relations such as the B and E cyclics used throughout these volumes. This is not surprising because by definition, the Pauli-Lubanski relation is a cyclic relation in four dimensions rather than three.

It has therefore been shown that the structure of both the Maxwell and Lorentz equations is the structure of the Pauli-Lubanski vector in four dimensions, and therefore part of the defining algebra of the Poincaré group of space-time. Therefore there is no necessity to regard the Lorentz equation as a separate empirical equation of motion.

References

[1] P. W. Atkins, *Molecular Quantum Mechanics* (Oxford University Press, Oxford, 1983). For a relativistic treatment see L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th edn. (Pergamon, Oxford, 1975).

[2] G. Hunter and R. P. L. Wadlinger, *Phys. Essays* 2(2), 158 (1989).

[3] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).

[4] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $\boldsymbol{B}^{(3)}$ (Kluwer Academic, Dordrecht, 1994).

[5] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).

[6] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).

[7] M. W. Evans, *Physica B* **182**, 227, 137 (1992); **183**, 103 (1993); *Physica B* **190**, 310 (1993); *Physica A* **214**, 605 (1995).

[8] M. W. Evans and S. Kielich, eds., *Modern Nonlinear Optics*, Vol. 85(2) of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, eds. (Wiley Interscience, New York, 1993).

[9] M. W. Evans, *The Photon's Magnetic Field* (World Scientific, Singapore, 1992); A. A. Hasanein and M. W. Evans, *The Photomagneton in Quantum Field Theory* (World Scientific, Singapore, 1994).

[10] M. W. Evans, *Found. Phys.* **24**, 892, 1519, 1671 (1994); **25**, 175, 383 (1995); M. W. Evans and S. Roy, *Found. Phys.*, in press.

[11] W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism.* (Addison-Wesley, Reading, Mass., 1962).

[12] B. L. van der Waerden, *Group Theory and Quantum Mechanics* (Springer-Verlag, Berlin, 1974).

[13] J. M. Normand, *A Lie Group: Rotations in Quantum Mechanics* (North-Holland, Amsterdam, 1980).

[14] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).

[15] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[16] D. V. Ahluwalia and D. J. Ernst, *Mod. Phys. Lett.* **A7**, 1967 (1992).

[17] V. V. Dvoeglazov, *Int. J. Theor. Phys.* **35**, 115 (1996).

[18] B. Lehnert, *Phys. Scripta* **53**, 204 (1996).

[19] A. Chubykalo and R. Smirnov-Rueda, *Phys. Rev. E*, **53**, 5373 (1996).

[20] E. Gianetto, *Lett. Nuovo Cim.* **44**, 140 (1985).

[21] M. Israelit, *Magneto Monopoles and Massive Photons in a Weyl Type Electrodynamics*, LANL Preprint 9611060 (1996).

Chapter 5. Practical Advantages of $B^{(3)}$ in Atomic and Molecular Spectroscopy

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The aim of this chapter is to describe the potential advantages of $B^{(3)}$ spectroscopy with the $B^{(3)}$ field in atoms and molecules. The major advantage is to be found in the proportionality of $B^{(3)}$ to I/ω^2, where I is the beam intensity and ω the angular frequency. In Chap. 1 and 2 of Vol. 3 it was shown that this advantage produces proton and electron resonance at frequencies very much higher than those available with contemporary apparatus, based on homogeneous superconducting magnets. This can be achieved with a pump beam homogeneous to the order of 10%, leading to nuclear and electronic resonance at infra-red or visible frequencies. In general, the $B^{(3)}$ field can interact with spin or orbital angular momentum, whose Z component can be defined by [1]

$$J_Z = L_Z + S_Z , \qquad S_Z = \hbar \frac{\sigma_Z}{2} , \tag{5.1}$$

so that the total magnetic dipole moment is

$$\boldsymbol{m}_Z = \gamma_L L_Z + \gamma_S S_Z . \tag{5.2}$$

The strong coupling between the conjugate product of the pump laser (Π) and the nuclear spin angular momentum $(\boldsymbol{I})$ was first expressed [2] as

$$\Delta H_{Em} = -\gamma_{Em} \boldsymbol{I} \cdot \Pi . \tag{5.3}$$

In Eqs. (5.2) and (5.3) γ_L, γ_S, and γ_{Em} are the appropriate gyromagnetic ratios and $\boldsymbol{L}$, $\boldsymbol{S}$, and $\boldsymbol{I}$ are the orbital and spin electronic, and nuclear spin angular momenta

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respectively. In general, $B^{(3)*} = -i(e/\hbar)\, A^{(1)} \times A^{(2)}$ is the phase free magnetic flux density for the electromagnetic radiation and is responsible for observable magneto-optic effects. In the non-relativistic approximation, the spin component of the optical Zeeman effect, for example, is described by the interaction energy,

$$\Delta H_S = \frac{e}{2m}\left(2 S^{(3)} \cdot B^{(3)*}\right). \tag{5.4}$$

This energy is generated by the interaction of the circularly polarized electromagnetic field with the spin angular momentum $S^{(3)}$ of an electron, or a spare electron in an atom. Equation (5.4) is therefore an example of a convenient calculation recipe in which the contribution of the electromagnetic field is always defined through

$$\begin{aligned} B^{(0)} &:= |B^{(3)}| = \frac{e}{\hbar} A^{(0)2} = \frac{e}{\hbar}\frac{c^2}{\omega^2} B^{(0)2} \\ &= \frac{e \mu_0 c}{\hbar} \frac{I}{\omega^2} = \left(4\pi \frac{\alpha}{e}\right) \frac{I}{\omega^2}, \end{aligned} \tag{5.5}$$

where

$$\alpha = \frac{e^2}{4\pi\hbar c \epsilon_0} = \frac{\mu_0 c e^2}{4\pi\hbar}, \tag{5.6}$$

is the fine structure constant $\left(\alpha = 7.297351 \times 10^{-3}\right)$. The only beam properties present in this formula are therefore I, the power density or intensity (W m^{-2}); and ω, the angular frequency (radians per second); α being a fundamental constant. In a Cartesian basis,

$$B^{(3)} := \left(4\pi \frac{\alpha}{e}\right) \frac{I}{\omega^2} k, \tag{5.7}$$

is a magnetic flux density in the Z axis, in which k is a unit vector. By maintaining the intensity (I) constant and using a low frequency beam the effects of $B^{(3)}$ can be amplified considerably over the equivalent at visible frequencies for any magneto-optic phenomenon in the non-relativistic limit. More generally, the interaction of electromagnetic radiation with matter can be treated in terms of $B^{(3)}$ when it comes to considering optical effects independent of phase.

In a fully relativistic treatment, the expectation value of the spin part of the interaction energy between fermion and classical field can be written as

$$\langle H_S \rangle = \frac{i e^2 c}{2 m_0 c + e A^{(0)}} \sigma^{(3)} \cdot A^{(1)} \times A^{(2)}, \tag{5.8}$$

and this determines the optical Zeeman and related effects from the spin term of the Dirac [3] equation. The limit of Eq. (5.4) is attained when $mc \gg ecB^{(0)}/2\omega$, a visible frequency limit [4]. In the opposite, radio-frequency, limit represented by $\omega \ll eB^{(0)}/2m$, Eq. (5.8) reduces to

$$\begin{aligned} \langle H_S \rangle &\rightarrow i \frac{ec}{A^{(0)}} \sigma^{(3)} \cdot A^{(1)} \times A^{(2)} \\ &= ec\, \sigma^{(3)} \cdot \frac{B^{(0)}}{\omega} e^{(3)*} \end{aligned} \tag{5.9}$$

Therefore in the high and low frequency limits there is respectively an inverse square and inverse frequency dependence of magneto-optic effects for a given I.

It has been assumed in the above development that beam intensity in these effects is such that it can be varied experimentally as well as frequency, so they become independently available variables. This is indeed the case experimentally, for example a 400 watt radio frequency generator is available commercially with a range from megahertz to gigahertz; while a visible frequency argon ion laser produces perhaps one watt of power. The latter is therefore much smaller at visible than at radio frequencies. For a given temperature, the beam intensity is governed by the Planck law, and for a coherent, monochromatic beam, the photon number states are Glauber states, as discussed in Vol. 1. For one photon in one quantum state of an electromagnetic oscillator, the charge quantization condition [5],

$$eA^{(0)} = \hbar\kappa, \tag{5.10}$$

can be applied to Eq. (5.9) to obtain the energy transfer equation.

$$\langle H_S \rangle = -\hbar\omega\, \sigma^{(3)} \cdot e^{(3)*}, \tag{5.11}$$

and in this case we see that the rotational photon energy has been transferred completely to the electron in the Zeeman effect.

5.1 Origin of the Conjugate Product in the Dirac Equation

The Dirac equation for a fermion in the classical electromagnetic field must produce the key inverse Faraday effect in the classical limit, i.e., when the spinor term is *missing*. In this limit the Dirac equation becomes the quantum wave equation [6],

$$\left(En - mc^2 + ecA_0\right)\psi = \left(\frac{1}{2m}\left(p + eA^*\right)\cdot\left(p + eA\right)\right)\psi, \tag{5.12}$$

where ψ is the wave function, $En - mc^2$ is the energy operator of the fermion, here mc^2 is its rest energy, m its mass and p its momentum. Unless A is complex, i.e., has the roto-translational character, for example, of a propagating plane wave, the inverse Faraday effect cannot be obtained from Eq. (5.12). The reason is that the dot product of A with its conjugate product is phase independent, as required for the static magnetization observed in the inverse Faraday effect, described by the term,

$$\hat{W}\psi \sim \frac{e^2}{2m}A\cdot A^*\psi, \qquad A\cdot A^* = A^{(1)}\cdot A^{(2)}. \tag{5.13}$$

If we now regard $A^{(1)}$ and $A^{(2)}$ as rotational in nature, the dot product $A^{(1)}\cdot A^{(2)}$ produces an eigenvalue $\frac{e^2c^2}{2m\omega^2}B^{(0)2}$, which is the interaction energy of the inverse Faraday effect, inversely proportional to the square of angular frequency. Therefore the Dirac equation produces the inverse Faraday effect for an electron in a classical electromagnetic field through the use of a complex $A^{(1)} = A^{(2)*}$. The same equation produces the spinor triple product $\sigma^{(3)}\cdot A^{(1)}\times A^{(2)}$ which is at the root of electron or proton spin resonance at infra red or visible frequencies [6]. Therefore the conjugate cross product $A^{(1)}\times A^{(2)}$ originates in the structure of the Dirac equation, and forms a non-zero interaction energy with the spinor. This is a first order interaction between $B^{(3)}$ and the permanent spin angular momentum of the electron (or proton).

Using Eq. (85) of Vol. 3, the energy eigenvalue from the Dirac equation of one fermion in the classical electromagnetic field can be written, using the relevant interaction terms as

$$\Delta W_{IFE} := En - mc^2 \sim -\frac{e^2}{2m}\left(A^{(1)}\cdot A^{(2)} + i\sigma^{(3)}\cdot A^{(1)}\times A^{(2)}\right), \tag{5.14}$$

where $\sigma^{(3)}$ is the third Dirac spinor in the basis ((1), (2), (3)). It can be shown as follows that this energy can be written as

$$\Delta W_{IFE} \sim \frac{e}{2m}\left(L^{(3)} + 2S^{(3)}\right)\cdot B^{(3)*}, \tag{5.15}$$

where $L^{(3)}$ and $S^{(3)}$ are respectively orbital and spin angular momenta. Therefore this calculation shows that the inverse Faraday effect can be written in the same way precisely as the Zeeman effect. The latter is part of the Faraday effect [3] as shown originally by Serber [8]. This result means that for a given magnetic flux density, $\boldsymbol{B}$, the Verdet constants for the forward and inverse Faraday effect must be the same. This was first shown experimentally by van der Ziel *et al.* [9], and therefore it follows that $B^{(3)}$ is a magnetic flux density, not an effective operator as in the received view [9].

The demonstration of Eq. (5.15) proceeds by using the operator identity developed in Vol. 3 [6],

$$A^{(1)}\cdot A^{(2)} + i\sigma^{(3)}\cdot A^{(1)}\times A^{(2)} = \left(\sigma^{(1)}\cdot A^{(2)}\right)\left(\sigma^{(2)}\cdot A^{(1)}\right), \tag{5.16a}$$

where the Pauli spinors are expressed in the basis ((1), (2), (3)). Equation (5.16a) shows that the dot product and cross product of $A^{(1)}$ with $A^{(2)}$ have the same topological origin. The dot product is now re-expressed as

$$A^{(1)}\cdot A^{(2)} = -ie^{(3)}\cdot A^{(1)}\times A^{(2)}, \tag{5.16b}$$

whereupon the energy from the Dirac equation becomes,

$$\Delta W_{IFE} = -i\frac{e^2}{2m}\left(e^{(3)} + \sigma^{(3)}\right)\cdot A^{(1)}\times A^{(2)} = \frac{e}{2m}\hbar\left(e^{(3)} + \sigma^{(3)}\right)\cdot B^{(3)*}. \tag{5.17}$$

Finally, this is re-expressed in the standard notation as

$$\Delta W_{IFE} = \frac{e}{2m}\left(L^{(3)} + 2S^{(3)}\right)\cdot B^{(3)*}, \tag{5.18}$$

using

$$L^{(3)} := \hbar e^{(3)}, \qquad S^{(3)} := \hbar \frac{\sigma^{(3)}}{2} . \tag{5.19}$$

Therefore the Dirac equation gives the interaction between a fermion such as an electron or proton in terms of the electromagnetic field, represented entirely by $B^{(3)*}$, and the orbital and spin angular momenta,

$$L^{(3)}(\text{orbital}), \qquad S^{(3)}(\text{spin}) . \tag{5.20}$$

The contribution of the electron (or proton) to this equation is given by the charge to mass ratio e/m; and the angular momentum $\hbar$ is donated by the electromagnetic field in the simplest case of one photon (one energy level of an electromagnetic oscillator). The quantity $\hbar$ enters through the equation $B^{(3)*} := -i(e/\hbar) A^{(1)} \times A^{(2)}$ defining the electromagnetic field's $B^{(3)*}$.

The Dirac equation therefore leads in a direct way to the equation (5.18), the equation of the Zeeman effect in its simplest form. Therefore the quantity $B^{(3)*}$ appearing in Eq. (5.18) is a magnetic flux density, not merely an operator that is an *effective* magnetic field. There is a difference however, between the meaning of $\hbar$ in Eq. (5.18) and its meaning when an electron is acted upon by an ordinary static magnetic field, as in the ordinary Zeeman effect discovered in Na vapor in 1896 [10]. In the treatment of the Zeeman effect and the Gerlach Stern experiment through the Dirac equation, $\hbar$ must be introduced through the correspondence principle of quantum mechanics [7], and the magnetic field must be defined as the curl of vector potential A. In deriving equation (5.18), however, $\hbar$ has been introduced as a property of the electromagnetic field through, the defining equation of $B^{(3)*}$, which is radiated and electromagnetic in nature. Usually, the spin angular momentum S is described as the half-integral spin of the fermion, but more accurately it is the topological quantity $\sigma/2$ multiplied by the Dirac constant $\hbar$. The latter does not actually originate in the fermion itself, but is introduced as a calculation recipe of quantum mechanics. It would be more accurate to say that the fermion interacts with the static magnetic field through $\hbar$. When the fermion interacts with the electromagnetic field through equation (5.18), $\hbar$ is introduced from the field as we have just argued. In this sense, $S^{(3)}$ and $L^{(3)}$ are both induced in the electron. The difference between them is that $S^{(3)}$ is described by the topological quantity $\sigma^{(3)}/2$, which has two topological states; whereas $L^{(3)}$ is described by the topological quantity $e^{(3)}$ with one state only. Resonance can occur between the two states of the spinor $\sigma^{(3)}$ but cannot occur through the agency of $e^{(3)}$.

Therefore Eq. (5.18) is anything but a trivial re-writing of the Dirac equation, because it defines the fundamentals of field fermion interaction. The equation (5.18) includes a factor $e/(2m)$ because it has used a particular approximation of the fully

relativistic expression given in Vol. 3 [6]. The factor 2 in the denominator of $\sigma^{(3)}/2$ has an exact, topological origin on the other hand, because the commutator of spinors is [6,7]

$$\left[\frac{\sigma^{(1)}}{2}, \frac{\sigma^{(2)}}{2}\right] = -\frac{\sigma^{(3)+}}{2}, \qquad \text{et cyclicum} . \tag{5.21}$$

These careful re-appraisals cannot be appreciated fully without using the Dirac equation. Clearly, the latter has been used here for the simplest case of one electron or one proton in the classical electromagnetic field, but the end result, Eq. (5.18), has the same form exactly as the equation of the Zeeman effect routinely used in textbooks for the interaction of a static magnetic field with atoms and molecules, as well as one fermion. In the next section, Eq. (5.18) is extended, approximately, to atoms and molecules.

5.2 Extension to Atoms and Molecules

The extension of the above one electron theory to atoms and molecules is a non-trivial task in general because the Dirac equation must be solved for many interacting electrons and protons in the presence of an electromagnetic field. An approximate approach is well worth trying, however, because RF ESR and RF NMR are expected to lead to a richer spectral structure than their conventional counterparts. At the rigorous level, quantum electrodynamics should be implemented both in the preceding one electron theory (or one proton theory) and in the atomic and molecular theory. In the semi-classical approach the electromagnetic field is treated classically, and the Dirac equation used to involve the all important spinor term, without which there is no resonance and no RF ESR or RF NMR spectra. The calculations should produce an indication of the way in which the nucleus and other electrons in an atom shifts the free electron resonance given by Eq. (5.18).

The latter can be re-expressed in spectroscopic notation by using the standard expressions [7],

$$\gamma_e = -\frac{\mu_B}{\hbar}, \quad \mu_B = \hbar\frac{e}{2m}, \quad g_e = 2(\text{Dirac}),$$
$$g_e = 2.002\,(q.e.d.) , \tag{5.22}$$

where γ_e is the magneto-gyric ratio; μ_B is the Bohr magneton; and g_e is the Landé factor of the electron. This is $g_e = 2$ from the Dirac equation and is modified to 2.002 by quantum electrodynamics, as is well known. As in standard ESR [7] spin-orbit coupling in atoms and molecules is a field independent internal term which exists in the absence of any external perturbation, but which shifts the free electron resonance [7]. This shift is also expected to be present in RF ESR, along with shifts caused by spin spin coupling, but in the following development is temporarily left out of consideration for the sake of simplicity and clarity.

The $B^{(3)}$ field is always defined in terms of the conjugate product of the electromagnetic field,

$$B^{(3)*} = -i\frac{e}{\hbar}A^{(1)} \times A^{(2)}, \tag{5.23}$$

and the calculation of RF ESR in an atom or molecule proceeds in time-dependent perturbation theory by expressing the interaction between $A^{(1)} \times A^{(2)}$ and its anti-symmetric polarizability [11], α''. From the one electron result, Eq. (5.18), α'' is defined as follows. The first term on the right hand side of Eq. (5.18) can be developed as

$$W_{IFE} = \left(\frac{e^2}{2m\hbar\omega^2}\right)\left(L^{(3)} + 2S^{(3)}\right) \cdot \left(-iE^{(1)} \times E^{(2)}\right), \tag{5.24}$$

so the antisymmetric polarizability is

$$\alpha''(\text{free electron}) = \frac{e^2}{2m\omega^2}. \tag{5.25}$$

For one electron this is an exact result, one which is comprised of a combination of electron properties e and m and a field property ω. It is a property which premultiplies the conjugate product of electric fields $E^{(1)} \times E^{(2)}$; its characteristic inverse square ω dependence arises from the relation between $A^{(1)}$ and $E^{(1)}$,

$$A^{(1)} = \frac{E^{(1)}}{\omega^2}. \tag{5.26}$$

At radio frequencies (order of 10^8 rad / sec), ω^2 is fifteen orders smaller than at visible frequencies (order of 10^{15} rad / sec), and so the antisymmetric polarizability is amplified by fifteen orders of magnitude for constant beam power density. RF ESR aims to take advantage of this amplification by irradiating atomic and molecular samples with a circularly polarized radio frequency field of the order of 1 to 100 watts per square centimeter. Such a beam is available commercially, the only slightly unusual specification being that it must be accurately circularly polarized with a device such as a log spiral antenna. Radio frequency generators are available commercially over the complete radio frequency range with power levels of the order 500 watts or more, and so RF ESR is feasible.

As discussed in Chap. 2 of Vol. 3 [6], resonance occurs in RF free electron spin resonance by a transition from one topological spinor state to the other. These energy

states are different because of the interaction of $A^{(1)} \times A^{(2)}$ with the spinor, one which occurs through $B^{(3)}$. The energy needed to produce this transition is absorbed from a probe beam at the frequency [6] given by Eq. (134) of Vol. 3,

$$f_{res} = \frac{\omega_{res}}{2\pi} = \left(\frac{e^2\mu_0 c}{2\pi\hbar m}\right)\frac{I}{\omega^2}. \tag{5.27}$$

The great advantage of RF ESR is that the resonance frequency can occur in the infra-red or visible range by adjusting the pump angular frequency (ω) and intensity (I). The resolution of the RF ESR spectrum is increased commensurably, and this is a major advantage over conventional ESR. The latter is essentially a form of microwave spectroscopy [7] carried out with homogeneous magnets. Fine structure at microwave frequencies is much more difficult to resolve than at visible frequencies, and by using $B^{(3)}$ we are effectively increasing the available magnetic field strength enormously. Furthermore, this can be achieved with relatively inexpensive radio frequency technology.

In the one electron theory, there is only one RF electron spin resonance frequency, given by Eq. (5.27). In atoms and molecules, the one electron antisymmetric polarizability is replaced by its atomic and molecular equivalent, through which interaction with the beam's $E^{(1)} \times E^{(2)}$ takes place. This is of course, interaction with the beam's $B^{(3)}$ field defined as in Eq. (5.23) and is a magnetic effect. In this respect, ESR and RF ESR are both magnetic in nature; the former is generated by a static magnetic field, the latter by the radiated phase free magnetic field, $B^{(3)}$.

The antisymmetric polarizability in atoms and molecules is given by the standard semi-classical perturbation theory based usually on the time-dependent Schrödinger equation [7], in which case the spinor term of the Dirac equation is however, missing. Therefore there is a need to extend the basis of semi-classical perturbation theory for use with the Dirac equation. This can probably be achieved with some accuracy in contemporary computer packages such as MOTECC [12]. In the following, some progress is made within time dependent perturbation theory by adapting the usual semi-classical expression for non-relativistic antisymmetric polarizability [13—18],

$$\alpha''_{\alpha\beta} = -\sum_n \frac{2}{\hbar}\frac{\omega}{\omega_{kj}^2 - \omega^2} Im\left(\langle j|\mu_\alpha|k\rangle\langle k|\mu_\beta|j\rangle\right). \tag{5.28}$$

This expression is well known to be a sum over the product of transition electric dipole moments; and in the denominator appears the transition frequency (ω_{kj}) from state k to j. Equation (5.28) therefore represents an atomic or molecular property tensor with a much richer structure than the one electron equivalent defined in Eq. (5.25). Therefore the RF ESR spectrum can be richer in atoms and molecules than in one electron. (Recall that we have not yet taken into account spin orbit and spin spin coupling, which enriches the spectrum further.)

By inspection of the well known definition (5.28) it is seen that when ω_{jk} approaches ω, the antisymmetric polarizability itself is amplified by *optical resonance* [7]. This is another potential advantage of RF ESR, which can be obtained by tuning the pump frequency (for a given intensity) to a natural microwave transition frequency of the atom or molecule, for example a rotational frequency [19]. The applied circularly polarized field then acts simultaneously as a generator of $\boldsymbol{B}^{(3)}$, and causes the antisymmetric polarizability to be amplified because the pump ω has been tuned to ω_{jk}. In order to avoid heating effects caused by this absorption, the applied radio frequency field can be pulsed and the resonance transiently detected.

The connection between the antisymmetric polarizability in an atom or molecule, as given approximately in perturbation theory by Eq. (5.28), and the equivalent in a free electron can be established by looking at Eq. (5.28) in the limit where there is no transition frequency present, so that $\omega_{jk} = 0$. In this limit the antisymmetric polarizability can be expressed as

$$\alpha'' \rightarrow \frac{e^2}{\omega}\left(\frac{2r^2}{\hbar}\right), \qquad (5.29)$$

where r^2 is an area and where the product of electric dipole moments has been expressed as e^2r^2 for simplicity. Equations (5.25) and (5.29) becomes the same if

$$\hbar = 4r^2 m\omega \qquad (5.30)$$

and identifying the moment of inertia of the electron in its atomic orbit as $8mr^2$ it is found that Eqs. (5.25) and (5.29) become identical if

$$\hbar\omega = \frac{1}{2}I\omega^2, \qquad (5.31)$$

i.e., if the rotational kinetic energy of electron bound in the atom or molecule becomes the rotational kinetic energy of the photon, $\hbar\omega$, at a given frequency ω. This absorption of the energy $\hbar\omega$ is thought to promote the electron inside the atom or molecule to a higher orbital. In the free electron a quantum jump from one orbital to the higher orbital is not possible, so the electron's rotational kinetic energy is simply increased by a factor $\hbar\omega$. The angular momentum given to the electron by the electromagnetic field is $\hbar$, and this transfer of angular momentum is the simplest type of inverse Faraday effect.

Equation (5.18) for the free electron can be adapted straightforwardly for the atom or molecule by substituting the gyroptic ratio γ_π for the magneto-gyric ratio. The gyroptic ratio can be calculated for atoms and molecules from the definition of the antisymmetric polarizability α'' as follows [13]. First express α''_k as

$$\alpha''_k = -\frac{2e^2\omega}{\hbar\left(\omega_{mn}^2 - \omega^2\right)}\epsilon_{ijk}r_{oi}r_{oj}, \qquad (5.32)$$

where the radii are defined in general as expectation values over appropriate wave-functions. The transition dipole moments in this case are therefore

$$r_{oi} = \int \psi_m^* r_i \psi_n d\tau, \qquad r_{oj} = \int \psi_n^* r_j \psi_m d\tau, \qquad (5.33)$$

and the quantity,

$$A_{ok} := \epsilon_{ijk}r_{oi}r_{oj}, \qquad (5.34)$$

is an electronic orbital area. Equation (5.33) defines the electric dipole moments as transition moments within the atom or molecule.

The classical definition of the magnetic dipole moment is [7]

$$|\boldsymbol{m}^{(c)}| := J^{(c)}Ar, \qquad (5.35)$$

where $J^{(c)} = ev/2\pi r$ is the charge per unit time (current) passing some point of an orbit r traversed by an electron at orbital speed v. For an assumed circular orbit,

$$|\boldsymbol{m}^{(c)}| = \left(\frac{e}{2m}\right)mrv = -\gamma_e|\boldsymbol{L}|, \qquad (5.36)$$

where $-\gamma_e$ is the gyromagnetic ratio. Here $|\boldsymbol{L}|$ is the magnitude of the electronic orbital angular momentum,

$$|\boldsymbol{L}| = mrv. \qquad (5.37)$$

If instead of taking the usual circular area A of the classical calculation we define the area,

$$A_{oz} := \epsilon_{ijz}r_{oi}r_{oj} := -A_1\alpha''_z, \qquad (5.38)$$

with [13]

$$A_1 := \frac{\hbar}{2c^2\omega}\left(\omega_{mn}^2 - \omega^2\right), \tag{5.39}$$

giving the effective magnetic dipole moment,

$$m_{oz} = I_1 A_{oz}, \tag{5.40}$$

and implying that

$$\gamma_e = -I_1 A_1 \gamma_\pi . \tag{5.41}$$

Choosing the current,

$$I_1 := e\left(\frac{v}{2\pi r_{av}}\right), \tag{5.42}$$

where r_{av} is the effective circular orbital radius, gives the desired expression for the gyroptic ratio,

$$\gamma_\pi = -\frac{2\pi r_{av}}{e v A_1}\gamma_e = \frac{\pi r_{av}}{m v}\cdot\frac{2e^2\omega}{\hbar\left(\omega_{mn}^2 - \omega^2\right)} \tag{5.43}$$

Therefore radio frequency induced ESR in atoms and molecules can be expressed in terms of the gyroptic ratio, provided that semi-classical perturbation theory is properly worked out with the Dirac equation rather than with the usual time-dependent Schrödinger equation. In an approximate treatment however, it is sufficient to rewrite equation (5.18) as

$$\Delta H = \gamma_\pi\left(L_i + 2S_i\right)E_{oi}^2, \tag{5.44}$$

whereupon it is seen that extra detail appears in the RF ESR spectrum as the result of the internal structure of the gyroptic ratio γ_π itself.

5.2.1 Time Independent Perturbation Theory

The antisymmetric polarizability in Eq. (5.28) is derived using time dependent perturbation theory [7]. The latter can be applied because the individual electric fields $E^{(1)}$ and $E^{(2)}$ in the conjugate product $E^{(1)} \times E^{(2)}$ are time dependent. Conjugate products such as $A^{(1)} \times A^{(2)}$ are however time independent, and appear as part of the energy eigenvalue of the Dirac equation at the one electron level [6]. It follows that time independent perturbation theory can be applied in atoms and molecules to calculate the effect of a conjugate product such as $A^{(1)} \times A^{(2)}$ on an electron spin within an atom or molecule. Such a calculation is formally identical with the calculation of a Zeeman effect or anomalous Zeeman effect provided that the external magnetic field is defined through Eq. (5.23). This is inversely proportional to the square of the angular frequency of a circularly polarized pump electromagnetic field, and so the effects due to $B^{(3)}$ are amplified greatly if radio frequencies are used instead of visible frequencies for a given I. These include ESR, NMR, Zeeman effects, Faraday rotation effects, circular dichroism, and bulk magnetization.

It is reasonable to develop the time independent perturbation theory (TIP) by using, from the one electron Dirac equation [6], the perturbation Hamiltonian operator,

$$V := -\frac{e^2}{2m}\left(\sigma^{(1)}\cdot A^{(2)}\right)\left(\sigma^{(2)}\cdot A^{(1)}\right), \tag{5.45}$$

defined in Eq. (5.16) between the classical electromagnetic field and the fermion spinor. In an atom or molecule this spinor is that of an unpaired electron which is simultaneously interacting with other fermions. The effect of this interaction is treated in this section using the standard spin Hamiltonians of ESR theory [7]. The TIP theory gives approximate solutions of the Dirac equation

$$H'\psi' = (H + V)\psi' = W'\psi, \tag{5.46}$$

where H is [14] the unperturbed atomic Hamiltonian, ψ' and W' are the perturbed atomic wavefunction and energy. The theory gives approximate expressions for the eigenfunctions ψ_j' and eigenvalues W_j' of the perturbed operator H' in terms of the unperturbed ψ_j and W_j of H [14].

For a non-degenerate wavefunction ψ_n, the perturbed energy eigenvalue is, to second order in the perturbation Hamiltonian operator [14] V,

$$W_n' = W_n + \langle n \mid V \mid n \rangle + \sum_{j \neq n} \frac{\langle n \mid V \mid j \rangle \langle j \mid V \mid n \rangle}{W_n - W_j} + \ldots , \tag{5.47}$$

and the perturbed wavefunction is

$$\psi_n' = \psi_n + \sum_{j \neq n} \frac{\langle j \mid V \mid n \rangle}{W_n - W_j} \psi_j + \ldots . \tag{5.48}$$

As is well known, the sum in these expressions extends [14] over the complete set of eigenfunctions with the exception of that of the ground state, n, denoted ψ_n. If there are no states other than the ground state, Eqs. (5.47) and (5.48) reduce to

$$W_n' - W_n = \langle n|V|n \rangle , \tag{5.49}$$

$$\psi_n' = \psi_n . \tag{5.50}$$

Equation (5.49) is Eq. (5.14), in which

$$\langle n|V|n \rangle = -\frac{e^2}{2m} \left(\boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)} + i\, \boldsymbol{\sigma}^{(3)} \cdot \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \right), \tag{5.51}$$

is the expectation value of V in the ground state, the only state for a free electron. Therefore the TIP gives the known exact result [6] for the free electron, as required.

In an atom or molecule, however, higher order terms in Eq. (5.47) are non-zero, because there are atomic electronic states other than the ground state ψ_n. The effect of these states is to introduce higher order corrections in TIP, and to shift the ESR line away from the frequency of resonance in the free electron [7]. The great advantage of using TIP with $\boldsymbol{B}^{(3)}$ is that it allows these useful spectral effects to be calculated as in the standard theory of ESR [7], except that the static magnetic field of standard ESR is replaced by the definition (5.23) from electromagnetic theory [4—6]. Standard expressions in the literature can be used directly with $\boldsymbol{B}^{(3)}$, leading in principle to many useful developments because of the fact that $\boldsymbol{B}^{(3)}$ is proportional to $1/\omega^2$ as discussed already. In order to illustrate these developments it is convenient to follow the revealing account given by Atkins [7], pp. 405 ff., using the same notation.

The expectation value of V is given by

$$E^{(1)} := \langle n|V|n \rangle = \frac{e}{2m} \left(\langle n | \boldsymbol{L}^{(3)} | n \rangle + \langle n | 2\boldsymbol{S}^{(3)} | n \rangle \right) \cdot \boldsymbol{B}^{(3)*} , \tag{5.52}$$

which can be rewritten in Atkins' standard spectroscopic notation as

$$E^{(1)} = -g_e \gamma_e \langle n|S_Z|n \rangle B^{(3)} - \gamma_e \langle n|L_Z|n \rangle B^{(3)} , \tag{5.53}$$

in which the only difference from standard theory is the replacement of the static magnetic field $\boldsymbol{B}$ by the time independent electromagnetic field $\boldsymbol{B}^{(3)}$. (If preferred, the latter can be thought of as the time independent conjugate product.) Atkins [7] also considers spin-orbit coupling, which is absent in a free electron, and is the effect in an atom or molecule of the Coulomb interaction with the nucleus, corrected by the Thomas precession. The spin orbit term is the one responsible for the shift in the resonance frequency of an electron bound in an atom compared with that of a free electron. The complete perturbation Hamiltonian to be considered is therefore

$$V = -g_e \gamma_e S_Z B^{(3)} - \gamma_e L_Z B^{(3)} + \lambda \boldsymbol{L} \cdot \boldsymbol{S} . \tag{5.54}$$

The first order correction [7] to the energy is the expectation value of V within the real and orbitally non-degenerate state $|n\rangle$. The only resonance producing contribution to $E^{(1)}$ is

$$V_{\text{spin}} := -g_e \gamma_e S_Z B^{(3)} , \tag{5.55}$$

which is the expectation value of the *spin Hamiltonian*.

The effect of spin-orbit coupling within the atom or molecule is to effectively change the electron g_e value from 2 (Dirac) or 2.0023 (quantum electrodynamics) to a value dependent on internal atomic or molecular structure and transitions. Atkins [7] conveniently gives an example of a single unpaired electron in a p_x orbital with an unoccupied p_y orbital an energy Δ above it. The effective Landé factor of the electron then becomes

$$g_{ZZ} = g_e - 2\frac{\lambda}{\Delta} \langle p_X | L_Z | p_Y \rangle \langle p_Y | L_Z | p_X \rangle , \tag{5.56}$$

with

$$\langle p_Y | L_Z | p_X \rangle = i\hbar . \tag{5.57}$$

This type of calculation can also be applied to Eq. (5.52) for the ground state in the circularly polarized electromagnetic field, because the same g_e factor enters into that equation as for the equation of an electron in a static magnetic field. However, $B^{(3)}$ must always be defined through Eq. (5.23), which controls the interaction of $B^{(3)}$ with matter.

5.3 Summary of Electromagnetically Induced ESR Effects

It is expected that the energy changes caused by the electromagnetic field in an atom or molecule will be governed by the equation,

$$W_n' = W_n - \frac{e^2}{2m} \langle n | \sigma^{(1)} \cdot A^{(2)} \sigma^{(2)} \cdot A^{(1)} | n \rangle - \alpha''_{\alpha\beta} E_\alpha E_\beta^* + \ldots , \tag{5.58}$$

in which there appears two perturbation terms, both to first order in $B^{(3)}$, provided the latter is defined in terms of Eq. (5.23). The first is deduced on the basis of time independent perturbation theory using the interaction Hamiltonian operator of Eq. (5.45), and the second is deduced on the basis of time dependent perturbation theory applied to each time dependent component of the antisymmetric product $E_\alpha E_\beta^*$ [14]. Here $\alpha''_{\alpha\beta}$ is the antisymmetric polarizability [14], which is non-zero in all atoms and molecules in the presence of a circularly polarized light beam. The presence of this second term should provide a mechanism for enriching the electromagnetically induced ESR spectrum.

Similar considerations to the above apply to electromagnetically induced NMR, and further details can be found in Vol. 3 [6], Chaps. 1 and 2.

It is possible to summarize the steps leading to the existence of electromagnetically induced ESR and NMR as follows. The inverse Faraday effect has been observed empirically [4—6] on several occasions, although it remains a difficult experiment. This effect for one electron must therefore be described by the Dirac equation [6] for one electron in the classical field,

$$W + ecA^{(0)} = \frac{e^2c^2 \left(\sigma^{(1)} \cdot A^{(2)} \sigma^{(2)} \cdot A^{(1)} \right)}{En + mc^2 + ecA^{(0)}} , \tag{5.59}$$

where W is defined as $En - mc^2$. The necessary spinor algebra in the basis ((1), (2), (3)) is given in Vol. 3. Without the use of a complex A in Eq. (5.59), it is not possible to predict an inverse Faraday effect, and since this equation contains all the information that the Dirac equation has to give on the time-independent interaction of a fermion with an electromagnetic field, it follows that there exist two interaction terms,

$$\sigma^{(1)} \cdot A^{(2)} \sigma^{(2)} \cdot A^{(1)} = A^{(1)} \cdot A^{(2)} + i\, \sigma^{(3)} \cdot A^{(1)} \times A^{(2)} , \tag{5.60}$$

one of which cannot exist without the other. This is a result of topology (Clifford algebra). Theories to date of the inverse Faraday effect have been based on classical considerations in which the spinor is missing, and so have been unable to predict the existence of electromagnetically induced resonance. Standard theories of the inverse Faraday effect such as that of Talin *et al.* [20] produce only the first term on the right hand side of Eq. (5.60). This term must also be developed in terms of the conjugate product and the antisymmetric polarizability for one electron in order to reproduce the observation [4] that the inverse Faraday effect changes sign with circular polarization. In other words, this first term must be written as

$$W_1 = \alpha''^{(3)} \cdot E^{(1)} \times E^{(2)} , \tag{5.61}$$

where

$$\alpha''^{(3)} := -i \frac{e^2}{2m\omega^2} e^{(3)} , \tag{5.62}$$

is the pure imaginary one electron antisymmetric polarizability multiplying the pure imaginary conjugate product to give a real interaction energy. Without the use of $\alpha''^{(3)}$ the experimental data cannot be described from the Dirac equation, which also provides the new spinor term responsible for electromagnetically induced E.S.R. and N.M.R. Incorporating the spinor terms reduces Eq. (5.59) to the familiar looking

$$W \sim \frac{e}{2m} \left(L^{(3)} + 2S^{(3)} \right) \cdot B^{(3)*} , \tag{5.63}$$

for $2mc \gg eA^{(0)}$. This equation can be written, equivalently, in terms of the conjugate product,

$$W \sim -\frac{i}{\hbar} \frac{e^2c^2}{2m\omega^2} \left(L^{(3)} + 2S^{(3)} \right) \cdot B^{(1)} \times B^{(2)} , \tag{5.64}$$

so that we use the one electron susceptibility,

$$\chi''^{(3)} = -\frac{ie^2c^2}{2m\omega^2\hbar} \left(L^{(3)} + 2S^{(3)} \right) . \tag{5.65}$$

If we make use of the positive spinor state in Eq. (5.65), it gives, interestingly, the same result as that obtained by Talin *et al.* [20] for the magnetization of the inverse Faraday effect in the non-relativistic limit,

$$M = -\frac{e^3}{2m^2\omega^3}E^{(0)2}, \tag{5.66}$$

but since both routes to equation (5.66) have used approximations, no particular significance can be attached to this agreement, except insofar as to show that neither route is very far off the mark. However, the route used by Talin *et al.* [20] does not use spinors, and cannot describe electromagnetically induced fermion resonance. In this respect some of our early work [3] also missed the key spinor term, which was first described in Ref. 6. It is now clear that the spinor term cannot be neglected if we are to use the Dirac equation, the rigorous equation of relativistic quantum theory. Therefore electromagnetically induced fermion resonance, which promises to be very useful [6], is a prediction of the Dirac equation.

References

[1] D. W. Davies, *The Theory of the Electric and Magnetic Properties of Molecules* (Wiley, New York, 1967).
[2] M. W. Evans, *J. Mol. Spec.* **146**, 143 (1991).
[3] M. W. Evans, *J. Phys. Chem.* **95**, 2256 (1991).
[4] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $\boldsymbol{B}^{(3)}$ (Kluwer Academic, Dordrecht, 1994).
[5] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).
[6] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).
[7] P. W. Atkins, *Molecular Quantum Mechanics* (Oxford University Press, Oxford, 1983). For a relativistic treatment see L. D. Landau and E. M. Lifshitz, *The Clasiical Theory of Fields*, 4th edn. (Pergamon, Oxford, 1975).
[8] L. D. Barron, *Molecular Light Scattering and Optical Activity*, (Cambridge University Press, Cambridge, 1982).
[9] J. P. van der Ziel, P. S. Pershan, and L. D. Malmstrom, *Phys. Rev.* **143**, 574 (1966).
[10] L. D. Barron, *Chem. Phys. Lett.*, **135**, 1 (1987).
[11] M. W. Evans, *Phys. Rev. A.* **41**, 4601, (1990).
[12] E. Clementi, ed., *MOTECC* volumes and software.
[13] M. W. Evans, *Phys. Lett. A.* **147**, 364 (1990).
[14] M. W. Evans, *J. Mod. Opt.*, **37**, 1655 (1990).
[15] G. Wagnière, *Phys. Rev. A.* **40**, 2437 (1989).
[16] G. Wagnière, *Z. Phys. D.* **8**, 229 (1988).

[17] M. W. Evans, *Phys. Rev. Lett.* **64**, 2909 (1990).
[18] M. W. Evans and G. Wagnière, *Phys. Rev. A.* **42**, 6732 (1990).
[19] M. W. Evans, G. J. Evans, W. T. Coffey, and P. Grigolini, *Molecular Dynamics* (Wiley, New York 1982).
[20] B. Talin, V. P. Kaftandjam, and L. Klein, *Phys. Rev. A.* **11**, 648 (1975).

Chapter 6. Physical Meaning of the Photon's Wavefunction According to Ettore Majorana

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6.1 Introduction

Following experiments by Aspect and by Rapisarda, renewed interest has developed in the foundations of quantum mechanics. Ettore Majorana [1—4], in his unpublished manuscripts of about 1928 to 1932, suggested a new approach to the photon wavefunction by writing down a Dirac-like equation for the photon. In this chapter we review and comment on this significant work.

In 1930, it was still the accepted wisdom that it is possible to write a Lorentz-covariant equation only for spin zero and half. To refute this, Majorana looked for covariant equations corresponding to different spin values, and in so doing included the photon spin in his development. He realized finally that it was possible to write down covariant equations describing an infinite series of particles with arbitrary spin, both integral and half integral. This was achieved using infinite dimensional unitary representations of the Lorentz group. Majorana published only this end result [5]. The following describes what he started to do with spin one particles.

M. W. Evans et al. (eds.), The Enigmatic Photon, Vol. 4, 137–145.

6.2 A Dirac-like Equation for the Photon [1]

In rationalized Gaussian units, let us introduce the quantities

$$\psi_i := E_i - i H_i , \quad (i = 1, 2, 3) . \tag{6.1}$$

Then the Maxwell equations may be written in the form,

$$\nabla \cdot \psi = \rho, \quad i\nabla \times \psi = j + \frac{\partial \psi}{\partial t}, \tag{6.2}$$

where

$$\psi := (\psi_1, \psi_2, \psi_3) . \tag{6.3}$$

In the absence of electric charges and currents we can *quantize* Eqs. (6.2) by using the correspondence principle $-i\partial/\partial x_i \to p_i$, $i\partial/\partial t \to W$ to obtain,

$$W\psi_1 + ip_2\psi_3 - ip_3\psi_2 = 0, \quad W\psi_2 + ip_3\psi_1 - ip_1\psi_3 = 0,$$
$$W\psi_3 + ip_1\psi_2 - ip_2\psi_1 = 0 . \tag{6.4}$$

Majorana introduced next the condition[1]

$$\boldsymbol{p} \cdot \psi = 0. \tag{6.5}$$

Equation (6.4) takes on the form of a Dirac equation [2],

$$(W + \boldsymbol{\alpha} \cdot \boldsymbol{p})\psi = 0 . \tag{6.6}$$

The matrices,

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[1] From the development in Chap. 4, we can no longer refer self-consistently to this as a transversality condition. It is, in covariant form, the condition (4.64) of that chapter, signifying the orthogonality of p_μ and ψ^μ in four dimensions.)

$$\alpha_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \alpha_2 := \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix},$$
$$\alpha_3 := \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{6.7}$$

satisfy the angular momentum *commutation rules*,

$$[\alpha_i, \alpha_k]_- = -i\epsilon_{ikl}\alpha_l, \quad (i, k, l = 1, 2, 3), \tag{6.8}$$

where ϵ_{ikl} as usual is the antisymmetric unit tensor in three dimensions.

Therefore ψ can now be considered as a *quantized* wavefunction for the photon, satisfying a Dirac equation without a mass term, Eq. (6.6). The eigenfunctions, ψ, of W and $\boldsymbol{p}$ in Eq. (6.6) are essentially plane waves in energy-momentum space, and the determinantal condition,

$$\begin{vmatrix} W & -ip_3 & ip_2 \\ ip_3 & W & -ip_1 \\ -ip_2 & ip_1 & W \end{vmatrix} = 0, \tag{6.9}$$

yields the correct energy-momentum relation for the photon [2],

$$W = \pm|\boldsymbol{p}|, \tag{6.10}$$

in addition to the solution $W = 0$. The three equations (6.6) may be rewritten as a Schrödinger equation with the Hamiltonian

$$H = -\boldsymbol{\alpha} \cdot \boldsymbol{p}. \tag{6.11}$$

By evaluating, as in standard procedure, the commutator of H with the *orbital* angular momentum $\boldsymbol{L}$,

$$[H, L]_{-} = -\alpha \times p , \tag{6.12}$$

the *total* angular momentum j follows as

$$j = L + \Sigma . \tag{6.13}$$

The quantity Σ is the *intrinsic (spin)* angular momentum for the photon,

$$\Sigma := -i\alpha \times \alpha. \tag{6.14}$$

Equation (6.14) is analogous to the expression,

$$\Sigma_D = -\frac{i}{2}\left(\alpha_D \times \alpha_D/2 \right), \tag{6.15}$$

for the spin operator of the Dirac electron, except for the factor $1/2$ which transforms into the factor 1 as expected. The operator (6.14) has eigenvalues $0,\ \pm 1$, signifying the presence of longitudinal and transverse components for the massless photon.[2] In definition (6.14), our momentum eigenfunctions ψ are also eigenstates of the photon helicity, the operator,

$$\epsilon := \Sigma \cdot \hat{p} = \alpha \cdot \hat{p}, \tag{6.16}$$

where, as usual, $\hat{p} := p'|p|$. As just described in Chap. 4, the angular momentum and helicity eigenstates are essentially the same; both have eigenvalues $0,\ \pm 1$.[3]

By introducing the complex conjugate quantities of Eq. (6.1),

$$\psi_i^* = E_i + iH_i \quad (i = 1,\ 2,\ ,3), \tag{6.17}$$

we obtain the complex conjugate of Eq. (6.6),

$$(W + \alpha \cdot p)\psi^* = 0 , \tag{6.18}$$

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[2] These components are interlinked through equations such as the A and B cyclics.

[3] In the standard view [7] however, the eigenvalue zero is, incorrectly, missing, and it is incorrectly asserted that there are only transverse modes. This is a major flaw in conventional electrodynamics.

which is identical to Eq. (6.6). Therefore massless photons and massless anti-photons are indistinguishable, even in the presence of longitudinal solutions.[4] Equations (6.18) may also be written

$$\bar{\psi}\,(W + \alpha \cdot p) = 0 , \tag{6.19}$$

which are analogous in form to the corresponding equations for the Dirac-adjoint wave-function, apart from sign. In Eqs. (6.19) the notation means that $p := i\overleftarrow{\partial}/\partial x$ and that $\bar{\psi}$ is the Hermitian conjugate (Vol. 2 and 3 [9,10]) of ψ,

$$\bar{\psi} = (\psi^*)^T . \tag{6.20}$$

By means of the formalism (6.1) and (6.2) one can reproduce classical (Maxwellian) electromagnetism, even when charges are present, i.e., one can describe wave irradiation, and irradiation (Vol. 3 [10]) of the $B^{(3)}$ field.

In Majorana-Openheimer's theory, Eqs. (6.1) and (6.2), the energy-momentum tensor can be defined by using the matrices [2],

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[4] This result shows that the $\hat{C}$ symmetry arguments [8] against the B cyclics are erroneous.

$$2\alpha_{00} := 1\,, \quad 2\alpha_{01} := \alpha_1\,, \quad 2\alpha_{02} := \alpha_2\,, \quad 2\alpha_{03} : \alpha_3\,,$$

$$2\alpha_{11} := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 2\alpha_{12} := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$2\alpha_{13} := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad 2\alpha_{22} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.21}$$

$$2\alpha_{23} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad 2\alpha_{33} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where

$$\alpha_{\mu\nu} = \alpha_{\nu\mu}\,, \quad (\mu\,, \nu = 0\,, 1\,, 2\,, 3)\,. \tag{6.22}$$

We can write the symmetric energy-momentum tensor $T_{\mu\nu} = T_{\nu\mu}$ as follows [2],

$$T_{\mu\nu} = \bar{\psi}\, \alpha_{\mu\nu}\, \psi\,. \tag{6.23}$$

Out of the nine matrices $\alpha_{\mu\nu}$ (besides the identity $2\alpha_{00}$) in Eqs. (6.21), only *eight* matrices are independent; and they form a basis for the fundamental representation of SU(3). The vector ψ is a vector of a *complex* three-dimensional space.

6.3 Comments

Macroscopically, and at a statistical level, the electric and magnetic field strengths $\boldsymbol{E}$ and $\boldsymbol{H}$ are known to be connected through the quantity $E^2 + H^2$ with the local mean number of equal photons. Majorana's idea seems to have been that of analogously expressing the probability quantum-function ψ of a photon in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$, thus giving it a more direct meaning than usual (i.e., than when introduced through the

electromagnetic four-potential). What is more, Majorana wrote for the photon an equation similar to Dirac's, and one which can be considered as part of his infinite component equation [5]. This suggests that an analogous, direct, interpretation can be made for the wave function of the electron and possibly, other elementary particles. The essential difference might be merely that the electromagnetic field possesses also an *intuitive* equivalent, since it is a long range field and affects our macroscopic experience; whilst the strong and weak (microscopic) fields do not possess any macroscopic (*intuitive*) equivalent.

Majorana realized the importance in electrodynamics of the complex quantity $\boldsymbol{E} - i\boldsymbol{H}$, whose fundamental meaning has been subsequently recognized by others [6].[5]

6.3.1 Ettore Majorana

A translation of a fragment from lecture notes made by Majorana for his undergraduate students follows below. The lecture notes have been known for some time and passed into the hands of Gilberto Bernardini and Edoardo Amaldi after Ettore Majorana left them with a student, S. Sciuti. They were published in facsimile (handwritten in Italian) in 1986 in Ref. [5c]. The English translation appears for the first time in this chapter. In these notes Majorana emphasizes the *guiding* role of the electromagnetic waves, and made much more interesting contributions in his unpublished scientific manuscripts, a catalogue of which appears in the book edited by Preziosi and Ricci [5c].

Science suffered the loss of the *omnium opus* of Majorana between 1933 and 1938, with the exception of one paper, introducing the Majorana spinors, mass and neutrino. If readers can think of a way to try to trace these lost manuscripts, it would be a discovery of the first magnitude for scholarship and also contemporary physics, which is taking a renewed interest in Majorana through theoretical physicists such as Dharam Ahluwalia and co-workers, and Valeri Dvoeglazov and co-workers.

The following fragment comes from the notes prepared by Majorana for undergraduates in the University of Naples, from which he disappeared in 1938, leaving them in the care of Sciuti, one of his students. These, and his much more important research notes, are regarded as among the most original of the Golden Age of Physics in the twentieth century.

They support the conclusions of this chapter and show Majorana's description for students of a theory similar to the pilot wave theory of Louis de Broglie. At the 1927 Solvay Conference, Louis de Broglie was *shouted down* (remark to MWE of Jean-Pierre Vigier) and temporarily adopted the Copenhagen philosophy of Bohr, Heisenberg, Pauli, Dirac and others. Here we see Majorana leaning towards the de Broglie-Einstein

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[5] For instance $\boldsymbol{B}^{(3)}$ emerges from the conjugate product.

philosophy favored by Schrödinger, Bohm, Vigier *et alteri*. Majorana was regarded by Fermi and others in having the ability to see around corners using a combination of imagination and technical prowess: an ability regarded by Fermi as being comparable with that of better known figures such as Newton and Einstein.

6.3.2 Majorana Lecture Notes Fragment: English translation[6]

Such a dualism between the wave and the particle representation of light (both are suggested, peremptorily, to be different sets of phenomena) can be resolved within the spirit of the new (quantum) mechanics by assuming that the light quanta can be GUIDED (*Majorana's emphasis*) by electromagnetic waves obeying Maxwell's equations: guided in such a way that the probability of finding a light quantum inside a certain spacetime region can be determined only by finding a solution of Maxwell's equations that takes account of the boundary conditions created by material objects (slits, mirrors, etc.) upon which interference phenomena depend. The fundamental idea of wave mechanics, due to Louis de Broglie (1924), is as follows. Light quanta are nothing but an intuitive aspect of a physical entity that in other cases manifests itself as a wave. Material points, electrons, or any other particles having a finite rest-mass and travelling at less than the speed of light can be associated with the plane waves of a field different in nature from the electromagnetic field.

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References

[1] R. Mignani, E. Recami, and M. Baldo, *Lett. Nuovo Cim.* **11**, 568 (1974); E. Giannetto, *Lett. Nuovo Cim.* **44**, 140 (1985); J. D. Edmonds, *Lett. Nuovo Cim.* **13** 185 (1975); S. Esposito, "Covariant Majorana formulation of Electrodynamics", submitted for publication.

Handwritten in the scientific papers of Ettore Majorana, Domus Galilaeana, Pisa, reproduced in facsimile by Preziosi and Ricci [5c].

[2] E. Majorana, Unpublished scientific manuscripts, deposited at the *Domus Galilaeana*, Pisa, Italy; notebooks **2**, 101/1; **3**, 11, 160; **15**, 16; **17**, 83, 159. These were written about 1930. See also J. R. Oppenheimer, *Phys. Rev.* **38**, 725 (1931).

[3] E. Amaldi, *La vita e l'opera di Ettore Majorana* (Accad. Naz. Lincei, Roma, 1966); Ettore Majorana, uomo e scionzato, *Giornale di Fisica* **9**, 300 (1968); and in *Strong and Weak Interactions* A. Zichichi, ed. (New York, 1966).

[4] E. Recami, *Scientia (Milano)* **110**, 589 (1975); and in *Tachyons, Monopoles and Related Topics*, E. Recami, ed. (North Holland Pub. Co., Amsterdam, 1978), Appendix, p. 267. See also E. Recami, *Il Caso Majorana: Lettere, Documenti, Testimonianze* (Mondadori, Milan, 1991).

[5] E. Majorana, *Nuovo Cim.* **9**, 335 (1932); see also D. M. Fradkin, *Am. J. Phys.* **34**, 314 (1966); B. Preziosi and R. A. Ricci (eds.), *E. Majorana, Lezioni all 'Università di Napoli* (Bibliopolis, Naples, 1986), p.82.

[6] See for example R. Mignani and E. Recami, *Nuovo Cim.* **30A**, 533 (1975); D. Weingarten, *Ann. Phys.* **76**, 510 (1973); J. M. Leinass, *Nuovo Cim.* **15A**, 740 (1973); L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields* (Moscow, 1966); S. Weinberg, in *1964 Lectures on Particles and Field Theory.* ed. S. Deser and K. W. Ford (Englewood Cliffs, New Jersey, 1965), p. 405 ff.

[7] P. W. Atkins, *Molecular Quantum Mechanics* (Oxford University Press, Oxford, 1983). For a relativistic treatment see L. D. Landau and E. M. Lifshitz, *The Clasiical Theory of Fields*, 4th edn. (Pergamon, Oxford, 1975).

[8] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $\boldsymbol{B}^{(3)}$ (Kluwer Academic, Dordrecht, 1994).

[9] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).

[10] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).

Chapter 8. The Conservation of Helicity

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In this chapter a geometrical proof of the existence of the vacuum $\boldsymbol{B}^{(3)}$ field component is given by constructing the Pauli-Lubanski pseudo four-vector (PL vector for short) from the geometrical three-manifold in four-space. The three-manifold is in general a tensor of rank three in four dimensions (4-D), antisymmetric in all three indices. The PL vector is dual to this three-tensor and so must have the same amplitude. The original three-tensor ($S^{\mu\nu\sigma}$) is the product of the normalized [4] generator of space-time translations (ϵ^{μ}) with the antisymmetric field strength tensor $G^{\nu\sigma}$, i.e.,

$$S^{\mu\nu\sigma} := \epsilon^{\mu}G^{\nu\sigma}. \tag{8.1}$$

The field strength tensor includes the $\boldsymbol{B}^{(3)}$ element. The background to this rigorous and general geometrical tensor theory is given in Ref. 1. The PL vector dual to $S^{\mu\nu\sigma}$ turns out to be the light-like invariant [4—6],

$$\tilde{B}^{\mu} = \left(B^{(3)},\ 0,\ 0,\ B^{(3)}\right), \tag{8.2}$$

in contra-variant notation. The PL vector $\tilde{B}^{\mu}$ cannot be defined self-consistently unless ϵ^{μ} is used as in Eq. (8.1). In contra-variant notation this polar unit four-vector is [4]

$$\epsilon^{\mu} = \left(1,\ 0,\ 0,\ 1\right), \tag{8.3}$$

representing a light-like vector proportional to $\tilde{B}^{\mu}$. Therefore if the axis of propagation is Z, the $\boldsymbol{B}^{(3)}$ vector is also in Z [4—6], i.e., is longitudinal and perpendicular to the transverse $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$, for example a plane wave.

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The above means that the Poincaré group must be used, i.e., the Lorentz group must be extended by the generator of space-time translation P^μ [4], which is directly proportional to the energy momentum polar four-vector p^μ in vacuo,

$$P^\mu = \frac{p^\mu}{\hbar}, \tag{8.4}$$

The three-tensor obtained from (8.2) by inverting the duality transformation gives a product of ϵ^μ with an antisymmetric field tensor $G_L^{\nu\sigma}$ whose only physical elements are $B^{(3)}$ and $-B^{(3)}$. This field tensor is missing from the usual theory [1]. The PL vector B^μ is proportional, (Chap. 3) to the consistently defined relativistic helicity of the photon as particle, and the helicity has eigenvalues $0, \pm 1$, those of a boson, irrespective of whether the photon is considered to have mass [4—6]. In the usual view, the 0 component of helicity is missing [1] and with it the $\boldsymbol{B}^{(3)}$ field. This produces a null PL vector [4],

$$\tilde{B}^\mu(\text{usual theory}) = (0,\, 0,\, 0,\, 0), \tag{8.5}$$

dual to the usual non-zero $G^{\mu\nu}$ and contradicts the basic fact [1] that a true dual pseudo-vector has the same magnitude as the three-tensor to which it is dual. The reason for this result in the usual theory (transverse components only) is that the contribution of the transverse components to the PL vector's X and Y components both cancel exactly by Maxwell's vacuum equations [4]; and since there is no longitudinal component by assertion in the usual view, the PL vector is null.

This paradox in the usual view means that the relativistic helicity is not defined self-consistently, i.e., is always zero because $\tilde{B}^\mu$ is null. In consequence, no particle interpretation can be built up from the classical field and in effect there is no photon. A hypothetical particle with zero helicity has no spin component in its axis of linear momentum, and is not spinning about this axis. This particle is not a boson and the analysis carried out in this way also contradicts the usual assertion [4] that the helicity of the photon without mass is ± 1. If the PL vector is null, the helicity must always be zero unless $\boldsymbol{B}^{(3)}$ is non-zero, when it acquires the additional values 1 and -1 from parity considerations (ϵ^μ is $\hat{P}$ negative, $\tilde{B}^\mu$ is $\hat{P}$ positive). The projection of the transverse components X and Y of $\tilde{B}^\mu$ on to the Z component of P^μ is always zero. The projection of the longitudinal component $\tilde{B}^\mu$ on to the direction of P^μ is either 1 or -1 after normalization. This is precisely what is indicated by a comparison of Eqs. (7.32) and (8.3) because helicity is defined [4] to be a projection on to the non-zero space component of p^μ — the longitudinal component in Z. Therefore $\boldsymbol{B}^{(3)}$ is the fundamental field component representing spin in the classical electromagnetic field. Intuitively, it is generated by the spinning and forward motion of the propagating transverse components.

A null dual pseudo-vector in 4-D geometry [1] means that the area is zero of the hyper-surface element represented by that pseudo-vector dual, normal by definition to the hyper-surface element and equal to it in magnitude. If the dual pseudo-vector is null, its space component is null, and therefore the space part of the equivalent hyper-surface element is also null, meaning that the usual theory is self inconsistent. A physical beam of light must have a finite cross-sectional area perpendicular to its propagation axis, in which $\boldsymbol{B}^{(3)}$ is defined. In this reasoning, if $\boldsymbol{B}^{(3)}$ were zero (as usually asserted in the standard theory), the area of the beam would vanish, and so would the beam itself.

Prior to Wigner [4—6], the P^μ vector, signifying the operator of space-time translation, was not known. The PL vector could not have been defined self-consistently. It appears that $\tilde{B}^\mu$ was first introduced in Vol. 1 of this series [4], Chap. 11, and in Ref. 7 (1995). The gap of more than half a century between the introduction of P^μ and that of $\tilde{B}^\mu$ is the result of the usual view that electrodynamics in vacuo has no longitudinal component. Self-consistent application of tensor theory, however, as shown in this chapter, leads to the conclusion that if $\boldsymbol{B}^{(3)}$ is not present in the usual electromagnetic four-tensor [1], the PL vector to which it is dual is a null vector, Eq. (8.5), meaning that the relativistic helicity vanishes. Similar considerations, amplified in the following, show that $\boldsymbol{B}^{(3)}$ is accompanied by its dual in the vacuum [4], the pure imaginary component $-i\boldsymbol{E}^{(3)}/c$.

8.1 The Dual Pseudo-Tensor

The dual pseudo-tensor of any antisymmetric tensor in four-space arises from the integral [1] over a two dimensional surface in four-space. In three-space, the projections of the area of a parallelogram formed from the infinitesimal line elements $d\boldsymbol{r}$ and $d\boldsymbol{r}'$ on the coordinate planes $x_i x_j$ are $dx_i dx_j' - dx_j dx_i'$. In four-space the infinitesimal element of surface is given by the antisymmetric tensor $df^{\mu\nu} = dx^\mu dx^{\nu'} - dx^\nu dx^{\mu'}$, its components being the projections of the element of area on the coordinate planes. In three-space it is always possible to define an axial pseudo-vector element $d\tilde{f}_i$ dual to the antisymmetric tensor df_{jk},

$$d\tilde{f}_i := \frac{1}{2}\epsilon_{ijk} df_{jk}\,. \tag{8.6}$$

The pseudo-vector element $d\tilde{f}_i$ represents the same surface element as df_{jk}, and geometrically, is a pseudo-vector normal to the surface element, *equal in magnitude to the area of the element*. In four-space, such a pseudo-vector cannot be constructed from an antisymmetric tensor such as $df_{\mu\nu}$. However, the dual pseudo-tensor can be defined by [1]

$$d\tilde{f}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}df_{\sigma\rho}, \tag{8.7}$$

where $\epsilon^{\mu\nu\sigma\rho}$ is the totally antisymmetric unit pseudo-tensor in four dimensions, with

$$\epsilon^{0123} = -\epsilon_{0123} = 1. \tag{8.8}$$

In geometrical terms, $d\tilde{f}^{\mu\nu}$ is an element of surface equal and *normal* to the element $df_{\sigma\rho}$. All segments in it [1] are orthogonal to all segments in $df_{\sigma\rho}$. This leads to the result [1]

$$d\tilde{f}^{\mu\nu}df_{\mu\nu} = 0. \tag{8.9}$$

In general therefore, an antisymmetric four-tensor is an element of surface in four-space.

8.1.1 Orthogonality of $\tilde{G}^{\mu\nu}$ and $G_{\mu\nu}$

Equation (8.9) means that in free space

$$\tilde{G}^{\mu\nu}G_{\mu\nu} = 0, \tag{8.10}$$

where

$$\tilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}G_{\sigma\rho}, \tag{8.11a}$$

$$G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}\tilde{G}^{\sigma\rho} \tag{8.11b}$$

In contra-variant covariant notation the field tensors are defined by [1] ($c = 1$),

$$G_{\mu\nu} := \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix},$$
$$G^{\mu\nu} := \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix}. \tag{8.12}$$

Therefore in calculating the dual we use,

$$\begin{aligned} &G_{00} = 0, \quad G_{01} = E_1, \quad G_{02} = E_2, \quad G_{03} = E_3, \\ &G_{10} = -E_1, \quad G_{11} = 0, \quad G_{12} = -B_3, \quad G_{13} = B_2, \\ &G_{20} = -E_2, \quad G_{21} = B_3, \quad G_{22} = 0, \quad G_{23} = -B_1, \\ &G_{30} = -E_3, \quad G_{31} = -B_2, \quad B_{32} = B_1, \quad G_{33} = 0, \end{aligned} \tag{8.13}$$

with $\epsilon^{0123} = -\epsilon_{0123} = 1$, $\epsilon^{0132} = -1$, $\epsilon^{0312} = 1$, and so on. If any two indices are the same, the tensor element is zero. Therefore, element by element,

$$\tilde{G}^{01} = -\tilde{G}^{10} = \frac{1}{2}\left(\epsilon^{0123}G_{23} + \epsilon^{0132}G_{32}\right) = -B^1,$$

$$\tilde{G}^{02} = -\tilde{G}^{20} = \frac{1}{2}\left(\epsilon^{0213}G_{13} + \epsilon^{0231}G_{31}\right) = -B^2,$$

$$\tilde{G}^{03} = -\tilde{G}^{30} = \frac{1}{2}\left(\epsilon^{0312}G_{12} + \epsilon^{0321}G_{21}\right) = -B^3, \quad (8.14)$$

$$G^{12} = -\tilde{G}^{21} = \frac{1}{2}\left(\epsilon^{1203}G_{03} + \epsilon^{1230}G_{30}\right) = E^3,$$

$$\tilde{G}^{13} = -\tilde{G}^{31} = \frac{1}{2}\left(\epsilon^{1302}G_{02} + \epsilon^{1320}G_{20}\right) = -E^2,$$

$$\tilde{G}^{23} = -\tilde{G}^{32} = \frac{1}{2}\left(\epsilon^{2301}G_{01} + \epsilon^{2310}G_{10}\right) = E^1,$$

giving the dual contra-variant pseudo-tensor,

$$\tilde{G}^{\mu\nu} = \begin{bmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{bmatrix}, \quad (8.15)$$

from which

$$\begin{aligned} \tilde{G}^{\mu\nu}G_{\mu\nu} &= \tilde{G}^{01}G_{01} + \tilde{G}^{02}G_{02} + \tilde{G}^{03}G_{03} \\ &+ \tilde{G}^{10}G_{10} + \tilde{G}^{12}G_{12} + \tilde{G}^{13}G_{13} \\ &+ \tilde{G}^{20}G_{20} + \tilde{G}^{21}G_{21} + \tilde{G}^{23}G_{23} \\ &= -3\left(B^1E_1 + B^2E_2 + B^3E_3\right), \end{aligned} \quad (8.16)$$

by geometry. Therefore the Lorentz invariant is

$$\boldsymbol{B} \cdot \boldsymbol{E} = 0. \quad (8.17)$$

In the usual theory [1], this geometrical result in four-space is consistent with the fact that the transverse $\boldsymbol{B}$ (e.g. a plane wave) is orthogonal to the transverse $\boldsymbol{E}$ in the vacuum propagation of electromagnetic waves. The invariant is then set to zero because the longitudinal components are set to zero. In the theory of these volumes [4—6] however, $\boldsymbol{B}^{(3)}$ occurs as a real, longitudinal and physical axial pseudo-vector in the Z axis of propagation. The invariant (8.17) reduces then to the T negative pseudo-scalar $\boldsymbol{E}^{(3)} \cdot \boldsymbol{B}^{(3)}$, which is not necessarily zero in vacuo. Experimental evidence shows conclusively that $\boldsymbol{B}^{(3)}$ is non-zero (inverse Faraday effect [4-6]), but there appears to be no evidence for a real longitudinal $\boldsymbol{E}^{(3)}$ in electro-optics. (There is however, a Coulombic $\boldsymbol{E}^{(3)}$.) There is therefore a degree of uncertainty about the traditional invariant (8.17), but as shown in Sec. 8.3.1, the invariant formed from the product of dual vectors is always identically zero for all $\boldsymbol{E}$ and $\boldsymbol{B}$. This illustrates that the traditional view lacks the translation generator. For the time being we proceed with the invariant (8.17). If there is no longitudinal $\boldsymbol{E}^{(3)}$, Eq. (8.17) then means that either $\boldsymbol{E}^{(3)} = \boldsymbol{0}$ or is pure imaginary, so that the real (i.e., physically significant) part of $\boldsymbol{E}^{(3)} \cdot \boldsymbol{B}^{(3)*} = 0$. The argument for $c\boldsymbol{B}^{(3)}$ being dual to $-i\boldsymbol{E}^{(3)}/c$ (S.I. units) was initiated in Vol. 1 [4]. Significantly, this choice allows the use of Majorana's complex wavefunction (Chaps. 4 and 6). Weinberg's equation for any spin [4] produces Majorana's equation (Chap. 6) for the electromagnetic sector with the same complex sum of magnetic and electric components. If this sum were not complex, Weinberg's equation would not reduce to Maxwell's equation and would not therefore be valid for any spin. Although $-i\boldsymbol{E}^{(3)}/c$ is unphysical at first order it produces the *acausal*, or zero, relativistic energy combination shown to be present in the generalization of Weinberg's equation given by Ahluwalia and Ernst. The link between the Ahluwalia-Ernst equation and the $\boldsymbol{B}^{(3)}$ field was pointed out by Dvoeglazov [18], who has also developed the theory significantly [19,20]. The relativistic energyless combination from $\boldsymbol{B}^{(3)}$ and $-i\boldsymbol{E}^{(3)}/c$ is given by

$$\begin{aligned} En(\text{"acausal"}) &= \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)*} - B^{(0)2}, \\ B^{(0)} = |\boldsymbol{B}^{(3)}| &= -i^2\boldsymbol{E}^{(3)} \cdot \boldsymbol{E}^{(3)} - E^{(0)2}, \\ E^{(0)} &= |-i\boldsymbol{E}^{(3)}| = 0. \end{aligned} \quad (8.18)$$

This result is in turn consistent with the quadratic light-like products [4] of PL vectors,

$$B^{\mu}B_{\mu}^{*} = E^{\mu}E_{\mu}^{*} = 0. \quad (8.19)$$

As shown in the next section, the only non-zero components of the PL vectors $\tilde{B}^{\mu}$

and $\tilde{E}^{\mu}$ are the longitudinal ((3)) and time-like ((0)) components. In $\tilde{B}^{\mu}$, $\boldsymbol{B}^{(3)}$ is real and physical; in $\tilde{E}^{\mu}$, the $iE^{(3)}/c$ component is unphysical at order one, but not mathematically zero. This condition is essential for a self-consistent definition of the energyless combination given in Eq. (8.18), i.e., is essential to define the acausal solution predicted by the Weinberg equation for any spin, generalized by Ahluwalia and Ernst. Conversely, the Weinberg and Ahluwalia-Ernst equations for the electromagnetic sector in vacuo are consistent with the existence of $\boldsymbol{B}^{(3)}$, as first pointed out by Dvoeglazov [18]. These various arguments converge on the conclusion that the symmetry of the electromagnetic sector in unified field theory is not U(1).

8.1.2.1. Another Interpretation of $\boldsymbol{E} \cdot \boldsymbol{B}$

Equation (8.17), i.e., $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ in the vacuum, is the direct result of geometry in four dimensions. In the ((1), (2), (3)) basis of these volumes it means that

$$I_2 = \boldsymbol{E}^{(1)} \cdot \boldsymbol{B}^{(1)*} + \boldsymbol{E}^{(2)} \cdot \boldsymbol{B}^{(2)*} + \boldsymbol{E}^{(3)} \cdot \boldsymbol{B}^{(3)*}, \tag{8.20}$$

and conceivably, transverse components may lose orthogonality experimentally if accompanied by longitudinal components in vacuo. For example if $\boldsymbol{B}^{(3)}$ is accompanied by the tiny, real and phase dependent $\boldsymbol{E}^{(3)}$ caused by photon mass [21—23] in the Proca equation in vacuo, or if $\boldsymbol{E}^{(3)}$ is taken to be the Coulombic contribution as discussed by Chubykalo and Smirnov-Rueda [15]. In this eventuality, the real $\boldsymbol{E}^{(3)}$ would be dual to an added, imaginary, part of $\boldsymbol{B}^{(3)}$, so that the longitudinal components would be complex in general, with real and imaginary components. This is precisely the conclusion reached by Múnera and Guzmán [24], who discuss it in terms of the recent experimental finding by Rikken and van Tiggelen [25] associated with temporal oscillations in the direction of propagation of light in vacuo, i.e., in terms of interaction effects between transverse and longitudinal components. These effects would be an experimental demonstration of non-U(1) sector symmetry. Equation (8.17) is a rigorous geometrical constraint in the theory of vacuum electromagnetism. It is usually interpreted to mean that both $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$ must vanish identically, but this conclusion is not supportable, Eq. (8.17) means that the complete vector $\boldsymbol{E}$ is orthogonal to the complete vector $\boldsymbol{B}$ in the vacuum.

Equation (8.17) also dictates the rigorous definition of boson (photon) helicity. The zero component is given by

$$\zeta_0 := \boldsymbol{p}^{(1)} \cdot \boldsymbol{J}^{(2)} = 0, \tag{8.21}$$

and the ± 1 components by

$$\zeta_{\pm} := \frac{\boldsymbol{p}^{(1)} \cdot \boldsymbol{J}^{(2)}}{|\boldsymbol{p}^{(1)} \cdot \boldsymbol{J}^{(2)}|} = \frac{\boldsymbol{p}^{(2)} \cdot \boldsymbol{J}^{(1)}}{|\boldsymbol{p}^{(2)} \cdot \boldsymbol{J}^{(1)}|} = \frac{\boldsymbol{p}^{(3)} \cdot \boldsymbol{J}^{(3)*}}{|\boldsymbol{p}^{(3)} \cdot \boldsymbol{J}^{(3)*}|} = \pm 1, \tag{8.22}$$

after application of $\hat{P}$ and magnitude normalization. The overall boson helicity is, as required, 0 and ± 1; for example the helicity of the photon with mass from the Proca equation [4—6,21,22]. Definition (8.21) allows for any type of inter-relation between the longitudinal and transverse components, an inter-relation which is dictated by the overall equation of motion. The most general equation of motion to date appears to have been derived by Vigier [23] and by Dvoeglazov [18]; work based on the Weinberg equation for any spin, including boson spin.

8.2 The Dual Pseudo-Vector $\tilde{B}^{\mu}$ and Dual Vector E^{μ}

The axial vector dual in four-space is constructed geometrically from the integral over a hyper-surface, or manifold, a rank three-tensor in four-space antisymmetric in all three indices [1]. In three dimensional space the volume of the parallelepiped spanned by three vectors is equal to the determinant of the third rank formed from the components of the vectors [1]. In four dimensions, the projections can be defined analogously of the volume of the parallelepiped (i.e., the areas of the hyper-surface) spanned by three vector elements dx^{μ}, dx'^{μ}, and dx''^{μ}. They are given by the determinant,

$$dS^{\mu\nu\sigma} = \begin{vmatrix} dx^{\mu} & dx'^{\mu} & dx''^{\mu} \\ dx^{\nu} & dx'^{\nu} & dx''^{\nu} \\ dx^{\sigma} & dx'^{\sigma} & dx''^{\sigma} \end{vmatrix}, \tag{8.23}$$

which forms a tensor of rank three, antisymmetric in all three indices. The axial four-vector element dS^{μ} dual to the tensor element $dS^{\mu\nu\sigma}$ is the element of integration over a hyper-surface in four dimensions [1],

$$d\tilde{S}^{\mu} = -\frac{1}{6}\epsilon^{\mu\nu\sigma\rho} dS_{\nu\sigma\rho}, \tag{8.24a}$$

$$dS_{\nu\sigma\rho} = \epsilon_{\mu\nu\sigma\rho} d\tilde{S}^{\mu}, \tag{8.24b}$$

so that $d\tilde{S}^{0} = dS^{123}$, $d\tilde{S}^{\prime} = dS^{023}$ and so on. The S^{0} component of S^{μ} is therefore equivalent to the S^{123} component of $S^{\nu\sigma\rho}$, normal to it and equal in magnitude. The PL vector is an example of an axial four-vector dual to the three-manifold in four-space. This result is derived from geometry in four dimensions and if the PL vector were null (i.e., if all its components were zero) it could not represent a non-zero hyper-surface in four dimensions. It is shown as follows that this result implies the existence of a non-zero $\boldsymbol{B}^{(3)}$ field in the vacuum. Geometrically, the dual vector S^{μ} is a four-vector equal in magnitude to the area of the hyper-surface to which it is dual, and is normal to this hyper-surface. It is therefore perpendicular to all lines drawn in the hyper-surface. In particular, the element $dS^0 = dXdYdZ$ is an element of three dimensional volume, dV, the projection of the hyper-surface element on to the hyper-plane x^0 = constant.

8.2.1 Geometrical Definition of the PL Vector for Vacuum Electromagnetism

The idea of a PL vector for vacuum electromagnetism was introduced in Vol. 1, Chap. 11, prior to that it had been used only in the particle interpretation of the photon, using generators of the Poincaré group (ten parameter, inhomogeneous Lorentz group). The PL vector is defined from geometry in this section through the three-manifold,

$$S^{\mu\nu\sigma} := \begin{vmatrix} \epsilon^{\mu} & \partial^{\mu} & A^{\mu} \\ \epsilon^{\nu} & \partial^{\nu} & A^{\nu} \\ \epsilon^{\sigma} & \partial^{\sigma} & A^{\sigma} \end{vmatrix}, \tag{8.25}$$

which involves an antisymmetric combination as required of the normalized space-time translation generator ϵ^{μ} and the potential four-vector A^{μ}. Equation (8.25) defines the fully antisymmetric rank three-tensor,

$$\begin{aligned} S^{\mu\nu\sigma} := {} & \epsilon^{\mu}\left(\partial^{\nu}A^{\sigma} - \partial^{\sigma}A^{\nu}\right) - \partial^{\mu}\left(\epsilon^{\nu}A^{\sigma} - \epsilon^{\sigma}A^{\nu}\right) \\ & + A^{\mu}\left(\epsilon^{\nu}\partial^{\sigma} - \epsilon^{\sigma}\partial^{\nu}\right), \end{aligned} \tag{8.26}$$

which consists of three terms, the first of which can be written as the product of ϵ^{μ} with the antisymmetric field tensor,

$$G^{\nu\sigma} := \partial^{\nu}A^{\sigma} - \partial^{\sigma}A^{\nu}. \tag{8.27}$$

This term gives the PL vector used in Vol. 1 through the duality definition,

$$\tilde{S}^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}S_{\nu\sigma\rho}, \tag{8.28}$$

in contra-variant-covariant notation. This definition (also used by Ryder [26]) differs from that given by Landau and Lifshitz [1]. The former uses a premultiplier $+1/2$, the latter a premultiplier $-1/6$. The second two terms of the sum in Eq. (8.26) can be eliminated using a combination of the free photon minimal prescription [4—6] and the quantum hypothesis,

$$\partial_{\mu} = -i\frac{e}{\hbar}A_{\mu}. \tag{8.29}$$

The manifold defined in Eq. (8.26) reduces precisely to

$$S^{\mu\nu\sigma} = \epsilon^{\mu}\left(\partial^{\nu}A^{\sigma} - \partial^{\sigma}A^{\nu}\right), \tag{8.30}$$

using Eq. (8.29) because the second two terms in Eq. (8.26) cancel exactly.

It is now possible to adopt the standard definition of the PL vector to the problem at hand to give

$$\tilde{G}^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}\epsilon_{\nu}G_{\sigma\rho}, \tag{8.31}$$

where

$$G_{\sigma\rho} = \partial_{\sigma}A_{\rho} - \partial_{\rho}A_{\sigma}. \tag{8.32}$$

In Eq. (8.31), $\tilde{G}^{\mu}$ is dual to the third rank $\epsilon_{\nu}G_{\sigma\rho}$ in four dimensions, and normal to it with the same magnitude. In the usual view [1], there is *nothing* normal to the purely transverse $G_{\sigma\rho}$ (U(1) field symmetry), and therefore $\tilde{G}^{\mu}$ cannot be consistently dual to $\epsilon_{\nu}G_{\sigma\rho}$. This is geometrically self-inconsistent at the most basic level, and inconsistent with the four dimensional Lie algebra of the Poincaré group [4]. The accepted sector symmetry of the electromagnetic field is therefore U(1) (two dimensions), while that of the electromagnetic particle (the photon), is the Poincaré group (four dimensions).

In Eq. (8.31) we have therefore discarded the accepted U(1) sector symmetry, and give as follows the details of the calculation of the four dimensional field pseudo vector $\tilde{B}^{\mu} := \tilde{G}^{\mu}$. For convenience $c = 1$.

$$\tilde{G}^{\mu} := \left(\tilde{G}^0, \tilde{G}^1, \tilde{G}^2, \tilde{G}^3\right), \qquad \epsilon_{\nu} := (1, 0, 0, -1),$$

$$\epsilon^{0123} = -\epsilon_{0123} = 1 \text{ etc.},$$

$$G_{00} = 0, \quad G_{01} = E_1, \quad G_{02} = E_2, \quad G_{03} = E_3,$$
$$G_{10} = -E_1, \quad G_{11} = 0, \quad G_{12} = -B_3, \quad G_{13} = B_2, \tag{8.33}$$
$$G_{20} = -E_2, \quad G_{21} = B_3, \quad G_{22} = 0, \quad G_{23} = -B_1,$$
$$G_{30} = -E_3, \quad G_{31} = -B_2, \quad G_{32} = B_1, \quad G_{33} = 0.$$

The non-zero components of $\tilde{G}^{\mu}$ are therefore

$$\tilde{G}^0 = \frac{1}{2}\left(\epsilon^{0312}\epsilon_3 G_{12} + \epsilon^{0321}\epsilon_3 G_{21}\right) = B^3,$$

$$\tilde{G}^3 = \frac{1}{2}\left(\epsilon^{3012}\epsilon_0 G_{12} + \epsilon^{3021}\epsilon_0 G_{21}\right) = B^3,$$

$$\tilde{G}^1 = \frac{1}{2}\left(\epsilon^{1023}\epsilon_0 G_{23} + \epsilon^{1032}\epsilon_0 G_{32} + \epsilon^{1320}\epsilon_3 G_{20} + \epsilon^{1302}\epsilon_3 G_{02}\right) = B^1 + E^2, \tag{8.34}$$

$$\tilde{G}^2 = \frac{1}{2}\left(\epsilon^{2013}\epsilon_0 G_{13} + \epsilon^{2031}\epsilon_0 G_{31} + \epsilon^{2310}\epsilon_3 G_{10} + \epsilon^{2301}\epsilon_3 G_{01}\right) = B^2 - E^1,$$

and the complete PL vector is in consequence,

$$\tilde{B}^{\mu} = \left(B^3,\ B^1 + E^2,\ B^2 - E^1,\ B^3\right). \tag{8.35}$$

The E^3 component does not appear in $\tilde{B}^{\mu}$ because E^3 is a component of a polar vector. As discussed in Chap. 11 of Vol. 1 [4], the $\tilde{B}^1$ and $\tilde{B}^2$ components vanish by using Maxwell's equations in vacuo. This result is easiest to see with the plane waves,

$$\boldsymbol{E} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi}, \qquad \boldsymbol{B} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}, \tag{8.36}$$

from which

$$B^1 = -E^2, \qquad B^2 = E^1, \tag{8.37}$$

with $1 = X$; $2 = Y$ (and with, for convenience, the non-S.I. $c = 1$). The PL vector is therefore in contra-variant notation,

$$\tilde{B}^{\mu} = \left(B^3,\ 0,\ 0,\ B^3\right), \tag{8.38}$$

(with $B^3 = B^{(3)}$). The relativistic helicity in the particle interpretation is usually defined [4] as the projection of the space-like components of the axial vector on to the direction of propagation of the polar vector, the result being normalized to a dimensional scalar. The helicity is therefore a pseudo-scalar and if we project the relevant components of B^{μ} on to the relevant component of ϵ^{μ} we obtain $0, \pm 1$. This is because the projection (dot product) of the 1 and 2 space components of B^{μ} on to the 3 component of ϵ^{μ} is always 0. The projection of the 3 component of B^{μ} on to the 3 component of ϵ^{μ} is 1, and this reverses sign to -1 upon application of $\hat{P}$ [4]. The classical relativistic helicity of the field, calculated in this way, is the helicity of a boson. The latter can be identified with the photon (the electromagnetic particle) after quantization. In the usual view [1] on the other hand we obtain $\tilde{B}^{\mu} = (0, 0, 0, 0)$ and the classical relativistic field helicity is $0, \pm 0$, which is not that of a boson. In this respect the received view is clearly self-inconsistent, because the relativistic helicity of the classical field is different from that of the quantized photon, also a fully relativistic particle which in the received view has no mass or rest frame. The received view also asserts [26] that the helicity of the field is ± 1, with the 0 component missing — this procedure is special pleading on behalf of U(1) field symmetry, and on behalf of $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$.

Ryder, for example, [26], in an excellent monograph, arrives at this result for the photon viewed as particle within the Poincaré group. He constructs the PL vector formed from the rotation generators of the Poincaré group for light-like angular momenta, and not field components. The PL vector corresponding to the photon's angular momentum corresponds in four-space ($c = 1$) to

$$\tilde{J}^{\mu} = \left(J^{(3)},\ 0,\ 0,\ J^{(3)}\right), \tag{8.39}$$

and the light-like photon linear four-momentum is ($c = 1$),

$$p^{\mu} = \left(p^{(3)}, 0, 0, p^{(3)}\right). \tag{8.40}$$

It is then asserted [26] that the helicity of the photon must be ±1 because J^{μ} is proportional to p^{μ}. The 0 component of the boson is left out of consideration. This particular definition of helicity is finally asserted to reflect U(1) field symmetry concomitant with the assumed massless photon. Whether the photon has mass or not, the relativistic field helicity is 0, ±1 if we systematically construct the PL field vector from the field tensor $G_{\mu\nu}$.

We therefore arrive at the key to our argument, *in that the concept of the Poincaré group has been applied to the photon as particle but not to its concomitant field.* The photon is defined using the generators [26] of the Poincaré group, the field is defined in terms of the U(1) group. This is self-inconsistent because there is, in consequence, no $\boldsymbol{B}^{(3)}$ component in the received view corresponding to the $\boldsymbol{J}^{(3)}$ component of the particle interpretation seen in Eq. (8.39). The source of the inconsistency is the assertion that Maxwell's equation cannot produce a longitudinal field component in the vacuum. This assertion has been shown repeatedly and independently [4—17] to be false, and in consequence the phaseless $\boldsymbol{B}^{(3)}$ field is an example of the set of longitudinal fields in vacuo that satisfy slightly more general than those of Maxwell. The latter have also been generalized [23,18] recently in the field-particle literature.

If we try to adopt a U(1) group symmetry for the spin of the photon, its helicity vanishes, because

$$\tilde{J}^{\mu} =? \,(0, 0, 0, 0). \tag{8.41}$$

The particle obtained within the U(1) group is not a boson, and therefore not a photon. If field-particle duality is accepted as the philosophical basis of quantum mechanics, the U(1) group produces an incorrect quantization of the electromagnetic field, produces a particle with no spin. In order to obtain the particulate photon the Poincaré group [4—6,27] must be used. It is not sufficient to use the Lorentz group, from which the space-time translation generator is missing. The Poincaré group was first applied in this way by Wigner [27] in 1939 to the particulate photon, but the undulatory field concomitant with the photon has dictated conventionally a U(1) group symmetry for the field, not the Poincaré group symmetry. Attempts at defining the photon in terms of spin and mass before Wigner's classic work of 1939 [27] were incomplete, and *the space-time translation generator was therefore missing from the electromagnetic field theory.* In consequence, the vector dual to the antisymmetric field tensor used by Lorentz, Poincaré and Einstein could not have been defined. It was first introduced in Vol. 1 [4], Chap. 11 by reinstating the translation generator. Only then does it become clear that the dual vector (Eq. (8.38)) contains only $\boldsymbol{B}^{(3)}$. In the next section it will be shown that conservation of the dual vector in vacuo leads back to the vacuum Maxwell equations.

Any attempt to assert that the latter are inconsistent with $\boldsymbol{B}^{(3)}$ is countered by the derivation of Maxwell's equations from $\tilde{B}^{\mu}$, (next section).

All particles were characterized by Wigner [27] in terms of the two Casimir invariants of the Poincaré group, and since photon and electromagnetic field are inseparable and complementary concepts the Poincaré group must also be used for the electromagnetic field. One does not have a Poincaré group if the translation generator is missing. This is true whether or not the photon is considered to have mass, because the translation generator is part of the structure of the group, i.e., is a group generator. The other group generators [4—6,26] are the boosts and rotations. Furthermore, Noether's theorem [26] is reduced to conservation of energy-momentum only with the use of the space-time translation generator. The latter is the energy-momentum vector itself within $\hbar$ [26]. The definition of the electromagnetic field includes the property of carrying energy-momentum, described by a four-vector, and this introduces automatically the space-time translation generator to which the energy-momentum four-vector is directly proportional. Otherwise the field does not carry energy-momentum. In the received view [26,27] energy-momentum is introduced using only the transverse components, but this procedure implies the *existence* of the translation generator (through its proportionality to the energy-momentum four-vector [26]) and therefore the existence of the dual vector (8.38). If $\boldsymbol{B}^{(3)}$ were zero, this dual vector would be null. Self-consistency in electromagnetic field theory requires a non-zero $\boldsymbol{B}^{(3)}$. Through the cyclic relations [4] the $\boldsymbol{B}^{(3)}$ field is implied by the existence of the $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ field, and is observed empirically in magneto-optics [4—6].

The nature of the dual vector (8.38) can be deduced without using any equation of motion, but the dual vector is a fundamental geometrical property in four dimensions. The electromagnetic field exists within special relativity, which assumes four dimensional space-time to be physically significant. The complete description of the electromagnetic field must then involve all the known generators — the boosts, rotations and translation generators. For light-like propagation in Z, the normalized translation generator is, as we have seen, the four unit vector $\epsilon^{\mu} = (1, 0, 0, 1)$. In the momentum representation [26] it is proportional to $\kappa(1, 0, 0, 1)$ where κ is the magnitude of the wavenumber in vacuo. If this were not the case, the photon would not have a linear momentum $\hbar\kappa$ from the quantum hypothesis of Louis de Broglie. Since ϵ^{μ} is a generator of the Poincaré group, it is a concept of space-time itself, and it is present, as just argued, in the electromagnetic field, otherwise the Poynting vector, for example, would vanish because there would be no linear field momentum. Field and particle are complementary concepts and are present within the same, Poincaré group, symmetry. This means that $\boldsymbol{B}^{(3)}$ is non-zero geometrically.

If the photon carries mass, the unit four-vector ϵ^{μ} can no longer be light-like, and the time-like component is no longer equal in magnitude to the space-like component. However, ϵ_{μ} is still orthogonal to $\tilde{B}^{\mu}$,

$$\epsilon_\mu \tilde{B}^\mu := 0, \tag{8.42}$$

whether or not the particle has mass, because Eq. (8.42) is a geometrical property of four dimensions. The photon as massive particle can, in theory, possess a rest frame, in which its linear momentum is zero, but in which its time-like component is non-zero. In the hypothetical photon rest frame the $\tilde{B}^\mu$ vector must be space-like by definition [26],

$$\tilde{B}^\mu(\text{rest frame}) = (0,\ 0,\ 0,\ B^3), \tag{8.43}$$

and this represents the magnetic field (or magnetic dipole moment) of the stationary photon, whose phase is zero. The cyclic equations [4—6] still hold in the rest frame, because $\boldsymbol{B}^{(3)}$ is phaseless. The cyclic equations in the rest frame are the equations of the ((1), (2), (3)) basis itself [4], i.e., the relations in the rest frame between the unit vectors $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$, and $\boldsymbol{e}^{(3)}$. The proper time for the massive photon is a Lorentz invariant [26,1] and does not vanish.

8.3 The Dual Four-Vectors of the Vacuum Electromagnetic Field

This section summarizes the main results of the chapter and lists for reference the dual vectors and tensors of the vacuum electromagnetic field. For convenience, we use the $c = 1$ notation, and make no assumptions about the nature of the B^3 and E^3 components. The dual pseudo-vectors are Pauli-Lubanski axial four-vectors, and as such contain no polar components, only axial components. Experimental evidence [4—6] points towards a real, axial $B^3 := B^{(3)}$ but there appears to be no electro-optic evidence for a real E^3. It is however possible that $i\boldsymbol{E}^{(3)}$ may interact with an imaginary atomic or molecular property tensor to form a real and physical interaction Hamiltonian or that E^3 may represent the Coulombic component [4—6]. The property tensor would, for example, be the imaginary part of an electric dipole moment in semi-classical perturbation theory [4—6]. The Proca equation also produces a minute but real and phase dependent $\boldsymbol{E}^{(3)}$ for the photon with mass. We use contra-variant-covariant notation and dual pseudo-tensors and pseudo-vectors are denoted with a tilde. It is always assumed that all components of the ordinary field tensor in vacuo are non-zero, i.e., may be real, imaginary or complex.

The field tensors in vacuo and their tensor duals are

$$\tilde{G}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}G_{\sigma\rho}, \qquad G^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\sigma\rho}\tilde{G}_{\sigma\rho},$$
$$\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}G^{\sigma\rho}, \qquad G_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\sigma\rho}\tilde{G}^{\sigma\rho}, \tag{8.44}$$

in which the minus signs are needed for self-consistency (this is not always made clear in the literature). In matrix form they are

$$G_{\sigma\rho} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix},$$
$$\tilde{G}^{\mu\nu} = \begin{bmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{bmatrix},$$
$$G^{\sigma\rho} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix},$$
$$\tilde{G}_{\mu\nu} = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{bmatrix}, \tag{8.45}$$

and it is seen that electric components become magnetic components with change of sign. The product of tensor and dual is a Lorentz invariant,

$$I_2 = G_{\mu\nu}\tilde{G}^{\mu\nu} = -\boldsymbol{E}\cdot\boldsymbol{B}, \tag{8.46}$$

a fundamental geometrical constraint. This is the result shown in Eq. (8.20). The received view asserts that both $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$ are zero, but this assertion leads to null dual vectors and pseudo-vectors and is self-inconsistent as discussed already. Furthermore, the assumption that $\boldsymbol{E}^{(3)}$ is zero leads similarly to two null dual vectors (Eqs. (8.50a) and (8.50b) as follows) and is similarly self-inconsistent.

The dual pseudo-vectors of the electromagnetic field in vacuo are

$$\tilde{B}^{\mu} = \tilde{G}^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}G_{\sigma\rho}\epsilon_{\nu}, \qquad \tilde{B}_{\mu} = \tilde{G}_{\mu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}G^{\sigma\rho}\epsilon^{\nu}, \tag{8.47a}$$

and the dual vectors are

$$E^{\mu} = G^{\mu} = -\frac{1}{2}\epsilon^{\mu\nu\sigma\rho}\tilde{G}_{\sigma\rho}\epsilon_{\nu}, \qquad E_{\mu} = G_{\mu} = -\frac{1}{2}\epsilon_{\mu\nu\sigma\rho}\tilde{G}^{\sigma\rho}\epsilon^{\nu}, \tag{8.47b}$$

where again two minus signs are needed for self-consistency. The dual pseudo-vectors can be displayed in covariant and contra-variant form,

$$\tilde{B}^{\mu} := \tilde{G}^{\mu} = \tilde{G}^{\mu\nu}\epsilon_{\nu} = \left(B^{3},\ B^{1}+E^{2},\ B^{2}-E^{1},\ B^{3}\right), \qquad \tilde{B}_{\mu} := \tilde{G}_{\mu} = \tilde{G}_{\mu\nu}\epsilon^{\nu} = \left(B_{3},\ -B_{1}-E_{2},\ E_{1}-B_{2},\ -B_{3}\right), \tag{8.48a}$$

$$E^{\mu} := G^{\mu} = G^{\mu\nu}\epsilon_{\nu} = \left(E^{3},\ E^{1}-B^{2},\ E^{2}+B^{1},\ E^{3}\right), \qquad E_{\mu} := G_{\mu} = G_{\mu\nu}\epsilon^{\nu} = \left(E_{3},\ B_{2}-E_{1},\ -E_{2}-B_{1},\ -E_{3}\right). \tag{8.48b}$$

Using the condition (8.37) (the vacuum Maxwell equations), the four dual vectors become

$$\tilde{B}^{\mu} = \left(B^{3},\ 0,\ 0,\ B^{3}\right), \tag{8.49a}$$

$$\tilde{B}_{\mu} = \left(B_{3},\ 0,\ 0,\ -B_{3}\right), \tag{8.49b}$$

$$E^{\mu} = \left(E^{3},\ 0,\ 0,\ E^{3}\right), \tag{8.50a}$$

$$E_{\mu} = \left(E_{3},\ 0,\ 0,\ -E_{3}\right), \tag{8.50b}$$

and this is the central result of this chapter, indeed of electromagnetic field theory in general.

Since $\tilde{B}^{\mu}$ and E^{μ} are conserved in vacuo, we have

$$\partial_{\mu}\tilde{B}^{\mu} = \partial_{\mu}E^{\mu} = 0. \tag{8.51}$$

If it is assumed that the vectors (8.48) are also conserved, we obtain the vacuum Maxwell equations *as a particular solution of the more general* (8.51),

$$\partial_{\mu}\tilde{B}^{\mu} = \partial_{\mu}\left(\tilde{G}^{\mu\nu}\epsilon_{\nu}\right) = \partial_{\mu}\tilde{G}^{\mu\nu} = 0, \tag{8.52a}$$

$$\partial_{\mu}E^{\mu} = \partial_{\mu}\left(G^{\mu\nu}\epsilon_{\nu}\right) = \partial_{\mu}G^{\mu\nu} = 0, \tag{8.52b}$$

because ϵ_{ν} is constant. Switching from the coordinate to the momentum representation (using de Broglie's hypothesis) it is found that

$$\partial_{\mu}\tilde{B}^{\mu} = \epsilon_{\mu}\tilde{B}^{\mu} = 0, \tag{8.53a}$$

$$\partial_{\mu}E^{\mu} = \epsilon_{\mu}E^{\mu} = 0, \tag{8.53b}$$

showing that ϵ_{μ} is orthogonal to the dual vector, as required geometrically. One can reverse the argument and assert that geometry implies the quantum hypothesis, a result which is consistent with the rule that all equations of physics are geometrical in nature [26].

It can be inferred from the above deduction of the Maxwell equations that if the dual vectors were null vectors there would be no field present, and this is a self-inconsistency of the standard field theory [26,1]. The Maxwell equations for the longitudinal B^{3} and E^{3} are obtained directly from Eq. (8.51) and state that the space and time derivatives of both B^{3} and E^{3} vanish. This is the condition of conservation itself. For example, the conservation equation for $\boldsymbol{B}^{(3)}$ is, from Eq. (8.51),

$$\partial_{\mu}\tilde{B}^{\mu} = \partial_{0}\tilde{B}^{0} + \partial_{3}\tilde{B}^{3} = \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}^{(3)} + \frac{\partial}{\partial Z}\boldsymbol{B}^{(3)} = 0, \qquad \text{i.e. } \frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = 0, \quad \frac{\partial \boldsymbol{B}^{(3)}}{\partial Z} = 0. \tag{8.54}$$

This condition allows, in general, for a phase dependence in $\boldsymbol{B}^{(3)}$, i.e.,

$$\frac{1}{c}\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = -\frac{\partial}{\partial Z}\boldsymbol{B}^{(3)}, \tag{8.55}$$

but if $\boldsymbol{B}^{(3)}$ is phaseless, as used in these volumes [4—6], the time and space derivatives in Eq. (8.55) vanish independently as in Eq. (8.54). This result means that $\boldsymbol{B}^{(3)}$ is explicitly time and space independent, but not implicitly so, because it is formed from the product of one phase $e^{i\phi}$ with its complex conjugate $e^{-i\phi}$. The $\boldsymbol{B}^{(3)}$ field must *always* by geometry, be defined in this way, otherwise it is not the $\boldsymbol{B}^{(3)}$ field. It is incorrect to refer to it as a *static* magnetic field. It is the fundamental, phaseless, spin field of electromagnetism in vacuo, and as such is conserved in vacuo. If the field interacts with a fermion, it is no longer conserved, because angular momentum is transferred from field to fermion, a process which is controlled by the Dirac equation [6].

8.3.1 Orthogonality of the Dual Vectors

The vector product of G_μ and $\tilde{G}^\mu$ is a null Poincaré invariant of the electromagnetic field under all conditions,

$$\tilde{G}_\mu G^\mu = G_\mu \tilde{G}^\mu := 0, \tag{8.56}$$

and for all E_1, E_2, E_3, B_1, B_2, B_3. The following are null invariants in the vacuum given the Maxwell equations for transverse components,

$$G_\mu G^\mu = \tilde{G}_\mu \tilde{G}^\mu = 0. \tag{8.57}$$

Equations (8.56) therefore represent the fundamental Poincaré invariant of the electromagnetic field,

$$\begin{aligned} \tilde{G}_\mu G^\mu &= B_3 E^3 - (B_1 + E_2)(E^1 - B^2) \\ &\quad + (E_1 - B_2)(E^2 + B^1) - B_3 E^3 \\ &= -(B_1 E^1 + E_2 E^1 - B_1 B^2 - E_2 B^2) \\ &\quad + (E_1 E^2 - B_2 E^2 + E_1 B^1 - B_1 B^2) := 0 . \end{aligned} \tag{8.58}$$

The dual pseudo-vector is always orthogonal to the dual vector, irrespective of any

equation of motion. The latter follow from conservation of $\tilde{G}^\mu$ or G^μ in vacuo, i.e., from

$$\partial_\mu \tilde{G}^\mu = \partial_\mu G^\mu := 0. \tag{8.59}$$

In the momentum representation they become

$$\epsilon_\mu \tilde{G}^\mu = \epsilon_\mu G^\mu := 0, \tag{8.60}$$

and are also geometrical identities for all E_1, E_2, E_3, B_1, B_2, B_3 under all conditions, in vacuo or in the presence of sources. The well known Maxwell equations follow from this geometrical identity of four dimensional space-time. As follows, it is shown how E_3 and B_3 enter into the equations of motion. Recall that these equations follow from the existence of dual vectors in vacuum electromagnetism and exemplify the rule that equations of physics are always geometrical.

By definition,

$$\partial_\mu G^\mu = \partial_\mu \left(G^{\mu\nu}\epsilon_\nu\right) := 0 . \tag{8.61}$$

Now sum over repeated indices to give

$$\begin{aligned} &\epsilon_0\left(\partial_0 G^{00} + \partial_1 G^{10} + \partial_2 G^{20} + \partial_3 G^{30}\right) \\ &\quad + \epsilon_1\left(\partial_0 G^{01} + \partial_1 G^{11} + \partial_2 G^{21} + \partial_3 G^{31}\right) \\ &\quad + \epsilon_2\left(\partial_0 G^{02} + \partial_1 G^{12} + \partial_2 G^{22} + \partial_3 G^{32}\right) \\ &\quad + \epsilon_3\left(\partial_0 G^{03} + \partial_1 G^{13} + \partial_2 G^{23} + \partial_3 G^{33}\right) = 0 . \end{aligned} \tag{8.62}$$

This is one single equation consisting of a sum of terms. A particular solution of this equation occurs when each term vanishes individually, giving four equations,

$$\begin{aligned} \partial_0 G^{00} + \partial_1 G^{10} + \partial_2 G^{20} + \partial_3 G^{30} &= 0, \\ \partial_0 G^{01} + \partial_1 G^{11} + \partial_2 G^{21} + \partial_3 G^{31} &= 0, \\ \partial_0 G^{02} + \partial_1 G^{12} + \partial_2 G^{22} + \partial_3 G^{32} &= 0, \\ \partial_0 G^{03} + \partial_1 G^{13} + \partial_2 G^{23} + \partial_3 G^{33} &= 0, \end{aligned} \tag{8.63}$$

which can be represented in tensor notation as

$$\partial_\mu G^{\mu\nu} = 0. \tag{8.64}$$

This is the inhomogeneous part of the vacuum Maxwell equations. However, the equation (8.62) is *more fundamental*, and gives more information, than the Maxwell equations: this is the result of constructing the Pauli-Lubanski field representation by introducing the translation generator ϵ_μ, without which there is no well-defined Poincaré group.

Similarly, the geometrical identity $\partial_\mu \tilde{G}^\mu = 0$ leads to the homogeneous Maxwell equations $\partial_\mu \tilde{G}^{\mu\nu} = 0$. The Maxwell equations are particular solutions of the more general geometrical identities, and are equations for one part, $\tilde{G}^{\mu\nu}$, of the complete PL vector $\tilde{G}^{\mu\nu}\epsilon_\nu$. Thus it has been shown that geometry dictates the structure of the field equations, a conclusion which is automatically gauge invariant, because physical (observable) fields are gauge invariant. Gauge transformations are then expected to produce optical Aharonov-Bohm effects [4—6], as discussed in Chap. 4. Equations (8.62) allow many more solutions than Eqs. (8.63), and similarly for their homogeneous equivalents. Specifically, Eqs. (8.62) allow phase independent and phase dependent longitudinal components in vacuo and in the presence of sources. These are related geometrically to the transverse ones. In the received view there are no longitudinal solutions in vacuo and obviously these are not related to the transverse solutions of Maxwell's particular solutions of Eq. (8.63). This is the result of not considering the translation generator, and of working within a U(1) group symmetry for fields, not a Poincaré symmetry. The longitudinal solutions can be obtained also from the particular Eqs. (8.63), a sub-set of Eqs. (8.62), as demonstrated by several authors independently [4—6]. These longitudinal solutions are also physical solutions, for example, the longitudinally directed Coulombic E^3, and the B^3 observed in magneto-optics [4—6]. (The rule used is that a physical field is observable.)

The received view therefore asserts, arbitrarily, that the Poincaré group cannot be used for fields, whereas it can be used for photons. The above development uses the Poincaré group both for photons and for fields. Dirac pointed out repeatedly [28] that the received view is inconsistent with itself, for example, a U(1) symmetry does not allow the Coulombic field perpendicular to the U(1) plane. It is not surprising in retrospect that the electromagnetic field cannot be quantized without serious difficulty [26].

8.4 The Fundamental Equation of Electrodynamics

The orthogonality properties of the Poincaré group are shown in this section to lead to a novel fundamental equation of electrodynamics under all conditions (i.e., absence or presence of sources). Using the transformation from momentum to coordinate representation [26] (the quantum condition),

$$\epsilon_\nu \rightarrow \frac{i}{\kappa}\partial_\nu, \tag{8.65}$$

the translation generator is transformed into the four-derivative, leading under all conditions to the conservation equation,

$$\partial_\mu G^\mu = \partial_\mu \tilde{G}^\mu = 0, \tag{8.66}$$

which becomes the *single* fundamental equation of classical electrodynamics. *The PL vector and pseudo-vector are conserved.* It is shown as follows that this geometrical condition leads to Maxwell's equations, together with equations for the novel fields $\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ which show that they are rigorously non-zero under all conditions.

The orthogonality of G_μ and $\tilde{G}^\mu$ under all conditions dictates the orthogonality condition,

$$\epsilon_\mu \tilde{\epsilon}^\mu = 0, \tag{8.67}$$

on the space-time translation generator. Without loss of generality we can use the unit generators,

$$\epsilon^\mu = \left(1, \frac{\mathrm{v}}{c}, -\frac{\mathrm{v}}{c}, \frac{\mathrm{v}}{c}\right), \tag{8.68a}$$

$$\tilde{\epsilon}^\mu = \left(\frac{\mathrm{v}}{c}, 1, 1, 1\right), \tag{8.68b}$$

where v is a linear velocity (m s^{-1}) and c the speed of light. Equation (8.68a) defines a unit energy-momentum four-vector which is orthogonal to the unit energy momentum four-vector in Eq. (8.68b). The existence of these generators signals the fact that the electromagnetic field has a simultaneous translation-rotation character, so forward momentum (Eq. (8.68a)) is always accompanied by an orthogonal transverse momentum (Eq. (8.68b)). Thus $\epsilon_\mu \tilde{\epsilon}^\mu = 0$, i.e., ϵ_μ is orthogonal to $\tilde{\epsilon}_\mu$. It turns out that this feature converts Eq. (8.66) into two field equations, which under empirical (i.e., observational) restriction, separate out into the four Maxwell equations under any conditions. If $\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ are non-zero, then the vacuum field is lost, the PL vector and pseudo-vector in Eq. (8.66) are both null. *This is plainly self-inconsistent.* Analogously, if we construct a theory of conservation of energy, and lose the energy (kinetic plus potential), that theory is just as inconsistent. Because of the simultaneous rotation and translation, the electromagnetic field is chiral in nature, and empirically this is supported by Arago's well known discovery in 1811 of right and left circular polarization. In consequence, there is no distinction between PL vector and PL pseudo vector. (In three dimensions, one cannot distinguish between polar and axial vector in a chiral group [29].) Henceforth we will refer to them simply as "PL vectors".

In general, therefore, ϵ^μ is not light-like (the condition $c = \mathrm{v}$) and the PL vector is, from Eqs. (8.48) and (8.68b),

$$G^\mu = G^{\mu\nu}\tilde{\epsilon}_\nu = \left(E^1 + E^2 + E^3,\ \frac{\mathrm{v}}{c}E^1 + B^3 - B^2,\ \frac{\mathrm{v}}{2}E^2 - B^3 + B^1,\ \frac{\mathrm{v}}{c}E^3 + B^2 - B^1 \right). \tag{8.69}$$

Using Eq. (8.66),

$$\partial_0\left(E^1 + E^2 + E^3\right) + \partial_1\left(\frac{\mathrm{v}}{c}E^1 + B^3 - B^2\right) + \partial_2\left(\frac{\mathrm{v}}{c}E^2 - B^3 + B^1\right) + \partial_3\left(\frac{\mathrm{v}}{c}E^3 + B^2 - B^1\right) = 0, \tag{8.70}$$

which in vector form (and S.I. units) is

$$\nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{c^2}\mathbf{v}\left(\nabla \cdot \boldsymbol{E}\right), \tag{8.71}$$

an equation which can be written as

$$\mu_0\epsilon_0 \frac{d\boldsymbol{E}}{dt} = \nabla \times \boldsymbol{B}, \tag{8.72}$$

where

$$\frac{d}{dt} := \left(\frac{\partial}{\partial t} + \mathbf{v}\,\nabla \cdot \right) \tag{8.73}$$

is the convective (or total) derivative. Equation (8.71) is a combination of the Gauss law,

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{8.74}$$

and the Ampère-Maxwell law (Ref. 29, Eq.(18.33)),

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{J}, \tag{8.75}$$

where

$$J^\mu := \left(c\rho,\ \mathbf{v}\rho\right), \tag{8.76}$$

is the current density-charge density four-vector [30].

Conservation of the PL vector G^μ leads to the inhomogeneous Maxwell equations in combined form, a result which is profound, and has many consequences. We discuss a few as follows. The conservation equation (8.66) is a form of Noether's theorem [26], the general conservation theorem. It is seen from the balance of terms in Eq. (8.71) that the notion of charge density enters into electrodynamics as the result of the conservation equation (8.66), and charge is field curvature, a geometrical result. Similarly, mass is curvature of the gravitational field (Chaps. 1 to 3). The novel Eq. (8.71) arises within the Poincaré group by inclusion of the unit translation generator ($\tilde{\epsilon}_\nu$), which is absent in the received field theory [26,1]. In the vacuum, the balance of terms in Eq. (8.71) may be achieved by using the notion that charge and current is absent (i.e., that there are no sources). In this limit, the unit translation generator is light-like, and if $\boldsymbol{E}^{(3)}$ =? **0**, G^μ =? 0, and the field is lost entirely. This is a self contradiction, because we are working within a field *conservation* theory. In order to keep the field, $\boldsymbol{E}^{(3)}$ must be non-zero [4—17]. It is observable empirically in Coulombic interactions between charges, however far apart. The $\boldsymbol{E}^{(3)}$ field does not enter into cyclic relations [4—6], however, simply because it is a polar vector. Without the translation generator, this result is not obtainable, and the dual PL vector is missing entirely. The received field theory [26,1] is developed therefore within the U(1) subgroup of the Lorentz group, whose Lie algebra is made up of boost and rotation generators only [26].

In the vacuum, conservation of the PL vector G^μ leads to the ordinary inhomogeneous Maxwell equations,

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}, \tag{8.77a}$$

$$\nabla \cdot \boldsymbol{E} = 0, \tag{8.77b}$$

but also leads to the novel continuity equation,

$$\nabla \times \boldsymbol{B}^{(3)} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(3)}}{\partial t} + \frac{\boldsymbol{u}}{c}\left(\nabla \cdot \boldsymbol{E}^{(3)}\right) := \boldsymbol{0}\,, \tag{8.78}$$

where $\boldsymbol{u}$ is a unit vector ($v = c$ in vacuo). The equations (8.77) and (8.78) are obtained with a light-like $\tilde{\epsilon}^\nu = (1,\ 0,\ 0,\ 1)$. (In the light-like condition, $\tilde{\epsilon}^\nu = \epsilon^\nu$.) The continuity equation (8.78) was first derived in Ref. 8 but is missing entirely from the received view [26,1]. It shows that the curl of $\boldsymbol{B}^{(3)}$ is identically zero in vacuo. Equation (8.78) is satisfied by a uniform, phaseless, electric field $\boldsymbol{E}^{(3)}$ in vacuo, or by a phase dependent longitudinal electric field in vacuo such as that obtained from the Proca equation [4]. The magnitude of $\boldsymbol{E}^{(3)}$ is not related to the magnitude of $\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*}$, and so $\boldsymbol{E}^{(3)}$ can be much smaller than $\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*}$, as expected in long range Coulombic interactions, or by use of the Proca equation. (Note that $\boldsymbol{B}^{(3)}$ in contrast is always related cyclically to $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ as demonstrated empirically [4—6] in magneto-optics.) The $\boldsymbol{B}^{(3)}$ field from Eq. (8.78) is irrotational (zero curl) as demonstrated elsewhere [4—6]. In the presence of sources, the longitudinal components $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$ are non-zero in general within the structure of Eq. (8.66). A significant feature of the analysis is that charges and currents do not enter into the fundamental field equation (8.71), which is balanced automatically by the presence of the translation generator in the definition (8.66). *The extension of the Lorentz group to the Poincaré group therefore eliminates the need to postulate charges and currents into electrodynamics when considered as a field theory.* In other words the Gauss law and Ampère-Maxwell law are balanced precisely in Eq. (8.71), which is Eq. (8.66) in vector form. Electrodynamics becomes a pure field theory, in which charge is a *consequence* of the fact that $\nabla \cdot \boldsymbol{E}$ is not zero; and current is a *consequence* of the fact that the combination $\nabla \times \boldsymbol{B} - \left(1/c^2\right)\partial \boldsymbol{E}/\partial t$ is not zero. If these terms are separately zero we obtain equations which are the traditional *vacuum equations* in which it is customary to assert the absence of sources. Classical electrodynamics happens to have developed historically the other way around, i.e., charges were known

before fields, and so it became natural to think of fields as emanating from charges. As in general relativity however, it is also possible to think of charge as field curvature, and because charge does not enter into the fundamental field equation (8.66), it becomes a field property. The charge and current needed to balance Eq. (8.71) can be positive or negative, and the charge conjugation does not affect the fundamental equation (8.66). (The latter works for positrons as well as for electrons.)

The homogeneous Maxwell equations under all conditions emerge from conservation of the PL vector $\tilde{G}^\mu$, with the generator ϵ^μ. Using Eqs. (8.48) and (8.68a) gives,

$$\tilde{G}^\mu = \tilde{G}^{\mu\nu}\epsilon_\nu = \left(-\frac{v}{c}B^1 + \frac{v}{c}B^2 - \frac{v}{c}B^3,\ B^1 - \frac{v}{c}E^3 - \frac{v}{c}E^2,\right.$$
$$\left. B^2 - \frac{v}{c}E^3 + \frac{v}{c}E^1,\ B^3 + \frac{v}{c}E^2 + \frac{v}{c}E^1\right), \tag{8.79}$$

and the conservation equation,

$$\frac{v}{c}\left(-\partial_0 B^1 + \partial_0 B^2 - \partial_0 B^3\right) + \partial_1\left(B^1 - \frac{v}{c}E^3 - \frac{v}{c}E^2\right)$$
$$+ \partial_2\left(B^2 - \frac{v}{c}E^3 + \frac{v}{c}E^1\right) + \partial_3\left(B^3 + \frac{v}{c}E^2 + \frac{v}{c}E^1\right) = 0 \tag{8.80}$$

which in vector form becomes (S.I. units)

$$\nabla \cdot \left(\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E}\right) = c^2 \nabla \cdot \boldsymbol{B}, \tag{8.81}$$

and is an exact balance of the Faraday law of induction and the law of divergence of a magnetic field. The equation holds under all conditions, and is again deeply significant. The homogeneous Maxwell equations are a special case of the more general Eq. (8.81), when

$$\nabla \cdot \boldsymbol{B} = 0, \tag{8.82a}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}. \tag{8.82b}$$

The separation of Eqs. (8.81) into the two Maxwell equations is therefore dictated by experimental observation, not by any theoretical constraint. A magnetic monopole appears never to have been observed in nature, and the Faraday law appears never to have been violated experimentally. It is of the utmost significance that the absence of a magnetic monopole is now balanced precisely by Faraday's empirical law. Conversely, violation of Faraday's law of induction, however tiny in magnitude, would mean *observation of the magnetic monopole* because of the exact balance of terms in Eq. (8.81). This inference is missing from the received view, because in that view the equations (8.82a) and (8.82b) are separated theoretically as a result of the neglect of the Wigner generator. They are balanced by the novel conservation Eq. (8.66) of the PL vector $\tilde{G}^{\mu}$, which in turn is defined through the translation generator ϵ^{μ}, which in turn extends Lorentz group symmetry to Poincaré group symmetry. It is deeply significant that the Poincaré group is the group of all particles, including the electromagnetic particle, the enigmatic photon. It seems entirely self-inconsistent to work within the Lorentz group for the electromagnetic field, and within the Poincaré group for the concomitant particle, yet this is the received view.

In the vacuum ($v = c$, $\epsilon^{\mu} = (1, 0, 0, 1)$ conservation of the PL pseudo-vector gives the continuity equation,

$$\nabla \times \boldsymbol{E}^{(3)} = -\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} - cu\nabla \cdot \boldsymbol{B}^{(3)} := \boldsymbol{0}, \tag{8.83}$$

and if $\boldsymbol{E}^{(3)}$ is central, as in Coulombic interactions [30], its curl is identically zero. The phaseless $\boldsymbol{B}^{(3)}$ field developed in Refs. 2—4 satisfies the continuity equation in the special case when the time derivative and divergence of $\boldsymbol{B}^{(3)}$ are both zero. If $\boldsymbol{B}^{(3)} =? \boldsymbol{0}$, then for a light-like ϵ^{μ}, $\tilde{G}^{\mu}$ is null and the field is lost completely. This is self-inconsistent because we started with a conservation theorem, Eq. (8.66) for the field.

If we accept for convenience or by (understandable) habit the notion of charge density, ρ, as a result of the precise balance of empirical laws represented by Eq. (8.71), the charge-current four-vector becomes definable (S. I. units) as

$$J^{\mu} = \rho c \epsilon^{\mu} = (\rho c, \rho \boldsymbol{v}), \tag{8.84}$$

under all conditions. In the received view, the charge density in vacuo is usually taken to be zero, but Eq. (8.66) allows for the plausible existence of a vacuum current.

The Lorentz force equation is then

$$f^{\mu} = G^{\mu\nu} J_{\nu}, \tag{8.85}$$

and it follows that the Lorentz force is conserved, i.e., action and reaction are equal and opposite in electrodynamics,

$$\partial_{\mu} f^{\mu} = 0, \tag{8.86}$$

because of Eq. (8.65). The Lorentz force four-vector is a Pauli-Lubanski four-vector. We have therefore unified the Maxwell equations and the Lorentz force equation. The theory of electrodynamics is therefore put on an equal footing with the theory of gravitation, in which Einstein's field equation (Chaps. 1 to 3) give the equation of motion, which is Newton's equation in the linear limit.

The development rests on Eq. (8.65), which is the quantum condition [26] within a factor $\hbar$. The translation generator [26] is the energy momentum four-vector within the same factor $\hbar$. In summary, the main equations of electrodynamics are derived from conservation of G^{μ} and $\tilde{G}^{\mu}$, a conservation law which requires $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$ to be rigorously non-zero in vacuo. The force equation of Lorentz is Eq. (8.85), which shows that the Lorentz force is conserved in the universe, the equivalent of Newton's third law of classical dynamics. Conservation of the PL vectors, Eq. (8.66), is an example of Noether's theorem [26], and Eq. (8.66) also conserves $\hat{C}$, $\hat{P}$, $\hat{T}$,, $\hat{C}\hat{P}$, $\hat{C}\hat{T}$, $\hat{P}\hat{T}$, and $\hat{C}\hat{P}\hat{T}$ [4—6]. It is therefore consistent with the $\hat{C}\hat{P}\hat{T}$ theorem [4] and is therefore Lorentz covariant.

8.5 Vacuum Limits

The vacuum limits of the conservation equation (8.66) provide the Maxwellian type equations for $\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ components. The vacuum limit is obtained when the translation generators become light-like, in which limit Eq. (8.66) provides

$$\frac{1}{c}\frac{\partial \boldsymbol{E}^{(3)}}{\partial t} + \frac{\partial \boldsymbol{E}^{(3)}}{\partial Z} = 0. \tag{8.87}$$

Now use

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\left(\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{v}(\nabla \cdot \boldsymbol{E})\right), \tag{8.88}$$

in the limit $|\boldsymbol{v}| \rightarrow c$, and take the (3) component to find

$$\nabla \times \boldsymbol{B}^{(3)} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(3)}}{\partial t} + \frac{\boldsymbol{v}}{c}\nabla \cdot \boldsymbol{E}^{(3)}. \tag{8.89}$$

Using Eq. (8.87) gives, in the vacuum,

$$\nabla \times \boldsymbol{B}^{(3)} = \boldsymbol{0}, \tag{8.90}$$

which is consistent with the phaseless, irrotational nature of $\boldsymbol{B}^{(3)}$ as given in the definition $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}$. Equation (8.87) describes the Coulombic field if

$$\frac{1}{c}\frac{\partial \boldsymbol{E}^{(3)}}{\partial t} = \frac{\partial \boldsymbol{E}^{(3)}}{\partial Z} = 0, \tag{8.91}$$

a special case. More generally, $\boldsymbol{E}^{(3)}$ can be phase dependent and longitudinal, e.g. Eq. (8.87) is satisfied by

$$\boldsymbol{E}^{(3)} = E^{(0)} \exp\left[i\omega\left(t - \frac{Z}{c}\right)\right] \boldsymbol{e}^{(3)}. \tag{8.92}$$

Equation (8.91) is a type of Maxwell equation.

Equation (8.66) in the vacuum limit provides the result

$$\frac{1}{c}\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} + \frac{\partial \boldsymbol{B}^{(3)}}{\partial Z} = 0, \tag{8.93}$$

which allows phase dependent and phase independent $B^{(3)}$ as for $E^{(3)}$. The monopole condition $\nabla \cdot \boldsymbol{B} = 0$, if accepted, reduces Eq. (8.93) to

$$\frac{1}{c}\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \frac{\partial \boldsymbol{B}^{(3)}}{\partial Z} = 0. \tag{8.94}$$

The (3) component of Eq. (8.81) is

$$\mathbf{v} \cdot \left(\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} + \nabla \times \boldsymbol{E}^{(3)} \right) = c\nabla \cdot \boldsymbol{B}^{(3)}, \tag{8.95}$$

which, with Eq. (8.94), implies

$$\nabla \times \boldsymbol{E}^{(3)} = \boldsymbol{0}. \tag{8.96}$$

The latter is always true for the central Coulombic field [1], and is also true for the phase dependent $\boldsymbol{E}^{(3)}$ defined in Eq. (8.92).

Therefore, both $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$ are irrotational in the vacuum [4—6].

Equation (8.66) conserves $\hat{C}$, $\hat{P}$, $\hat{T}$, $\hat{C}\hat{P}$, $\hat{P}\hat{T}$, $\hat{C}\hat{T}$ and $\hat{C}\hat{P}\hat{T}$, and therefore so do its vacuum limits, Eqs. (8.87) and (8.95) for the $\boldsymbol{E}^{(3)}$ and $\boldsymbol{B}^{(3)}$ components. Since $\hat{C}(e) = -e$, the $\hat{C}$ operator conserves Eq. (8.66) with $\hat{C}(0) = 0$. This can be

checked directly in Eq. (8.88), for example, using

$$\hat{C}(\boldsymbol{E}) = -\boldsymbol{E}, \quad \hat{C}(\boldsymbol{B}) = -\boldsymbol{B}, \quad \hat{C}(\nabla) = \nabla, \quad \hat{C}(t) = t, \tag{8.97}$$

and the same equation (8.88) is recovered intact after operating with $\hat{C}$. Similarly for $\hat{P}$ and $\hat{T}$ using [4]

$$\begin{aligned} &\hat{T}(\boldsymbol{E}) = \boldsymbol{E}, \quad \hat{T}(\nabla) = \nabla, \quad \hat{T}(t) = -t, \\ &\hat{T}(\boldsymbol{B}) = -\boldsymbol{B}, \quad \hat{T}(\mathbf{v}) = -\mathbf{v}, \end{aligned} \tag{8.98}$$

$$\begin{aligned} &\hat{P}(\boldsymbol{E}) = -\boldsymbol{E}, \quad \hat{P}(\nabla) = -\nabla, \quad \hat{P}(t) = t, \\ &\hat{P}(\boldsymbol{B}) = \boldsymbol{B}, \quad \hat{P}(\mathbf{v}) = -\mathbf{v}. \end{aligned} \tag{8.99}$$

The same result is obtained for $\hat{C}\hat{P}$, $\hat{C}\hat{T}$ and $\hat{P}\hat{T}$. Finally, Eq. (8.66) is again unchanged on application of $\hat{C}\hat{P}\hat{T}$. The $\hat{C}\hat{P}\hat{T}$ theorem [4] then implies that the field theory represented by Eq. (8.66) is Lorentz covariant under all conditions. Equation (8.66) itself is of course written in covariant form, because otherwise it would be relativistically incorrect. (Recall that the Newton equation $\boldsymbol{F} = m\boldsymbol{g}$ conserves $\hat{C}\hat{P}\hat{T}$, but is not covariant. The correctly covariant equation in special relativity is the Minkowski equation [30], in which $\boldsymbol{F}$ is replaced by the Minkowski four-force K^{μ} and $m\boldsymbol{g}$ by $dp^{\mu}/d\tau$, where p^{μ} is the relativistic momentum and τ the Lorentz invariant proper time.)

8.6 Stokes' Theorem

The fundamental spin of the electromagnetic field is represented by

$$\tilde{G}^{\mu} = \left(B^{(3)},\ 0,\ 0,\ B^{(3)}\right), \tag{8.99}$$

which means that the hyper-surface integral over $\tilde{G}^{\mu}$ is always zero in space-time. This is a result of the four dimensional Stokes theorem [1],

$$\oint \tilde{G}_{\mu} dx^{\mu} = \frac{1}{2}\int \left(\frac{\partial \tilde{G}_{\nu}}{\partial x^{\mu}} - \frac{\partial \tilde{G}_{\mu}}{\partial x^{\nu}} \right) dS^{\mu\nu}, \tag{8.100}$$

written for the vector $\tilde{G}_{\mu}$. *Equation (8.100) has no equivalent in U(1) electrodynamics, a key result.* Using Eq. (8.99) we obtain

$$\oint \tilde{G}_\mu dx^\mu = 0 \,. \tag{8.101}$$

In three dimensions, the equivalent of Eq. (8.101) implies that any line integral over $\boldsymbol{B}^{(3)}$ is zero in vacuo, because $\nabla \times \boldsymbol{B}^{(3)}$ is zero, i.e., because $\boldsymbol{B}^{(3)}$ is irrotational.

Similarly,

$$\oint G_\mu dx^\mu = \frac{1}{2}\int \left(\frac{\partial \tilde{G}_\nu}{\partial x^\mu} - \frac{\partial \tilde{G}_\mu}{\partial x^\nu} \right) dS^{\mu\nu}, \tag{8.102}$$

and in the vacuum,

$$\oint G_\mu dx^\mu = 0, \tag{8.103}$$

because

$$G^\mu = \left(E^{(3)},\ 0,\ 0,\ E^{(3)} \right). \tag{8.104}$$

Equations (8.100) and (8.104) provide twelve new equations based on

$$\frac{\partial \tilde{G}_\nu}{\partial x^\mu} = -\frac{\partial \tilde{G}_\mu}{\partial x^\nu}, \tag{8.105a}$$

$$\frac{\partial G_\nu}{\partial x^\mu} = -\frac{\partial G_\mu}{\partial x^\nu}, \tag{8.105b}$$

i.e., based on the antisymmetry of the four-curl. These contain no direct reference to charge and current, and hold under all conditions in general. From Eq. (8.105a),

$$\begin{gathered} \partial_0 \left(B_1 + \frac{\mathrm{v}}{c^2} E_2 \right) = \frac{\mathrm{v}}{c} \partial_1 B_3 \,, \\ \partial_2 \left(B_1 + \frac{\mathrm{v}}{c^2} E_2 \right) = \partial_1 \left(\frac{\mathrm{v}}{c^2} E_1 - B_2 \right), \\ \partial_0 \left(\frac{\mathrm{v}}{c^2} E_1 - B_2 \right) = \frac{\mathrm{v}}{c} \partial_2 B_3 \,, \quad \partial_1 B_3 = \partial_3 \left(B_1 + \frac{\mathrm{v}}{c^2} E_2 \right), \\ \partial_0 B_3 = \frac{\mathrm{v}}{c} \partial_3 B_3 \,, \quad \partial_2 B_3 = \partial_3 \left(\frac{\mathrm{v}}{c^2} E_1 - B_2 \right), \end{gathered} \tag{8.106}$$

and from Eq. (8.105b),

$$\begin{gathered} \partial_0 \left(\frac{\mathrm{v}}{c^2} E_1 - B_2 \right) = \frac{1}{c} \partial_1 E_3 \,, \\ \partial_1 \left(\frac{\mathrm{v}}{c^2} E_2 + B_1 \right) = \partial_2 \left(B_2 - \frac{\mathrm{v}}{c^2} E_1 \right), \\ \partial_0 \left(B_1 + \frac{\mathrm{v}}{c^2} E_2 \right) = \frac{1}{c} \partial_2 E_3 \,, \quad \frac{\mathrm{v}}{c^2} \partial_1 E_3 = \partial_3 \left(B_2 - \frac{\mathrm{v}}{c^2} E_1 \right), \\ \frac{\mathrm{v}}{c^2} \partial_0 E_3 = \frac{1}{c} \partial_3 E_3 \,, \quad -\frac{\mathrm{v}}{c^2} \partial_2 E_3 = \partial_3 \left(B_1 + \frac{\mathrm{v}}{c^2} E_2 \right). \end{gathered} \tag{8.107}$$

8.7 Stress-Energy-Momentum Tensor

The stress-energy-momentum tensor $T^{\mu\nu}$ is symmetric and contains the Maxwell stress tensor, the momentum and the energy of the field [1]. The force of radiation per unit volume is

$$\frac{F^\nu}{V} = \frac{1}{\mu_0} \partial_\mu T^{\mu\nu}, \tag{8.108}$$

and the radiation pressure is

$$\boldsymbol{P} = \int_0^\infty \frac{F^\nu}{V} dx_\nu \,. \tag{8.109}$$

The latter can be measured experimentally (see Chap. 9). In this section, it will be shown that vacuum electrodynamics in terms of the Pauli-Lubanski vectors gives the correct equation of state for isotropically distributed radiation,

$$\boldsymbol{P}V = \frac{U}{3} = kT, \tag{8.110}$$

where P, V and U are the radiation pressure, volume and energy. Equation (8.110) is the rare gas equation of state combined with the Equipartition theorem. The thermodynamic result (8.110) is not given properly by the received view (Chaps. 7 and 9) because the latter is dimensionally flawed. There cannot be kT per degree of freedom in a pure transverse theory of vacuum radiation; and in such a theory, the Pauli-Lubanski vector is not defined.

If the radiation is isotropically distributed and circularly polarized (e.g. is made up of an ensemble of isotropically distributed left spinning photons in a cavity) then the Pauli-Lubanski vectors in the X, Y and Z axes are

$$\tilde{G}^{\mu}_{(Z)} = \left(B^{(0)}, 0, 0, B^{(0)}\right), \qquad \tilde{G}^{\mu}_{(Y)} = \left(B^{(0)}, 0, B^{(0)}, 0\right),$$
$$\tilde{G}^{\mu}_{(X)} = \left(B^{(0)}, B^{(0)}, 0, 0\right). \tag{8.111}$$

The magnetic contribution to the stress-energy-momentum density is then

$$\frac{T^{\mu}_{\nu}}{V} = \frac{1}{\mu_0} g^{\mu\alpha} \tilde{G}_{\alpha} \tilde{G}^{\sigma} g_{\sigma\nu}, \tag{8.112}$$

which is a symmetric tensor. Here $g_{\mu\nu}$ is the metric tensor. Element by element, we have

$$\frac{T^{\mu}_{\nu}}{V} = \frac{1}{\mu_0} \begin{bmatrix} 3B^{(0)2} & B^{(0)2} & B^{(0)2} & B^{(0)2} \\ B^{(0)2} & -B^{(0)2} & 0 & 0 \\ B^{(0)2} & 0 & -B^{(0)2} & 0 \\ B^{(0)2} & 0 & 0 & -B^{(0)2} \end{bmatrix}. \tag{8.113}$$

Defining

$$U = cp = \frac{1}{\mu_0} \int 3B^{(0)2} dV, \tag{8.114}$$

where p is linear momentum, the tensor (8.113) becomes

$$U^{\mu}_{\nu} = \begin{bmatrix} U & U/3 & U/3 & U/3 \\ U/3 & -U/3 & 0 & 0 \\ U/3 & 0 & -U/3 & 0 \\ U/3 & 0 & 0 & -U/3 \end{bmatrix}, \tag{8.115}$$

which is symmetric, as required, and whose $(0, 0)$ element is the total radiation energy U. Its $(0, i) = (i, 0)$ elements $(i = 1, 2, 3)$ denote the linear momentum components $cp/3$, and

the other elements are the non-zero elements of the stress tensor of the isotropically distributed radiation. Since each component of the momentum and stress tensor is $U/3$, the correct equation of state (8.110) is obtained from the PL representation of isotropic radiation. As shown in Chaps. 7 and 9, this leads to the correct Planck law. Evidence for the Planck law may therefore be cited as evidence for longitudinal components in vacuo.

The electric field's contribution to T^{ν}_{μ} may be evaluated similarly using the PL vectors,

$$G^{\mu}_{(Z)} = \left(E^{(0)}, 0, 0, E^{(0)}\right), \qquad G^{\mu}_{(Y)} = \left(E^{(0)}, 0, E^{(0)}, 0\right),$$
$$G^{\mu}_{(X)} = \left(E^{(0)}, E^{(0)}, 0, 0\right), \tag{8.116}$$

and the total energy is a combination of magnetic and electric components. From Noether's theorem the conservation law of stress-energy- momentum is

$$\partial_{\mu} T^{\mu}_{\nu} = 0. \tag{8.117}$$

In the presence of matter, the PL vectors no longer have the simple vacuum form (8.111) and (8.116) and off-diagonal elements appear in the Maxwell stress tensor. However, the same equation (8.112) can be used for defining T^{μ}_{ν}. Comparing Eqs. (8.108) and (8.117), there is no net radiation pressure in the vacuum. The radiation acts on a confining vessel in order to produce a measurable pressure.

8.8 Lie Algebra of the Relativistic Field Helicity Vectors

The PL vector, as we have seen, is defined within the underlying Poincaré group symmetry by [26,1]

$$\tilde{W}_{\mu} = \tilde{J}_{\mu\nu} \boldsymbol{P}^{\nu}, \tag{8.118}$$

where

$$\tilde{J}_{\mu\nu} := \begin{bmatrix} 0 & J_1 & J_2 & J_3 \\ -J_1 & 0 & K_3 & -K_2 \\ -J_2 & -K_3 & 0 & K_1 \\ -J_3 & K_2 & -K_1 & 0 \end{bmatrix}, \tag{8.119}$$

so that

$$\tilde{W}_0 = J_1P^1 + J_2P^2 + J_3P^3,$$
$$\tilde{W}_1 = -J_1P^0 + K_3P^2 - K_2P^3,$$
$$\tilde{W}_2 = -J_2P^0 - K_3P^1 + K_1P^3, \qquad (8.120)$$
$$\tilde{W}_3 = -J_3P^0 + K_2P^1 - K_1P^2.$$

If the above relations are regarded as linear operator relations, and if P^0, P^1, P^2, and P^3 in Eqs. (8.120) denote numbers (eigenvalues of operators), we obtain

$$\left[P^0, \tilde{W}^\mu\right] = 0, \qquad (8.121)$$

showing that the Hamiltonian operator $H := P^0$ commutes with the *complete* vector $\tilde{W}^\mu$ under all conditions. Equation (8.121) means that

$$\partial_\mu \tilde{W}^\mu = 0 \qquad (8.122)$$

as derived in Eq. (8.66). *Relativistic helicity is conserved.* From Eqs. (8.120) we obtain the closed Lie algebra,

$$\left[\tilde{W}^1, \tilde{W}^2\right] = i\left(p^0\tilde{W}^3 + p^3\tilde{W}^0\right),$$
$$\left[\tilde{W}^2, \tilde{W}^3\right] = i\left(p^0\tilde{W}^1 + p^1\tilde{W}^0\right),$$
$$\left[\tilde{W}^3, \tilde{W}^1\right] = i\left(p^0\tilde{W}^2 + p^2\tilde{W}^0\right),$$
$$\left[\tilde{W}^0, \tilde{W}^1\right] = i\left(p^3\tilde{W}^2 - p^2\tilde{W}^3\right), \qquad (8.123)$$
$$\left[\tilde{W}^0, \tilde{W}^2\right] = i\left(p^1\tilde{W}^3 - p^3\tilde{W}^1\right),$$
$$\left[\tilde{W}^0, \tilde{W}^3\right] = i\left(p^2\tilde{W}^1 - p^1\tilde{W}^2\right).$$

The Jacobi identity is obeyed by the above Lie algebra, for example,

$$\left[\tilde{W}^1,\left[\tilde{W}^2, \tilde{W}^3\right]\right] + \left[\tilde{W}^2,\left[\tilde{W}^3, \tilde{W}^1\right]\right] + \left[\tilde{W}^3,\left[\tilde{W}^1, \tilde{W}^2\right]\right] = 0, \qquad (8.124)$$

and so forth. This shows that $\tilde{W}^\mu$ is a valid group generator of the Poincaré group. It

also forms an invariant, $\tilde{W}_\mu\tilde{W}^\mu$, and with P^μ, is one of the two fundamental generators of the Poincaré group.

To develop electromagnetic field theory from first principles, we replace $\tilde{W}^\mu$ by $\tilde{G}^\mu$, the relativistic field helicity vector. Thus, Eq. (8.123) is also the fundamental Lie algebra in electrodynamics, for the free field, and also for field-matter interaction. As we have seen, the

$$\partial_\mu \tilde{G}^\mu = 0, \qquad (8.125)$$

and there is no reason, from first principles, to assert that $B^{(3)}$ (or $E^{(3)}$) is zero.

If we consider a light beam propagating at c in Z, then

$$p^1 = p^2 = 0, \quad p^0\tilde{W}^3 = p^3\tilde{W}^0, \qquad (8.126)$$

and we obtain from Eq. (8.123) an E(2) Lie algebra,

$$\left[\tilde{W}^1, \tilde{W}^2\right] = 0, \qquad (8.127a)$$
$$\left[\tilde{W}^2, \tilde{W}^3\right] = ip^0\tilde{W}^1, \qquad (8.127b)$$
$$\left[\tilde{W}^3, \tilde{W}^1\right] = ip^0\tilde{W}^2, \qquad (8.127c)$$
$$\left[\tilde{W}^1, \tilde{W}^2\right] = ip^0\tilde{W}^3, \qquad (8.127d)$$
$$\left[\tilde{W}^2, \tilde{W}^3\right] = ip^0\tilde{W}^1, \qquad (8.127e)$$
$$\left[\tilde{W}^3, \tilde{W}^1\right] = ip^0\tilde{W}^2, \qquad (8.127f)$$

and similarly for $\tilde{G}^\mu$ In terms of field components, Eq. (8.127b) gives (in $c = 1$ units)

$$\left[B^2 - E^1, B^3\right] = iB^{(0)}B^1, \qquad (8.128)$$

which is satisfied by

$$\left[B^2, B^3\right] = iB^{(0)}B^1, \qquad (8.129a)$$
$$\left[B^3, E^1\right] = iB^{(0)}E^2. \qquad (8.129b)$$

These are two of the field cyclic relations introduced elsewhere [4—6]. Similarly, Eq. (8.127c) gives

$$\left[B^3, B^1 + E^2\right] = iB^{(0)}\left(B^2 - E^1\right), \tag{8.130}$$

which is satisfied by

$$\left[B^3, B^1\right] = iB^{(0)}B^2, \tag{8.131a}$$

$$\left[B^3, E^2\right] = -iB^{(0)}E^1, \tag{8.131b}$$

Equation (8.127a) gives

$$\left[B^1 + E^2, B^2 - E^1\right] = 0, \tag{8.132}$$

which is satisfied by

$$\left[B^1, B^2\right] = \left[E^1, E^2\right], \tag{8.133a}$$

$$\left[E^2, B^2\right] = \left[E^1, E^1\right]. \tag{8.133b}$$

From the commutator relations of boost and rotation generators we know that both sides in Eq. (8.133a) are non-zero. So from Eqs. (8.129a), (8.131a), and (8.133a)

$$\left[B^1, B^2\right] = iB^{(0)}B^3, \quad \left[B^2, B^3\right] = iB^{(0)}B^1, \quad \left[B^2, B^1\right] = iB^{(0)}B^2. \tag{8.134}$$

The B cyclics are therefore obtained as a sub-algebra of E(2) when it is assumed that light travels at c in Z. If we had used the more general light-like condition,

$$p_0^2 = p_1^2 + p_2^2 + p_3^2, \tag{8.135}$$

the E(2) structure is *not* obtained in general. We must use the more general structure (8.123).

This proves the existence of the B cyclics from first principles for light propagating at c in Z. If we do not make these assumptions, the overall Lie algebra is not E(2). The assumption that light propagates in one axis, Z is the same as assuming that translation of space-time takes place only in one axis. If we use Eq. (8.135), *isotropic expansion* of space-time is implied, and the Lie algebra (8.123) is not that of E(2), even in the light-like condition (8.135). So E(2) is a group generated by the assumption that light propagates at c in one axis, and this is the physical meaning of E(2). An example of this is the plane wave, which gives, self-consistently [4—6], the B cyclics.

An O(3) group structure is obtained from the Lie algebra (8.123) when

$$p_1 = p_2 = p_3 = 0, \tag{8.136}$$

which gives the rest frame Lie algebra,

$$\left[\tilde{W}^1, \tilde{W}^2\right] = ip^0\tilde{W}^3, \quad \left[\tilde{W}^2, \tilde{W}^3\right] = ip^0\tilde{W}^1, \quad \left[\tilde{W}^3, \tilde{W}^1\right] = ip^0\tilde{W}^2, \tag{8.137}$$

which is

$$\left[J^1, J^2\right] = iJ^3, \tag{8.138}$$

et cyclicum, the Lie algebra of the Poincaré group rotation generators. There is, self-consistently, no linear momentum in this state. We again obtain the B cyclics if $\tilde{G}^\mu$ is substituted for $\tilde{W}^\mu$, but this time in a rest frame corresponding to that of the photon with mass [4—6].

It is concluded that the B cyclics occur in the rest frame of the massive, spinning, photon, and also when the photon is translating at c in any *one* axis. A photon with mass cannot attain the speed c if we accept the Lorentz transform, and accept mass as a scalar Lorentz invariant. A photon without mass has no rest frame. In both cases, it is *wholly* incorrect to conclude that a massless particle has only two degrees of polarization, because the generators J^1, J^2, and J^3 are all three non-zero in the Lie algebra (8.123). From first principles, the Lie algebra (8.123) is obtained, and this contains cyclic field relations as a sub-algebra. In other words, $\boldsymbol{B}^{(3)}$ and J^3 are not zero if we accept the Poincaré group as the underlying group for electrodynamics. The use of the U(1) sub-algebra [26] is *not* indicated from first principles. Magneto-optics give empirical evidence for $\boldsymbol{B}^{(3)}$ from experiments such as the inverse Faraday effect [4—6]. The Lie algebra (8.123) can reduce to both E(2) and to O(3) and this appears to be confirmed by the independent work by Kim and Wigner [31] in Wigner's last papers. In other words, we essentially throw away information from the Poincaré group to arrive at the E(2) and O(3) structures.

If the isotropic rotation generator is used the Maxwell equations are obtained in the vacuum through conservation of relativistic field helicity as follows:

$$\tilde{G}^\mu = \begin{bmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \partial_\mu \tilde{G}^\mu = 0, \tag{8.139}$$

giving the homogeneous equations,

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} - \sqrt{3}\, c \,(\nabla \cdot \boldsymbol{B})\, \boldsymbol{u} = 0, \tag{8.140}$$

and

$$G^{\mu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \partial_{\mu} G^{\mu} = 0, \tag{8.141}$$

giving the inhomogeneous equations,

$$\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} - \frac{\sqrt{3}}{c}(\nabla \cdot \boldsymbol{E})\, \boldsymbol{u} = 0, \tag{8.142}$$

it is to be noted that the $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$ components are included in the overall structure, which is a conservation equation. The empirical evidence from magneto-optics suggests that $\boldsymbol{B}^{(3)}$ is phase free and real and that $\boldsymbol{E}^{(3)}$ is in consequence pure imaginary [4—6]. The most general case leads to Eqs. (8.140) and (8.142) in the vacuum. In those, the complete fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are sums over components in the Cartesian basis or in the circular basis,

$$\boldsymbol{B} = \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \boldsymbol{B}^{(3)}, \qquad \boldsymbol{E} = \boldsymbol{E}^{(1)} + \boldsymbol{E}^{(2)} + \boldsymbol{E}^{(3)}, \tag{8.143}$$

so in general equations (8.140) and (8.142) apply, and the cyclic field relations interrelate the circular components. This is enough to define the nature of $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$ in general.

References

[1] P. W. Atkins, *Molecular Quantum Mechanics* (Oxford University Press, Oxford, 1983). For a relativistic treatment see L. D. Landau and E. M. Lifshitz, *The Clasiical Theory of Fields*, 4th edn. (Pergamon, Oxford, 1975).

[2] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill Book Co., New York, 1980).

[3] K. Huang, *Quarks, Leptons and Gauge Fields* (World Scientific, Singapore, 1992).

[4] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $\boldsymbol{B}^{(3)}$ (Kluwer Academic, Dordrecht, 1994).

[5] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).

[6] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).

[7] M. W. Evans, *Physica B* **182**, 227, 137 (1992); **183**, **103** (**1993**); *Physica B* **190**, 310 (1993); *Physica A* **214**, 605 (1995).

[8] M. W. Evans and S. Kielich, eds., *Modern Nonlinear Optics,* Vol. **85**(2) of *Advances in Chemical Physics,* I. Prigogine and S. A. Rice, eds. (Wiley Interscience, New York, 1993).

[9] M. W. Evans, *The Photon's Magnetic Field* (World Scientific, Singapore, 1992); A. A. Hasanein and M. W. Evans, *The Photomagneton in Quantum Field Theory* (World Scientific, Singapore, 1994).

[10] M. W. Evans, *Found. Phys.* **24**, 892, 1519, 1671 (1994); **25**, 175, 383 (1995); M. W. Evans and S. Roy, *Found. Phys.*, in press.

[11] M. W. Evans, *Found. Phys. Lett.* 7, 67, 209, 379, 437, 577, 591 (1994); ibid. **8**, 83, 187, 253, 279 (1995).

[12] F. Farahi, Y. Aktas, and M. W. Evans, *J. Mol. Structure* **285**, 47 (1993); M. W. Evans and F. Farahi, ibid., **300**, 435 (1993); M. W. Evans, *Mod. Phys. Lett.* **B7**, 1247 (1993); M. W. Evans, S. Roy and S. Jeffers, *Il Nuovo Cim. D* (1995).

[13] M. W. Evans in A. van der Merwe and A. Garuccio, eds., *Waves and Particles in Light and Matter* (Plenum, New York, 1994).

[14] M. W. Evans, *Found. Phys. Lett.* and *Found. Phys.*, in press (1996 to 1997); M. W. Evans and S. Roy, *Found. Phys.*, submitted.

[15] A. Chubykalo and R. Smirnov-Rueda, *Phys. Rev. E*, **53**, 5373 (1996); also V. Dvoeglazov, *Phys. Rev. D*, submitted for publication (Feb., 1996).

[16] H. Múnera and O. Guzmán, *Found. Phys. Lett.*, submitted (Feb. 1996).

[17] G. Hunter and R. P. L. Wadlinger, *Phys. Essays* **2(2)**, 158 (1989).

[18] See Chap. 12.

[19] V. V. Dvoeglazov, *Int. J. Theor. Phys.* **35**, 115 (1996).

[20] V. V. Dvoeglazov, Yu. N. Tyukhtyaev and S. V. Khudyakov, *Russ. Phys. J.* **37**, 898 (1994).

[21] L. Bass and E. Schrödinger, *Proc. Roy. Soc.* **232A,** 1 (1955).

[22] M. Moles and J.-P. Vigier, *Compt. Rend.* **276**, 697 (1973).

[23] J.-P. Vigier, *Il Nuovo Cim.* **88B**, 20 (1985).

[24] H. A. Múnera and O. Guzmán, Found. Phys. Lett. **10**, 31 (1997).

[25] H. A. Múnera, D. Buritica, O. Guzmán and J.I. Vallejo, *Rev. Col. Fis*, **27**, 215 (1995).

[26] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).

[27] E. P. Wigner, *Ann. Math.* **40,** 149 (1939).

[28] P. A. M Dirac, in *Directions in Physics*,. H. Hora and J.R. Shepanski, eds. (Wiley, New York, 1978).
[29] W. Pauli, *General Principles of Quantum Mechanics* (Springer, Berlin, 1980).
[30] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).
[31] Y. S. Kim, email communications (1995 to present).

Chapter 9. On the Pressure of Electromagnetic Radiation

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Abstract. Light pressure is calculated in different physical systems, e.g. in the case of reflection of plane waves and in a black body cavity. It is shown that the equation of state of a photon gas ($p = u/3$) has not been confirmed experimentally, and we call for new measurements for its determination. Moreover, it will be proved that the $p = u/3$ connection is meaningless without the longitudinal electromagnetic fields introduced by Evans in 1992.

9.1 Introduction

In this paper the pressure of electromagnetic radiation will be investigated with purely electrodynamical methods. The idea of light pressure came from Maxwell [1], and it was developed by Bartoli [2]. The first successful light pressure experiment was made by Lebedew [3]. It seems that Planck solved the thermodynamic problem of the radiation field at the beginning of the twentieth century [4]. His result was the starting point of quantum mechanics, which is the most successfully confirmed and verified theory of modern physics. Several contradictions of Big-Bang cosmology have been uncovered by Mészáros and Molnár [5], some of which have implications for thermodynamic laws of classical electromagnetic radiation. It will be shown here that there are several open questions about the basic concepts in thermal radiation, and these will appear as the result of our investigation of pure electromagnetic radiation without charge and current.

Using Gaussian units the properties of the electromagnetic field are determined by the Maxwell equations [6],

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\dot{\boldsymbol{B}}, \tag{9.1a}$$

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Appendix A. Reduction of Equation (1.29) to Equation (1.32)

M. W. Evans

J.R.F., 1975
Wolfson College, Oxford

Define the field strength tensor as

$$G^{\nu\rho} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix}, \tag{A1}$$

whose Cartesian components are in general complex. The first term on the right hand side of Eq. (1.29) is

$$E^2A^1 = -\frac{1}{2}\left(\epsilon_{3021}G^{02}A^1 + \epsilon_{3201}G^{20}A^1\right). \tag{A2}$$

Similarly, the second term is $-E^1A^2$ and

$$A^0 G_3 = E^2A^1 - E^1A^2 . \tag{A3}$$

In general this is a product of complex components, and in the particular case of the product of conjugates we can use the plane waves [4—6],

$$\begin{aligned} \boldsymbol{E}^{(1)} &= \boldsymbol{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi}, \\ \boldsymbol{A}^{(1)} &= \boldsymbol{A}^{(2)*} = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi} . \end{aligned} \tag{A4}$$

Therefore the components in the general case, Eq. (A3), become for the conjugate product,

$$A^0G_3 = E^2A^{1*} - E^1A^{2*}\,, \tag{A5}$$

showing that

$$A^0G_Z = E_X^{(1)}A_Y^{(2)} - E_Y^{(1)}A_X^{(2)}\,. \tag{A6}$$

Therefore introducing Cartesian vector notation into Eq. (1.29),

$$A^0G_Z = |\,\boldsymbol{E}^{(1)} \times \boldsymbol{A}^{(2)}\,| = \kappa^{-1}|\,\boldsymbol{E}^{(1)} \times \boldsymbol{B}^{(2)}\,|\,, \tag{A7}$$

and using the free space plane wave relations [4—6],

$$\boldsymbol{B}^{(1)} = \nabla \times \boldsymbol{A}^{(1)}, \qquad \boldsymbol{B}^{(2)} = \nabla \times \boldsymbol{A}^{(2)}\,. \tag{A8}$$

Finally, using

$$G_3 := cB_3\,, \qquad \kappa = \frac{\omega}{c}\,,$$

$$A^0 := \frac{B^{(0)}}{\kappa}\,, \tag{A9}$$

we obtain

$$iB^{(0)}\boldsymbol{B}^{(3)*} = \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}\,, \tag{A10}$$

which is Eq. (1.32) with G_3 defined as $|\,B^{(3)}\,|$. This shows that Eq. (1.21) produces the cyclic relations when the conjugate product of plane waves is considered. Therefore the cyclic relations are fundamental invariant equations of special relativity in which P^σ of Eq. (1.19) is replaced by A^σ of Eq. (1.21), and $J^{\nu\rho}$ of Eq. (1.19) is replaced by $G^{\nu\rho}$ of Eq. (1.21). The latter replacement is equivalent to Eq. (1.17), and makes electromagnetism a theory of general relativity.

Appendix B. Four Dimensional Cyclics

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On page 16 of the text we considered a helicity equation which is very similar to that proposed recently by Afanasev and Stepanovsky [1], following work on relativistic helicity [2]. The equation is

$$A^0\tilde{G}_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\nu\rho}A^\sigma\,. \tag{B1}$$

Part of this appendix illustrates the reduction of this equation to a three dimensional cyclic form. Before doing so, however, we give a simple proof of the topological inconsistency of standard electrodynamics in which the $\boldsymbol{B}^{(3)}$ and $\boldsymbol{E}^{(3)}$ fields are missing.

The proof is based on the topological definition of the vector dual of the ordinary antisymmetric field tensor of classical electrodynamics,

$$\tilde{G}_\mu = \tilde{G}_{\mu\nu}\epsilon^\nu\,, \tag{B2}$$

where $\tilde{G}_{\mu\nu}$ is the tensor dual, defined by

$$\tilde{G}_{\mu\nu} = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{bmatrix}. \tag{B3}$$

This definition is consistent with the structure of the Poincaré group as discussed in the text. The vector dual $\tilde{G}_\mu$ is orthogonal to the unit vector ϵ^μ, which is light-like in the vacuum,

$$\left.\begin{aligned} \epsilon_\mu \epsilon^\mu &= 0 \\ \tilde{G}_\mu \epsilon^\mu &= 0 \end{aligned}\right\} \quad \textbf{(B4)}$$

The vector dual is a topological invariant, and is proportional to the relativistic helicity. For this reason it is phase free and non-zero. The unit light-like vector ϵ^μ is proportional to the potential four-vector A^μ, a polar vector proportional through the minimal prescription to the energy momentum four-vector. There is freedom to choose ϵ^μ as long as it is light-like, because we are considering the fields associated with the photon propagating in the vacuum. These are the fundamental fields $\boldsymbol{E}$ and $\boldsymbol{B}$ from which $\boldsymbol{D}$ and $\boldsymbol{H}$ are constructed when light interacts with matter.

This freedom in choosing ϵ^μ is related to gauge freedom for A^μ, which is defined up to a choice of gauge as is well known. However, the following reasoning excludes the transverse gauge because the latter leads to a null dual vector $\tilde{G}_\mu$. The choice B_3 =? 0 and E_3 =? 0 also leads to a null dual vector. For example, the following choice of a purely translational A^μ is light-like,

$$A^\mu = \left(A^0, 0, 0, A^3\right), \qquad A^0 = A^3, \quad \textbf{(B5)}$$

but so is the choice of a rototranslational A^μ, whose transverse components are related to $\boldsymbol{B}$ by the usual $\boldsymbol{B} = \text{curl } \boldsymbol{A}$. This choice is also light-like,

$$A^\mu = \left(A^0, i\frac{A^0}{\sqrt{2}} e^{i\phi}, \frac{A^0}{\sqrt{2}} e^{i\phi}, A^3 \right), \quad \textbf{(B6)}$$

and satisfies the d'Alembert equation in vacuo. The above choice includes an arbitrary phase factor in the transverse components, arbitrary because for any choice of phase, A^μ is light-like. The arbitrariness of phase means that we can also choose the light-like

$$A^\mu = \left(A^0, iA^0, A^0, A^0\right) := A^0 \epsilon^\mu, \quad \textbf{(B7)}$$

and this proves to be convenient for the following development, in which

$$\epsilon^\mu = (1, i, 1, 1). \quad \textbf{(B8)}$$

From Eqs. (B3) and (B8) the vector dual is given by

$$\begin{aligned} \tilde{G}^\mu = \big(& iB_1 + B_2 + B_3,\ B_1 - E_3 + E_2, \\ & B_2 + iE_3 - E_1,\ B_3 - iE_2 + E_1 \big). \end{aligned} \quad \textbf{(B9)}$$

The vacuum Maxwell equations mean, however, that

$$\begin{aligned} iB_1 &= -B_2, & B_1 &= -E_2, \\ B_2 &= E_1, & E_1 &= iE_2. \end{aligned} \quad \textbf{(B10)}$$

These are the relations between components of ordinary transverse plane waves in the vacuum. The vector dual, topologically equivalent to the original antisymmetric field tensor, therefore reduces in the vacuum to

$$\tilde{G}^\mu = \left(B_3, -E_3, iE_3, B_3\right), \quad \textbf{(B11)}$$

and consists only of B_3 and E_3 components. It is orthogonal to the unit polar vector ϵ^μ,

$$\tilde{G}_\mu \epsilon^\mu = B_3 - B_3 + iE_3 - iE_3 = 0. \quad \textbf{(B12)}$$

If the longitudinal fields are asserted to be zero, the vector dual vanishes. This means that the relativistic helicity and the dual vector are both null vectors, a topologically incorrect result. Furthermore, experimental evidence for B_3 is available in magneto-optics of all kinds, and for E_3 in the Coulomb field, which is longitudinal, not transverse.

This simple but profoundly important topological argument gives two of the Maxwell equations because the vector dual is conserved in the vacuum, because helicity is a conserved topological quantity in electrodynamics [3]. Thus,

$$\partial_\mu \tilde{G}^\mu = 0, \quad \textbf{(B13)}$$

or in vector form,

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} + (\nabla \cdot \boldsymbol{B})\boldsymbol{u} = \boldsymbol{0}. \quad \textbf{(B14)}$$

This result is obviously consistent with the experimental equations of electrodynamics:

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = \boldsymbol{0}, \quad \textbf{(B15)}$$

as discussed in Chap. 8 of this volume.

Use of the transverse gauge however, destroys this argument because in the transverse gauge only two out of four components of A^μ are used. Thus A^μ (and ϵ^μ) are not completely covariant. Thus, in the received view,

$$\tilde{G}_\mu = \begin{bmatrix} 0 & B_1 & B_2 & 0 \\ -B_1 & 0 & 0 & -E_2 \\ -B_2 & 0 & 0 & E_1 \\ 0 & E_2 & -E_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ i \\ 1 \\ 0 \end{bmatrix} \tag{B16a}$$

$$= \left(iB_1 + B_2, 0, 0, iE_2 - E_1\right) \tag{B16b}$$

$$= (0, 0, 0, 0), \tag{B16c}$$

and the conventional view leads to the topologically inconsistent result (B16c).

Similarly, if choice is made of a gauge that keeps all four components of ϵ^μ while discarding B_3 and E_3 from the outset, we obtain from Eq. (B9) a null four-vector $\tilde{G}^\mu$ after use of the Maxwell equations. This result is self-inconsistent because the tensor dual constructed from a null four-vector is a null tensor. In the particle interpretation, $\tilde{G}_\mu$ is the Pauli-Lubanski vector, and if this were null, there would be no finite helicity, and no properly defined particle, in this case the photon.

The above analysis is a particular case of

$$\tilde{G}_\mu := \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon^0 \\ \epsilon^1 \\ \epsilon^2 \\ \epsilon^3 \end{bmatrix}, \tag{B17}$$

and

$$G_\mu := \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon^0 \\ \epsilon^1 \\ \epsilon^2 \\ \epsilon^3 \end{bmatrix}, \tag{B18}$$

where

$$\epsilon^\mu := \left(\epsilon^0, \boldsymbol{\epsilon}\right), \tag{B19}$$

is a four dimensional polar unit vector. The conservation equation,

$$\partial_\mu G^\mu = \partial_\mu \tilde{G}^\mu = 0, \tag{B20}$$

in vector form is therefore

$$c\epsilon^0 \nabla \cdot \boldsymbol{B} + \boldsymbol{\epsilon} \cdot \left(\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} \right) = 0, \tag{B21a}$$

$$\frac{\epsilon^0}{c} \nabla \cdot \boldsymbol{E} + \boldsymbol{\epsilon} \cdot \left(\frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{B} \right) = 0. \tag{B21b}$$

The usual vacuum Maxwell equations are special cases of Eq. (B21) but the latter also allow other solutions, such as the Lehnert current discussed in Chap. 10, in which the divergence in vacuo of the electric field is not zero. If the photon has mass, then ϵ^μ is not light-like in the vacuum.

It is now possible to show straightforwardly that the Maxwell equations, in particular the Faraday law, and the B cyclics originate in the same topology and are closely similar in structure. One is obtained from the other by use of the de Broglie-Einstein hypothesis and the minimal prescription. This conclusion is illustrated by using the orthogonality condition,

$$\epsilon_\mu G^\mu = \epsilon_\mu \tilde{G}^\mu = 0, \tag{B22}$$

which is a topological identity as discussed in Chap. 8. Equation (B20) gives solutions such as

$$\partial_0 B_3 = -\left(\partial_1 E_2 - \partial_2 E_1\right), \tag{B23a}$$

and Eq. (B22) gives

$$\epsilon_0 B_3 = -\left(\epsilon_1 E_2 - \epsilon_2 E_1\right). \tag{B23b}$$

Equation (B23a) is a component of the Faraday law, one of the earliest laws of electrodynamics, whereas Eq. (B23b) can be put into the form of the B cyclics by using the minimal prescription and by defining the polar potential four-vector,

$$A^\mu := A^{(0)} \epsilon^\mu. \tag{B24}$$

Equation (B23b) becomes

$$A_0 B_3 = -\left(A_1 E_2 - A_2 E_1\right), \tag{B25}$$

which is equivalent to,

$$B_0 B_3 = -i\left(B_1 B_2^* - B_2 B_1^*\right), \tag{B26}$$

i.e., to

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \tag{B27}$$

in the circular basis. Finally, if $p^\mu = eA^\mu$ is a four-momentum, then the Faraday law (B23a) becomes identical with the B cyclic relation (B27) by use of the Einstein-de Broglie relation.

References

[1] G. N. Afanasiev and Yu. P. Stepanovsky, *Nuovo Cim. A.* **109**, 271 (1996).
[2] A. F. Ranada, *Eur. J. Phys.* **13**, 70 (1992); *J. Phys. A.* **25**, 1621 (1992); G. N. Afanasiev, *J. Phys. A.* **27**, 2143 (1994); **26**, 731 (1993).
[3] H. Bacry, *Helv. Phys. Acta*, **67**, 632 (1994); H. K. Moffat, *Nature*, **347** (1990).

Appendix C. Complete Lie Algebra of the Poincaré Group

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If it is accepted that a theory must be developed on the basis of its underlying symmetry, then it becomes important to have available for reference the complete Lie algebra of the group of special relativity, the Poincaré group. Part of this Lie algebra is given by Ryder [1], in a condensed form. In this Appendix it is written out for clarity, and completed by the use of the Pauli-Lubanski pseudo-vector. The theory of the electromagnetic field should then be based on this symmetry.

The Lie algebra that follows is the commutator algebra of the infinitesimal generators of the Poincaré group:

J_μ, the generators of rotations in spacetime.
K_μ, the boost generators.
P_μ, the generators of spacetime translation.
W_μ, the Pauli-Lubanski generators.

The matrix of boost and rotation generators used by Ryder is first written out in full

$$J_{\mu\nu} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{bmatrix}, \tag{C1}$$

and is seen to be an antisymmetric matrix with the same symmetry as the matrix of electromagnetic field components; boost generators replacing electric field components; rotation generators replacing magnetic field components. The matrix (C1) is one made up of operator components, whose Lie algebra is part of the complete Lie algebra. Matrix (C1) also exists in the Lorentz group, which does not include P_μ.

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The J, K Lie Algebra

This is the Lie algebra of components of (C1), that of the Lorentz group, a sub group of the Poincaré group, which contains P_μ. It is seen that the original development of special relativity from the Maxwell equations could not have been completely self-consistent, because it did not take into consideration the P_μ and W_μ generators. This is a key point when we come to discuss $\boldsymbol{B}^{(3)}$. In contrast, the theory of particle mass and spin, developed by Wigner in 1939 [3,1], rests on the recognition of W_μ and P_μ as the basic generators of the Poincaré group. Particles classified in this way include the photon, and so if we accept wave particle dualism, the concomitant electromagnetic field should also be developed within the Poincaré group. This leads directly to longitudinal structure such as $\boldsymbol{B}^{(3)}$. The theory of field helicity depends on a non-zero $\boldsymbol{B}^{(3)}$.

The following rules [2] are needed for commutators, which are defined equally well in classical and quantum physics.

$$
\begin{aligned}
[A,B] &= -[B,A], \\
[A,BC] &= [A,B]C + B[A,C], \\
[AB,C] &= A[B,C] + [A,C]B, \\
[DE,BC] &= [D,B]EC + D[E,B]C \\
&\quad + B[D,C]E + BD[E,C].
\end{aligned} \tag{C2}
$$

The operator components in what follows are either 4 x 4 matrices or differential operators, as defined by Ryder [1] or in earlier volumes of this series [3—5]. The Lie algebra of the Lorentz group is therefore

$$
\begin{aligned}
[J_1, J_2] &= iJ_3 = -[K_1, K_2], \\
[J_2, J_3] &= iJ_1 = -[K_2, K_3], \\
[J_3, J_1] &= iJ_2 = -[K_3, K_1],
\end{aligned} \tag{C3a}
$$

$$
\begin{aligned}
[J_1, K_2] &= iK_3 = [K_1, J_2], \\
[J_2, K_3] &= iK_1 = [K_2, J_3], \\
[J_3, K_1] &= iK_2 = [K_3, J_1],
\end{aligned} \tag{C3b}
$$

$$
\begin{aligned}
[J_1, K_1] &= 0, \\
[J_2, K_2] &= 0, \\
[J_3, K_3] &= 0,
\end{aligned} \tag{C3c}
$$

and is a series of cyclic relations between J and K commutators. This algebra was first applied to the electromagnetic field in Refs. [3—5] by replacing J components by B components and K components by E components. This method leads to the conclusion that $\boldsymbol{B}^{(3)}$ is non-zero because if one component of J (e.g. J_3) is set to zero in the Lie algebra the following occurs. If $J_3 =? 0$, then J_1 and J_2 become zero from Eq. (C3a); K_2 becomes zero from Eq. (C3b); and K_1 and K_3 become zero from Eq. (C3a). So if one generator is arbitrarily set to zero we lose all the generators. If we accept the first principle that the symmetry of the electromagnetic field is the symmetry of special relativity, and if we set $\boldsymbol{B}^{(3)}$ (proportional to J_3) to zero, we lose the electromagnetic field entirely, a self-inconsistency. This is true even within the Lorentz group.

Therefore $J_3 =? 0$ implies the loss of all the group generators; and in the field interpretation, $\boldsymbol{B}^{(3)} =? 0$ implies the complete loss of the field. In the received view, $\boldsymbol{B}^{(3)} =? \mathbf{0}$, and so the received view is not consistent with the Lie algebra even of the restricted Lorentz group.

Poincaré Group Lie Algebra

The Lorentz group is extended to the Poincaré group by adjoining the generator P_μ, which is used to define [1] W_μ. The P_μ operator is defined classically as a differential operator,

$$
\boldsymbol{P}_\mu := i\partial_\mu , \tag{C4}
$$

with

$$\partial_\mu := \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z} \right) , \tag{C5}$$

and so in the quantum theory $\hbar P_\mu$ becomes the energy momentum operator in the the coordinate representation. In the classical theory it is an infinitesimal generator of the Poincaré group, and generates translations in spacetime.

The Pauli-Lubanski pseudo four-vector completes the definition of the Poincaré group, and is defined by Ryder [1] as

$$W_\mu := -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} \boldsymbol{P}^\sigma , \tag{C6}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the fully antisymmetric unit four tensor in four dimensions,

$$\begin{aligned} \epsilon_{0123} &= -1 , \\ \epsilon_{0213} &= 1 , \\ \epsilon_{2013} &= -1 , \quad \text{etc.} \end{aligned} \tag{C7}$$

Therefore,

$$\begin{aligned} W_0 &= -J_1P_1 - J_2P_2 - J_3P_3 , \\ W_1 &= J_1P_0 + K_2P_3 - K_3P_2 , \\ W_2 &= J_2P_0 + K_3P_1 - K_1P_3 , \\ W_3 &= J_3P_0 + K_1P_2 - K_2P_1 , \end{aligned} \tag{C8}$$

and the Pauli-Lubanski (PL) pseudo four-vector is made up of sums of quadratic products of operators. Its W_0 component is the scalar helicity operator in particle physics. The P and W vectors form the two Casimir invariants [1] of the Poincaré group, the mass and spin invariants. All particles, including the photon, were classified by Wigner in terms of these invariants.

Ryder gives that part of the Lie algebra which depends on P_μ, but does not give the part dependent on W_μ. The complete Lie algebra is needed to have available the complete information that is present in the underlying symmetry of special relativity, which according to the first principle mentioned already, is the complete symmetry of the electromagnetic field itself, because that field is developed within the same underlying

symmetry, that of special relativity.

We first give the Lie algebra defined by P_μ and W_μ. It turns out that all components commute as follows:

$$[P_\mu, P_\nu] = 0, \quad [P_\mu, W_\nu] = 0, \tag{C9a}$$

i.e.

$$\begin{aligned} &[P_0, W_0] = [P_1, W_1] = [P_2, W_2] = [P_3, W_3] = 0, \\ &[P_0, W_1] = i[P_2, P_3] = 0, \quad [P_1, W_0] = i[P_2, P_3] = 0, \\ &[P_0, W_2] = i[P_3, P_1] = 0, \quad [P_1, W_2] = i[P_3, P_0] = 0, \\ &[P_0, W_3] = i[P_1, P_2] = 0, \quad [P_1, W_3] = i[P_0, P_2] = 0, \end{aligned} \tag{C9b}$$

$$\begin{aligned} &[P_2, W_0] = i[P_3, P_1] = 0, \quad [P_3, W_0] = i[P_2, P_1] = 0, \\ &[P_2, W_1] = i[P_0, P_3] = 0, \quad [P_3, W_1] = i[P_2, P_0] = 0, \\ &[P_2, W_3] = i[P_1, P_0] = 0, \quad [P_3, W_2] = i[P_0, P_1] = 0, \end{aligned} \tag{C9c}$$

This sub algebra shows that all components of the generator P_μ commute with all other components of P_μ, and also with all components of W_μ. Within this structure occur, formally, cyclic relations such as

$$\begin{aligned} [P_1, W_2] &= i[P_3, P_0], \\ [P_2, W_3] &= i[P_1, P_0], \\ [P_3, W_1] &= i[P_2, P_0]. \end{aligned} \tag{C10}$$

The Hamiltonian operator P_0 commutes with the three components of W, and with the three components of momentum, giving two conservation laws.

It is noted that equations (C9a) to (C10) must be evaluated using the commutator rules (C2), because we are always dealing with operators, i.e., infinitesimal generators. The Lie algebra $([P_\mu, P_\nu])$ is given by Ryder [1], but the Lie algebra $([P_\mu, W_\nu])$ appears be novel. Equations (C9a) are important conservation laws which show that the ndamental vectors are P_μ and W_μ.

If we accept the idea that the electromagnetic field is a physical entity whose iderlying symmetry is that of special relativity, we are led to the conclusion that there ists the equivalent of P_μ and W_μ for the field as well as the photon, and that these are n-zero. Otherwise there is no photon in the particle interpretation, and we lose the field tirely. This principle leads directly to the conclusion that the $\boldsymbol{B}^{(3)}$ component is non-ro, and that there are Maxwell equations for $\boldsymbol{B}^{(3)}$ as well as for the transverse $\boldsymbol{B}^{(1)}$ d $\boldsymbol{B}^{(2)}$.

The algebra (C3) and (C9) are supplemented by other types of commutators which eatly enrich the Lie algebra of the Lorentz group. Ryder gives only one type, his Eq. .187) [1],

$$\left[\boldsymbol{P}_\mu, J_{\rho\sigma}\right] = i\left(g_{\mu\rho}\boldsymbol{P}_\sigma - g_{\mu\sigma}\boldsymbol{P}_\rho\right), \tag{C11}$$

ıere $g_{\mu\rho}$ is the metric tensor,

$$g_{\mu\rho} = g^{\mu\rho} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{C12}$$

is worth writing Eq. (C11) out in full for clarity of reference, and to show the existence various cyclic relations:

$$\begin{aligned}
&[P_0, K_1] = iP_1, \quad [P_0, J_1] = 0, \\
&[P_0, K_2] = iP_2, \quad [P_0, J_2] = 0, \\
&[P_0, K_3] = iP_3, \quad [P_0, J_3] = 0, \\
&[P_1, K_1] = iP_0, \quad [P_1, J_1] = 0, \\
&[P_1, K_2] = 0, \quad [P_1, J_2] = iP_3, \\
&[P_1, K_3] = 0, \quad [P_1, J_3] = -iP_2, \\
&[P_2, K_1] = 0, \quad [P_2, J_1] = -iP_3, \\
&[P_2, K_2] = iP_0, \quad [P_2, J_2] = 0, \\
&[P_2, K_3] = 0, \quad [P_2, J_3] = iP_1, \\
&[P_3, K_1] = 0, \quad [P_3, J_1] = iP_2, \\
&[P_3, K_2] = 0, \quad [P_3, J_2] = -iP_1, \\
&[P_3, K_3] = iP_0, \quad [P_3, J_3] = 0.
\end{aligned} \tag{C13}$$

The Hamiltonian P_0 commutes with the rotation generator components, which constitute a constant of motion which in the particle interpretation becomes angular momentum. The latter is therefore conserved as required. The Hamiltonian does not, in contrast, commute with the boost generators. There are cyclic relations such as,

$$\begin{aligned}
&[P_1, J_2] = iP_3, \\
&[P_2, J_3] = iP_1, \\
&[P_3, J_1] = iP_2,
\end{aligned} \tag{C14}$$

which again must have their equivalents in the classical electromagnetic field.

The above Lie algebra is supplemented for the first time in this Appendix by,

$$\left[W_\mu, J_{\rho\sigma}\right] = i\left(g_{\mu\sigma}W_\rho - g_{\mu\rho}W_\sigma\right), \tag{C15}$$

which has a sign change from Eq. (C11). This Lie algebra gives information on commutators of W_μ with J_μ and K_μ, as follows:

$$
\begin{aligned}
&[W_0, K_1] = -iW_1, \quad [W_0, J_1] = 0,\\
&[W_0, K_2] = -iW_2, \quad [W_0, J_2] = 0,\\
&[W_0, K_3] = -iW_3, \quad [W_0, J_3] = 0,\\
&[W_1, K_1] = -iW_0, \quad [W_1, J_1] = 0,\\
&[W_1, K_2] = 0, \quad [W_1, J_2] = iW_3,\\
&[W_1, K_3] = 0, \quad [W_1, J_3] = -iW_2,\\
&[W_2, K_1] = 0, \quad [W_2, J_1] = -iW_3,\\
&[W_2, K_2] = -iW_0, \quad [W_2, J_2] = 0,\\
&[W_2, K_3] = 0, \quad [W_2, J_3] = iW_1,\\
&[W_3, K_1] = 0, \quad [W_3, J_1] = iW_2,\\
&[W_3, K_2] = 0, \quad [W_3, J_2] = -iW_1,\\
&[W_3, K_3] = -iW_0, \quad [W_3, J_3] = 0.
\end{aligned} \tag{C16}
$$

There are cyclic relations such as

$$
\begin{aligned}
&[W_1, J_2] = iW_3,\\
&[W_2, J_3] = iW_1,\\
&[W_3, J_1] = iW_2,
\end{aligned} \tag{C17}
$$

which can be applied to the structure of the electromagnetic field. If we arbitrarily set the longitudinal W_3 to zero, all the generators commute, and the physical entity being described is lost entirely, a self-inconsistency.

Using the foregoing results, we finally arrive at the Lie algebra of components of W_μ,

$$
\begin{aligned}
&[W_0, W_1] = -i(W_2P_3 - W_3P_2) = i(P_3W_2 - P_2W_3),\\
&[W_0, W_2] = -i(W_3P_1 - W_1P_3) = i(P_1W_3 - P_3W_1),\\
&[W_0, W_3] = -i(W_1P_2 - W_2P_1) = i(P_2W_1 - P_1W_2),\\
&[W_1, W_2] = i(W_3P_0 + W_0P_3) = i(P_0W_3 + P_3W_0),\\
&[W_2, W_3] = i(W_1P_0 + W_0P_1) = i(P_0W_1 + P_1W_0),\\
&[W_3, W_1] = i(W_2P_0 + W_0P_2) = i(P_0W_2 + P_2W_0).
\end{aligned} \tag{C18}
$$

Special Cases

In the lightlike condition,

$$P^\mu = (P_0, 0, 0, P_3), \qquad P_3 = P_0, \tag{C19}$$

and

$$
\begin{aligned}
W_0 &= -J_3P_3,\\
W_1 &= J_1P_0 + K_2P_3,\\
W_2 &= J_2P_0 - K_1P_3,\\
W_3 &= J_3P_0.
\end{aligned} \tag{C20}
$$

In this condition, therefore,

$$[W_1, W_2] = P_0[J_1 + K_2, J_2 - K_1] = 0, \tag{C21}$$

which is consistent with the Lie algebra of the Lorentz group given earlier,

$$[J_1, J_2] + [K_1, K_2] + [K_2, J_2] + [K_1, J_1] = 0. \tag{C22}$$

Equation (C21) is also consistent with

$$J_1 + K_2 = 0, \qquad J_2 - K_1 = 0, \tag{C23}$$

as in Ryder's Eq. (2.212) with his L_2 replaced by our $K_2 + J_1$; his L_1 replaced by $K_1 - J_2$.

The E(2) Structure

The complete Lie algebra of the Poincaré group contains an E(2) structure, namely,

$$[W_1, W_2] = i(W_3P_0 + W_0P_3),$$
$$[J_3, W_1] = iW_2, \tag{C24}$$
$$[W_2, J_3] = iW_1,$$

which in the lightlike condition becomes

$$[W_1, W_2] = 0,$$
$$[J_3, W_1] = iW_2, \tag{C25}$$
$$[W_2, J_3] = iW_1,$$

the planar Euclidean group. The latter is the little group in the lightlike condition [1]. It is seen that J_3 is non-zero, and in the field interpretation, $B^{(3)}$ is non-zero in the E(2) group.

The Rest Frame

If there is a rest frame,

$$p^\mu = (p_0, 0, 0, 0), \tag{C26}$$

and Eq. (C18) becomes an O(3) structure,

$$[W_1, W_2] = iP_0W_3,$$
$$[W_2, W_3] = iP_0W_1, \tag{C27}$$
$$[W_3, W_1] = iP_0W_2.$$

In the rest frame, however,

$$W_0 = 0,$$
$$W_1 = J_1P_0,$$
$$W_2 = J_2P_0, \tag{C28}$$
$$W_3 = J_3P_0,$$

and the O(3) structure (C27) becomes the cyclic Lie algebra of the rotation generators of the Lorentz group.

A complete knowledge of the Lie algebra shows that the E(2) and O(3) groups can be generated as sub algebra of the Poincaré group's Lie algebra.

Helicity in the Lightlike Condition

The complete PL vector in this condition is

$$W^\mu = P_0(-J_3, 0, 0, J_3), \tag{C29}$$

if we accept the constraints

$$J_1 = -K_2, \quad J_2 = K_1, \tag{C30}$$

which in the field interpretation are given by the Faraday law of induction, i.e.,

$$cB_1 = -E_2, \quad cB_2 = E_1, \tag{C31}$$

is satisfied by

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0. \tag{C32}$$

This is a simple illustration of the fact that the experimentally verified Faraday law of induction leads to the conclusion that a non-zero $\boldsymbol{B}^{(3)}$ is needed for non-zero field helicity. Since helicity is a topological invariant, $\boldsymbol{B}^{(3)}$ is non-zero topologically.

If we accept the first principle that all theories in special relativity are based on the underlying Poincaré group, we should proceed logically by deriving the equations of the electromagnetic field from the group structure, as just illustrated for the Faraday law. Similarly, the equations of the electroweak field can be obtained, by using the methods described elsewhere [4]. The Lie algebra includes E(2) and O(3), and applies to all physical entities and theories within special relativity, using vectors and spinors. The notion of the relativistic helicity of the classical electromagnetic field is based on the existence of P_μ and W_μ, and leads to the existence of $\boldsymbol{B}^{(3)}$ as a topological invariant. The same group structure shows that $\boldsymbol{B}^{(3)}$ must always be related to $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ topologically, and this determines the way in which $\boldsymbol{B}^{(3)}$ interacts with a fermion [5] as in the inverse Faraday effect. Dvoeglazov has recently developed several theories based on field and particle helicity and chirality, as described in this volume.

Any generalization of the Maxwell equations must also take place within the Poincaré group if we proceed within special relativity. In general relativity the underlying symmetry group becomes the Einstein group.

It is particularly important to note that it is not possible to arbitrarily remove a generator without completely removing the physical entity being considered, be this particle or field. The arbitrary removal of $\boldsymbol{B}^{(3)}$ is usually accepted however in the received view in order to use a $U(1)$ group for the electromagnetic sector. This is clearly inconsistent with the struture not only of the Poincaré group, but also of the Lorentz group, yet this is the received view. Therefore $U(1)$ is not a group symmetry of special relativity, surprising as this may seem.

References

[1] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).

[2] P. W. Atkins, *Molecular Quantum Mechanics* (Oxford University Press, Oxford, 1983). For a relativistic treatment see L. D. Landau and E. M. Lifshitz, *The Clasiical Theory of Fields*, 4th edn. (Pergamon, Oxford, 1975).

[3] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $\boldsymbol{B}^{(3)}$ (Kluwer Academic, Dordrecht, 1994).

[4] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).

[5] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).

Appendix D. Standard Expressions for Generator Matrices

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The generator matrices used in this volume are summarized in this appendix, with definitions of duals. Their structure is seen to be identical with that of the corresponding electromagnetic field tensors, with boost components K_i replacing electric field components E_i and rotation generator components J_i replacing magnetic field components B_i.

The contravariant generator matrix is

$$J^{\mu\nu} := \begin{bmatrix} 0 & -K^1 & -K^2 & -K^3 \\ K^1 & 0 & -J^3 & J^2 \\ K^2 & J^3 & 0 & -J^1 \\ K^3 & -J^2 & J^1 & 0 \end{bmatrix}, \qquad \textbf{(D1)}$$

which has the same structure as given, for example, in Ryder's Eq. (2.223) [1].

The covariant generator matrix is therefore

$$J_{\mu\nu} := \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & -J_3 & J_2 \\ -K_2 & J_3 & 0 & -J_1 \\ -K_3 & -J_2 & J_1 & 0 \end{bmatrix}, \qquad \textbf{(D2)}$$

as given for example in Barut [5].

The contravariant dual is defined by

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$$\tilde{J}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\rho\sigma} \tag{D3}$$

$$= \begin{bmatrix} 0 & -J^1 & -J^2 & -J^3 \\ J^1 & 0 & K^3 & -K^2 \\ J^2 & -K^3 & 0 & K^1 \\ J^3 & K^2 & -K^1 & 0 \end{bmatrix}, \tag{D4}$$

as in for example Ryder's Eqs. (2.234) and (2.235).

The covariant dual is defined by

$$\tilde{J}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\rho\sigma} = \begin{bmatrix} 0 & J_1 & J_2 & J_3 \\ -J_1 & 0 & K_3 & -K_2 \\ -J_2 & -K_3 & 0 & K_1 \\ -J_3 & K_2 & -K_1 & 0 \end{bmatrix}. \tag{D5}$$

Comparing Eqs. (D1) and (D3) it is possible to write the dual transform *formally* as

$$\boldsymbol{K} \rightarrow \boldsymbol{J}, \qquad \boldsymbol{J} \rightarrow -\boldsymbol{K}, \tag{D6}$$

with the understanding that if the original element is zero, e.g. $K_3 = J_{03} = 0$, the dual element is also zero. Therefore the dual of the zero element $\boldsymbol{E}^{(3)}$ [2—4] is also zero, and the dual of the non-zero element $\boldsymbol{B}^{(3)}$ is non-zero. For example, the dual of the K_3 element from Eq. (D3) is

$$\tilde{J}^{12} := \frac{1}{2}\left(\epsilon^{1203}J_{03} + \epsilon^{1230}J_{30}\right) = K^3, \tag{D7}$$

which is the same element, but situated in a different place in the dual matrix. In the first three volumes [2—4] the dual was understood to mean the formal replacement summarized symbolically in Eq. (D6), by which is meant the formal replacement of an element of the original matrix by an element situated in the same place in the dual matrix. This usage has crept into the literature but is rather confusing. It is clearer to use the rigorous definitions (D3) and (D5) given in this Appendix. These show that the non-zero $\boldsymbol{B}^{(3)}$ remains non-zero in the dual matrix. Any attempt to use the *dual transform* to assert that $\boldsymbol{B}^{(3)}$ must be zero is incorrect, for example in the debate between van Enk [6] and Evans [7].

References

[1] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).

[2] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $\boldsymbol{B}^{(3)}$ (Kluwer Academic, Dordrecht, 1994).

[3] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).

[4] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).

[5] A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (MacMillan, New York, 1964).

[6] S. J. van Enk, *Found. Phys. Lett.* **9**, 183 (1996).

[7] M. W. Evans, ibid, p. 191.

Appendix E. Poincaré Group, Maxwell Equations and Cyclics

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The Lie algebra of the Poincaré group gives the structure of the Maxwell equations and cyclic relations between field components under any conditions that are compatible with special relativity. This process is exemplified in this appendix starting from, for example,

$$
\begin{aligned}
\left[P_2, J_3\right] &= iP_1 ,\\
\left[P_3, J_2\right] &= -iP_1 ,\\
\left[P_0, K_1\right] &= iP_1 .
\end{aligned}
\tag{E1}
$$

By definition,

$$P_\mu = i\partial_\mu , \tag{E2}$$

so Eq. (E1) becomes

$$\left(\left[\partial_2, J_3\right] - \left[\partial_3, J_2\right] - \left[\partial_0, K_1\right]\right) \psi = P_1 \psi , \tag{E3}$$

where ψ is an eigenfunction. Equation (E3) can be rewritten as

$$\left(\partial_2 J_3 - \partial_3 J_2 - \partial_0 K_1 - \left(J_3 \partial_2 - J_2 \partial_3 - K_1 \partial_0\right)\right) \psi = P_1 \psi , \tag{E4}$$

which is a relation between operators on ψ. Now use

$$
\begin{aligned}
J_3 \psi &= j_3 \psi ,\\
J_2 \psi &= j_2 \psi ,\\
K_1 \psi &= k_1 \psi ,
\end{aligned}
\tag{E5}
$$

where lower case letters denote eigenvalues. We have,

$$\partial_2 (j_3 \psi) = (\partial_2 j_3)\psi + j_3(\partial_2 \psi),$$
$$\partial_3 (j_2 \psi) = (\partial_3 j_2)\psi + j_2(\partial_3 \psi), \qquad \text{(E6)}$$
$$\partial_0 (k_1 \psi) = (\partial_0 k_1)\psi + k_1(\partial_0 \psi).$$

We assume that

$$J_3(\partial_2 \psi) + J_2(\partial_3 \psi) + K_1(\partial_0 \psi)$$
$$= j_3(\partial_2 \psi) + j_2(\partial_3 \psi) + k_1(\partial_0 \psi), \qquad \text{(E7)}$$

which is compatible with

$$(\partial_2 + \partial_3 + \partial_0)\psi = \text{constant } \psi. \qquad \text{(E8)}$$

Equations (E4) to (E8) give the eigenvalue relation,

$$\partial_2 j_3 - \partial_3 j_2 - \partial_0 k_1 = p_1, \qquad \text{(E9)}$$

which is one component of

$$\nabla \times \boldsymbol{j} - \frac{1}{c}\frac{\partial \boldsymbol{k}}{\partial t} = \boldsymbol{p}. \qquad \text{(E10)}$$

If we write

$$\psi := e^{i\phi}\psi. \qquad \text{(E11)}$$

where ϕ is a phase factor, then,

$$J_3\psi = J_3(e^{i\phi}\psi_0) = j_3^{(0)} e^{i\phi}\psi_0 := j_3\psi, \qquad \text{(E12)}$$

and so on. Therefore the eigenvalues appearing in Eq. (E10) are phase dependent in general. It is clear that the structure of Eq. (E10) is that of one of the inhomogeneous Maxwell equations.

The complete set of operator relations leading to this equation is

$$([\partial_1, J_2] - [\partial_2, J_1] - [\partial_0, K_3])\psi = P_3\psi,$$
$$([\partial_2, J_3] - [\partial_3, J_2] - [\partial_0, K_1])\psi = P_1\psi, \qquad \text{(E13)}$$
$$([\partial_3, J_1] - [\partial_1, J_3] - [\partial_0, K_2])\psi = P_2\psi.$$

Similarly, the Lie algebra,

$$([\partial_2, K_3] - [\partial_3, K_2] + [\partial_0, J_3])\psi = 0 \qquad \text{(E14)}$$

and so forth leads to the eigenvalue relation,

$$\nabla \times \boldsymbol{k} + \frac{1}{c}\frac{\partial \boldsymbol{j}}{\partial t} = 0, \qquad \text{(E15)}$$

which has the structure of one of the homogeneous Maxwell equations.

The Lie algebra,

$$([\partial_1, J_1] + [\partial_2, J_2] + [\partial_3, J_3])\psi = 0, \qquad \text{(E16)}$$

gives

$$((\partial_1 J_1 - J_1\partial_1) + (\partial_2 J_2 - J_2\partial_2) + (\partial_3 J_3 - J_3\partial_3))\psi = 0. \qquad \text{(E17)}$$

Using

$$J_1\psi = j_1\psi,$$
$$\partial_1(j_1\psi) = j_1(\partial_1\psi) + (\partial_1 j_1)\psi, \qquad \text{(E18)}$$

and assuming

$$J_1(\partial_1\psi) + J_2(\partial_2\psi) + J_3(\partial_3\psi)$$
$$= j_1(\partial_1\psi) + j_2(\partial_2\psi) + j_3(\partial_3\psi), \qquad \text{(E19)}$$

leads to

$$\partial_1 j_1 + \partial_2 j_2 + \partial_3 j_3 = 0 \quad i.e.\ \nabla\cdot\boldsymbol{j} = 0. \qquad \text{(E20)}$$

Appendix F. The Lehnert Current

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The Lehnert current is a non-zero divergence in vacuo of the electric field, and is sometimes known as the vacuum current. The structure of the Poincaré group gives, for generator eigenvalues,

$$\nabla \cdot \boldsymbol{k} = 3 p_0 , \tag{F1}$$

$$\nabla \cdot \boldsymbol{j} = 0 , \tag{F2}$$

$$\nabla \times \boldsymbol{k} + \frac{1}{c} \frac{\partial \boldsymbol{j}}{\partial t} = 0 , \tag{F3}$$

$$\nabla \times \boldsymbol{j} - \frac{1}{c} \frac{\partial \boldsymbol{k}}{\partial t} = \boldsymbol{p} . \tag{F4}$$

The Lehnert current is therefore intrinsic within the structure of the Poincaré group, but *not* of the Lorentz group, in which $\boldsymbol{p}$ is undefined. It is given by identifying $\boldsymbol{p}$ with $\mu_0 \boldsymbol{I}/e$, where μ_0 is the vacuum permeability and $\boldsymbol{I}$ the vacuum current. The structure (F1) to (F4) exists under *all* conditions, including the vacuum. It allows for the Lehnert current [1], but not for the magnetic monopole, because $\nabla \cdot \boldsymbol{j} = 0$ by symmetry. This shows that the existence of $\boldsymbol{B}^{(3)}$ does not imply the existence of the magnetic monopole. Significantly, $\boldsymbol{B}^{(3)}$ is observed in magneto optics, but the magnetic monopole has not been observed experimentally. These findings are consistent with special relativity, and so is the Lehnert current, which should therefore be observable experimentally. Finally, the structure (F1) to (F4) is consistent with the novel vacuum current proposed by Chubykalo et al [2] as a consequence of longitudinal structure typified by $\boldsymbol{B}^{(3)}$. The Poincaré group allows for the existence of the longitudinal $\boldsymbol{E}^{(3)}$ in the vacuum, for

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example the Coulombic field, through the symmetry (F1).

References

[1] B. Lehnert, Chap. 10, this volume.
[2] A. E. Chubykal and R. Smirnov-Rueda, Chap. 11, this volume.

Appendix G. Some Useful Relations in Spherical Polar Coordinates

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The spherical polar coordinates are defined by

$$\left.\begin{aligned} X &= r\sin\phi\cos\theta \\ Y &= r\sin\phi\sin\theta \\ Z &= r\cos\phi \end{aligned}\right], \qquad \textbf{(G1)}$$

so that the unit vectors in the Cartesian and spherical polar system are inter related by

$$\left.\begin{aligned} \boldsymbol{i} &= \sin\phi\cos\theta\,\boldsymbol{e}_r + \cos\phi\cos\theta\,\boldsymbol{e}_\phi - \sin\theta\,\boldsymbol{e}_\theta \\ \boldsymbol{j} &= \sin\phi\sin\theta\,\boldsymbol{e}_r + \cos\phi\sin\theta\,\boldsymbol{e}_\phi + \cos\theta\,\boldsymbol{e}_\theta \\ \boldsymbol{k} &= \cos\phi\,\boldsymbol{e}_r - \sin\phi\,\boldsymbol{e}_\phi \end{aligned}\right], \qquad \textbf{(G2)}$$

$$\left.\begin{aligned} \boldsymbol{e}_r &= \sin\phi\cos\theta\,\boldsymbol{i} + \sin\phi\sin\theta\,\boldsymbol{j} + \cos\phi\,\boldsymbol{k} \\ \boldsymbol{e}_\phi &= \cos\phi\cos\theta\,\boldsymbol{i} + \cos\phi\sin\theta\,\boldsymbol{j} - \sin\phi\,\boldsymbol{k} \\ \boldsymbol{e}_\theta &= -\sin\theta\,\boldsymbol{i} + \cos\theta\,\boldsymbol{j} \end{aligned}\right]. \qquad \textbf{(G3)}$$

There exists an O(3) symmetry as follows:

$$\left.\begin{aligned} \boldsymbol{e}_\phi \times \boldsymbol{e}_\theta &= \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \times \boldsymbol{e}_r &= \boldsymbol{e}_\phi \\ \boldsymbol{e}_r \times \boldsymbol{e}_\phi &= \boldsymbol{e}_\theta \end{aligned}\right]. \qquad \textbf{(G4)}$$

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The ((1), (2), (3)) system is related to the spherical polar system by

$$e^{(1)} = e^{(2)*} = \frac{1}{\sqrt{2}}\big(\sin\phi(\cos\theta - i\sin\theta)e_r + \cos\phi(\cos\theta - i\sin\theta)e_\phi - (\sin\theta + i\cos\theta)e_\theta\big), \tag{G5}$$

$$e^{(3)} = e^{(3)*} = \cos\phi\, e_r - \sin\phi\, e_\phi\,. \tag{G6}$$

Therefore the $B^{(3)}$ field in spherical polar coordinates is

$$B^{(3)} = B^{(0)}\left(\cos\phi\, e_r - \sin\phi\, e_\phi\right). \tag{G7}$$

The curl of B is

$$\nabla\times B^{(3)} = \frac{1}{r^2\sin\phi}\begin{vmatrix} e_r & re_\phi & r\sin\phi\, e_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\theta} \\ B_r & rB_\phi & r\sin\phi\, B_\theta \end{vmatrix} = \mathbf{0}\,. \tag{G8}$$

Spherical polar coordinates are useful in the evaluation of $B^{(3)}$ from dipole or higher multipole radiation theory, where the far zone magnetic flux density has a characteristic $1/r$ dependence,

$$B_\alpha^{dipole} = \frac{\mu_0\omega^2}{4\pi rc}\epsilon_{\alpha\beta\gamma}\, n_\beta\, \mu_\gamma^{(0)} e^{i(\kappa r - \omega t)}\,. \tag{G9}$$

Here μ_0 is the vacuum permeability, ω the angular frequency, r the radial coordinate, c the conventional speed of light, $\mu_\gamma^{(0)}$ the electric dipole moment, κ the wavevector, and where the electromagnetic phase is $\kappa r - \omega t$ at instant t.

In order to calculate $B^{(0)}$ we first find the total radiated power by integrating the square amplitude of B_α^{dipole}

$$B_{dipole}^2 = \left(\frac{\mu_0^2\mu^{(0)2}}{16\pi^2 r^2 c^2}\right)\omega^4, \tag{G10}$$

over a spherical surface. The surface is expressed in spherical polar coordinates to give

$$W(\text{watts}) = \frac{c}{\mu_0}\int_0^{2\pi}\int_0^{\pi} B_{\text{dipole}}^2 r^2 \sin\phi\, d\phi\, d\theta = \left(\frac{\mu^{(0)2}\mu_0}{6\pi c}\right)\omega^4\,. \tag{G11}$$

Finally,

$$B^{(0)2} = \frac{\mu_0}{c}I, \tag{G12}$$

where I, the total radiated power density, or intensity, is W divided by the r independent beam cross section. Therefore the observable $B^{(0)}$ is independent of r, and $B^{(3)}$ in consequence is divergentless. Thus $B^{(0)}$ and not B_α^{dipole} is used in the B cyclic theorem.

Invariance of the B Cyclics

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Using relativistic angular momentum theory, it is shown that the B cyclic theorem is invariant under a Z axis Lorentz boost, as is both $\boldsymbol{B}^{(3)}$ and its amplitude $B^{(0)}$. This result is consistent with the definition of $B^{(0)}$ in terms of the quantum of power, and with the fact that $\boldsymbol{B}^{(3)}$ is a phaseless and longitudinal magnetic field. If aligned in Z it is therefore invariant under the Z axis Lorentz boost, as for any magnetic field. Therefore $\boldsymbol{B}^{(3)}$ and the B cyclic equations are novel invariants of the electromagnetic field in Poincaré group electrodynamics.

1. Introduction

Poincaré group electrodynamics [1—12] develops the vacuum electromagnetic field as a four dimensional physical entity, and allows the existence of longitudinal field components in vacuo. The subject has developed within five years from several independent directions [6—12] following the inference of the $\boldsymbol{B}^{(3)}$ field in 1992 [1]. This is the fundamental *field spin*, and $\boldsymbol{B}^{(3)}$ is a magnetic flux density in the propagation axis, observable empirically in magneto-optics [14—17]. It is defined through the B cyclic theorem,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \qquad et\ cyclicum, \tag{1a}$$

which is a spin angular momentum relation between the three components of the complete magnetic flux,

$$\boldsymbol{B} = \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \boldsymbol{B}^{(3)}, \tag{1b}$$

in the complex basis [1—6] ((1), (2), (3)). In this basis (1) corresponds to the lowering angular momentum operator in quantum mechanics [18] and (2) to its complex conjugate, the raising operator. Therefore (3) can be aligned for simplicity with the Z axis [18]. The B cyclics have an O(3) (space) symmetry, the same symmetry as that of the little group of a particle with mass in the Wigner theory [19]. This suggests that the photon can be treated as a particle with mass. In this letter, standard relativistic angular momentum theory is used to show that the B cyclic relations (1) are invariant under the Lorentz transformation, in the sense that the product $B^{(1)} \times B^{(2)}$ does not change its magnitude or direction under a Z axis Lorentz boost, for example, and neither does the product $iB^{(0)}B^{(3)*}$. *Individually*, the fields $B^{(1)}$ and $B^{(2)}$ are not Lorentz invariant. The method used to demonstrate this starts in Sec. 2 with the properties of relativistic angular momentum under a Lorentz boost. It is shown that the infinitesimal rotation generators in the rest frame appear again unchanged in the light-like condition, as part of an E(2) structure [19] that contains the O(3) structure of the rest frame. Sec. 3 shows that the quantum of flux $B^{(0)}$ is within a factor the quantum of angular momentum $\hbar$, so that the B cyclics are spin angular momentum relations. This allows the results of Section 2 to be applied to the B cyclics, with the result that they are found to be Lorentz invariant. Finally, Sec. 4 checks this result by direct calculation, using the well known rules for Lorentz transform of magnetic and electric fields.

2. Relativistic Angular Momentum

The theory of angular momentum in special relativity is based on the Pauli-Lubanski pseudo four-vector [19]. Without this construct the theory cannot be expressed in vector form. The Pauli-Lubanski vector dual to the angular momentum in classical relativity can be defined as [20]

$$W^{\mu} = -\frac{1}{2}\epsilon^{\lambda\mu\nu\rho}p_{\mu}J_{\nu\rho}, \tag{2}$$

where $J_{\nu\rho}$ is the antisymmetric tensor form of classical relativistic spin angular momentum,

$$J_{\nu\rho} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & -J_3 & J_2 \\ -K_2 & J_3 & 0 & -J_1 \\ -K_3 & -J_2 & J_1 & 0 \end{bmatrix}, \tag{3}$$

where the scalar J and K elements are those of spin angular momentum [20]. The four-momentum p_{μ} appears in Eq. (2) because it is not possible otherwise to define a four axial vector dual to the antisymmetric tensor $J_{\nu\rho}$. Thus Eq. (2) is the direct result of four dimensional topology, without any intervening assumption. If we assume that the angular momentum is conserved, we have

$$\partial_{\nu}\tilde{J}^{\mu\nu} = 0, \tag{4}$$

and

$$\partial^{\lambda}J^{\mu\nu} + \partial^{\mu}J^{\nu\lambda} + \partial^{\nu}J^{\lambda\mu} = 0, \tag{5}$$

which are equations analogous to the Maxwell equations for the field tensor. Here $\tilde{J}^{\mu\nu}$ is the dual of $J_{\sigma\rho}$. Thus W^{λ} is the axial four vector dual to the antisymmetric three tensor $p_{\mu}J_{\nu\rho}$ in four-space. Analogously, in three-space, the axial vector is dual to the antisymmetric tensor.

These equations can be used unchanged for the infinitesimal generators of the Poincaré group [20], as is well known. Thus K, J and P become *operators*, respectively the generators of boost, rotation and translation in four dimensions. The commutator algebra of these operators is the Lie algebra of the Poincaré group [21], which is automatically covariant. Barut [20] shows that the commutator of W^{λ} operators is given by

$$\left[W^{\lambda}, W^{\mu}\right] = -i\epsilon^{\mu\nu\sigma\rho}p_{\sigma}W_{\rho}, \tag{6}$$

$$\left[P_{\lambda}, W_{\mu}\right] = 0, \tag{7}$$

which is a cyclical, covariant equation of operators. In the rest frame the vector p^{μ} becomes

$$p^{\mu} = \left(p^{0}, 0, 0, 0\right), \tag{8}$$

and in the light-like condition it can be written as

$$p^{\mu} = \left(p^{0}, 0, 0, p^{0}\right). \tag{9}$$

Therefore the covariance of the equation (6) can be studied by boosting conveniently from the rest frame (8) to the light-like (9). In Cartesian notation the Pauli-Lubanski pseudo four-vector from Eq. (2) is

$$W_{\mu} = \left(p_Z J_Z, p_Z K_Y + p_0 J_X, -p_Z K_X + p_0 J_Y, p_0 J_Z\right), \tag{10}$$

so its elements are products of translation and rotation generators (operators). In the rest frame, Eq. (10) reduces with Eq. (8) to

$$W_\mu(\text{rest-frame}) = p_0\left(0, J_X, J_Y, J_Z\right). \tag{11}$$

Replacing the translation operators by their eigenvalues we obtain the correct rest frame Lie O(3) algebra of rotation generators,

$$\left[J_X, J_Y, \right] = i J_Z, \qquad et\ cyclicum, \tag{12}$$

in the Cartesian basis. Repeating this process in the light-like condition, using Eq. (9), we obtain the Lie algebra of the E(2) little group [20,21],

$$\begin{aligned} \left[J_X + K_Y, J_Y - K_X\right] &= i\left(J_Z - J_Z\right) = 0, \\ \left[J_Y - K_X, J_Z\right] &= i\left(K_Y + J_X\right), \\ \left[K_Y + J_X, J_Z\right] &= i\left(K_X - J_Y\right). \end{aligned} \tag{13}$$

This algebra contains Eq. (12) unchanged, along with part of the Lie algebra of the boost commutators,

$$\left[K_X - K_Y\right] = -i J_Z, \tag{14}$$

so the E(2) group contains two O(3) groups as sub symmetries. In particular the Lie algebra of the rotation generators is the same in the rest frame and light like condition. A particular solution of Eq. (4) is the plane wave solution,

$$\boldsymbol{J}^{(1)} = \boldsymbol{J}^{(2)*} = \frac{J^{(0)}}{\sqrt{2}}\left(i\boldsymbol{i} + \boldsymbol{j}\right) e^{i(\omega t - \kappa Z)}, \tag{15}$$

obtained from the conservation of angular momentum, Eqs. (4) and (5). By changing basis to ((1), (2), (3)) and by replacing rotation generators by angular momentum operators in the usual way [21], it is proven that the angular momentum cyclics,

$$\boldsymbol{J}^{(1)} \times \boldsymbol{J}^{(2)} = i\hbar \boldsymbol{J}^{(3)*}, \tag{16}$$

are Lorentz invariants.

3. Angular Momentum to Magnetic Field

The angular momentum cyclics (16) can be converted directly to cyclics in magnetic fields through the relation between $B^{(0)}$ and $\hbar$, the quantum of angular momentum [1—12],

$$B^{(0)} = \frac{\hbar\omega^2}{ec^2} = \frac{(\hbar\omega)\omega}{ec^2}, \tag{17}$$

where e is the quantum of charge and c the speed of light in vacuo. This is a Lorentz invariant relation because $\hbar\omega$ transforms as the quantum of energy and ω as a frequency,

$$En = \hbar\omega \rightarrow \gamma En', \qquad \omega \rightarrow \gamma^{-1}\omega, \tag{18}$$

where γ is the factor $\left(1 - (v^2/c^2)\right)^{-1/2}$. Here v is the interframe velocity in the Z axis for a Lorentz boost in Z. Therefore the quantum of power $\hbar\omega^2$ is a Lorentz invariant if we accept Eqs. (18). If however $\hbar\omega$ is regarded as transforming as frequency, $B^{(0)}$ is no longer an invariant quantity but is still covariant. Whether or not $\hbar\omega$ transforms as energy or frequency is one of the fundamental paradoxes [1—12] of the relativistic quantum field, first recognized by de Broglie. If we take $B^{(0)}$ to be an invariant, the angular momentum relation (16) goes directly over to the B cyclic theorem,

$$\begin{gathered} \boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \\ \updownarrow \\ \boldsymbol{J}^{(1)} \times \boldsymbol{J}^{(2)} = i\hbar\boldsymbol{J}^{(3)*}, \end{gathered} \tag{19}$$

because $B^{(0)}$ is directly proportional to $\hbar$, the quantum of angular momentum. In this case the B cyclic theorem itself is a Lorentz invariant construct.

4. Check on the Lorentz Invariance of the B Cyclics

If we use the usual rules for the Lorentz transform of a magnetic field, and boost the product $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ along Z from the hypothetical rest frame to the speed of light, we obtain the result,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} \rightarrow \left(\frac{0}{0}\right)\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}, \tag{20}$$

ə. we obtain in frame K', moving at c with respect to frame K, the original $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ ıultiplied by an indeterminate $0/0$. Similarly, the same Lorentz transform carried out on $B^{(0)}$ produces $B^{(0)}$ if we regard it as an invariant, and $\boldsymbol{B}^{(3)}$ is transformed to $\boldsymbol{B}^{(3)}$ herefore this checks that the B cyclics are invariant if $B^{(0)}$ is invariant. If however, $\hbar\omega$ ·ansforms as frequency rather than energy, the B cyclics are covariant rather than ıvariant. The problem is that relativistic quantum theory is paradoxical when it comes to ansforming $\hbar\omega$, as described in a recent reference [6]. This paradox carries through into ıe Pauli-Lubanski four-vector [1—12,19—21] ($B^{(0)}$, $\boldsymbol{B}^{(3)}$), and whether or not this is ıvariant depends on whether or not $B^{(0)}$ is invariant.

We conclude that the B cyclic theorem is a fundamentally new property of field ıeory which is determined by topology. The individual components of the Theorem are ı general determined by an equation of motion, usually the Maxwell equation. The latter ıtroduces the Lorentz invariant phase into the B cyclic theorem.

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₹eferences

l] M. W. Evans, *Physica B* **182**, 227, 237 (1992).

2] M. W. Evans and S. Kielich eds., *Modern Nonlinear Optics*, Vols. 85(1), 85(2), and 85(3) of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, eds., 3rd printing (Wiley, New York, 1997).

3] M. W. Evans, *The Photon's Magnetic Field* (World Scientific, SIngapore, 1992); A. A. Hasanein and M. W. Evans, *The Photomagneton in Quantum Field Theory* (World Scientific, Singapore, 1994).

4] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Volume One: The Field* $\boldsymbol{B}^{(3)}$ (Kluwer, Dordrecht, 1994); M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Volume Two: Non-Abelian Electrodynamics* (Kluwer, Dordrecht, 1995).

5] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Volume Three: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer, Dordrecht, 1996); M. W. Evans, J.-P. Vigier, S. Roy, and G. Hunter, eds., *The Enigmatic Photon, Volume Four: New Directions* (Kluwer, Dordrecht, 1997), with contributed review articles from leading specialists; M. W. Evans, *Found. Phys.* **24**, 1671 (1994); ibid., *Physica* **214A**, 605 (1995); L. D. Barron, *Physica B* **190**, 307 (1993); M. W. Evans, ibid., p. 310; A. D. Buckingham and L. Parlett, *Science* **264**, 1748 (1994); A. D. Buckingham, *Science* **266**, 665 (1994); A. Lakhtakia, *Found. Phys. Lett.* **8**, 183 (1995); M. W. Evans, ibid., **8**, 187 (1995); A. Lakhtakia, *Physica B* **191**, 362 (1993); D. M. Grimes, ibid., p. 367; M. W. Evans, *Found. Phys. Lett.*, **8**, 563 (1995); S. J. van Enk, ibid., **9**, 183 (1996); M. W. Evans, ibid., p. 191; G. L. J. A. Rikken, *Opt. Lett.* **20** 846 (1995); M. W. Evans, *Found. Phys. Lett.* **9** 61 (1996); E. Comay, *Chem. Phys. Lett.* **261**, 601 (1996); M. W. Evans and S. Jeffers, *Found. Phys. Lett.*, **9**, 587 (1996); E. Comay, *Found. Phys. Lett.*, in press (1997); M. W. Evans, ibid., in press (1997); V. V. Dvoeglazov, ibid., in press (1997); A. E. Chubykalo, R. Smirnov-Rueda and M. W. Evans, ibid., in press (1997).

[8] A. E. Chubykalo and R. Smirnov-Rueda, *Phys. Rev. E*, **53**, 5373 (1996); ibid., *Int. J. Mod. Phys.* in press (rev.); review in Ref. 6, Vol. 4 of *The Enigmatic Photon*; monograph, World Scientific, in prep.

[9] V. V. Dvoeglazov, *Int. J. Theor. Phys.* **34**, 2467 (1995); *Rev. Mex. Fis.* **41**, 159 (1995); *Helv. Phys. Acta*, in press; *Nuovo Cim. A*, **108**, 1467 (1995); review in Ref. 6, Vol. 4 of *The Enigmatic Photon*.

[10] H. A. Múnera, D. Buritica, O. Guzmán and J. I. Vallejo, *Rev. Colomb. Fis.* **27**, 215 (1995); H. A. Múnera and O. Guzmán, *Found. Phys. Lett.*, in press (1997); ibid., part B, submitted, *Magnetic Potentials, Longitudinal Currents and Magnetic Properties of the Vacuum, All Implicit in Maxwell's Equations*; ibid., *Phys. Rev. Lett.*, submitted, *A Symmetric Formulation of Maxwell's Equations*.

[11] B. Lehnert, *Phys. Scripta* **53**, 204 (1996); *Optik* **99**, 113 (1995); *Spec. Sci. Tech.* **17**, 259, 267 (1994); ibid., a review in Ref. 6, Vol. 4 of *The Enigmatic Photon.*; B. Lehnert and S. Roy, a monograph in prep. for World Scientific.

[12] M. Mészaros, P. Molnár, T. Borbely and Z. G. Esztegar, in Ref. 6, Vol. 4 of *The Enigmatic Photon*, two review articles.

[13] P. S. Pershan, *Phys. Rev.* **130**, 919 (1963); J. P. van der Ziel, P. S. Pershan, and L. D. Malmstrom, *Phys. Rev. Lett.* **15**, 190 (1965); *Phys. Rev.* **143**, 574 (1966).

[14] M. W. Evans, *J. Phys. Chem.* **95**, 2256 (1991); W. S. Warren, S. Mayr, D. Goswami, and A. P. West, Jr., *Science* **255**, 1683 (1992); ibid., **259**, 836 (1993); R. A. Harris and I. Tinoco, Jr., *Science* **259**, 835 (1993); ibid., *J. Chem. Phys.* **101**, 9289 (1994); M. W. Evans, in Ref. 6, Chaps. 1 and 2 of Vol. 3 of *The Enigmatic Photon*.

[15] S. Jeffers, M. Novikov, and G. Hathaway, in S. Jeffers, S. Roy, J.-P. Vigier, and G. Hunter, eds., *The Present Status of the Quantum Theory of Light*, pp. 127—139, (Kluwer, Dordrecht, 1997); S. Jeffers, a review in Ref. 6, Vol. 3 of "*The Enigmatic Photon*; R. Zawodny, a review in Vol. 85(1) of Ref. 2; S. Wozniak, M. W. Evans and G. Wagniere, *Mol. Phys.* **75**, 81, 99 (1992).

[16] J. Deschamps, M. Fitaire, and M. Lagoutte, *Phys. Rev. Lett.* **25**, 1330 (1970); ibid., *Rev. Appl. Phys.* 7, 155 (1972).

[17] T. W. Barrett, H. Wohltjen and A. Snow, *Nature* **301**, 694 (1983).

[18] P. W. Atkins, *Molecular Quantum Mechanics*, 2nd edn., (Oxford University Press, Oxford, 1983).

[19] E. P. Wigner, *Ann. Math.* **140**, 149 (1939).

[20] A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (Macmillan, New York, 1964).
[21] L. H. Ryder, *Quantum Field Theory*, 2nd edn., (Cambridge University Press, Cambridge, 1987).

The B Cyclic Theorem for Multipole Radiation

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The B Cyclic Theorem is developed for the multipole expansion of plane wave radiation. It is shown that the $\boldsymbol{B}^{(3)}$ field is irrotational, divergentless and fundamental for each multipole component.

Key words: B cyclic theorem, multipole radiation.

1. Introduction

The original B cyclic theorem [1—5] was developed using plane wave radiation and interlinks the transverse and longitudinal components of electromagnetic radiation. In this communication we develop the theorem straightforwardly for multipole radiation and demonstrate that for each multipole component (l) the $\boldsymbol{B}^{(3)}$ field is divergentless and irrotational.

2. Multipole Expansion of the Plane Wave

The magnetic components of the plane wave are defined, using Silver's notation [6] as

$$\left.\begin{aligned} \boldsymbol{B}_1 &:= B^{(0)} e^{i(kz-\omega t)} \boldsymbol{e}_1 , \\ \boldsymbol{B}_{-1} &:= B^{(0)} e^{-i(kz-\omega t)} \boldsymbol{e}_{-1} , \\ \boldsymbol{B}_0 &:= B^{(0)} \boldsymbol{e}_0 . \end{aligned}\right\} \tag{1}$$

where the basis vectors in spherical representation are related by

$$\boldsymbol{e}_{-1} \times \boldsymbol{e}_1 = -i\boldsymbol{e}_0 , \tag{2}$$

in cyclic permutation.

The electromagnetic phase is

$$\phi := kz - \omega t , \tag{3}$$

/here k is the magnitude of the wavevector at point z and ω the angular frequency at ıstant t. The B cyclic theorem [1—5] in this notation is therefore

$$\boldsymbol{B}_{-1} \times \boldsymbol{B}_{1} = -iB^{(0)}\boldsymbol{B}_{0}, \tag{4}$$

/here the space and time independent magnitude $B^{(0)}$ is defined by [1-5],

$$I := \frac{c}{\mu_0} B^{(0)2}. \tag{5}$$

Iere I is the total beam power density in watts per square meter, c the conventional peed of light, and μ_0 the vacuum permeability.

In order to develop Eq. (4) for multipole radiation, we use Silver's expressions 17.1) and (29.1),

$$e^{ikz} = \sum_l i^l (2l+1) j_l(kz) P_l(\cos\theta) \tag{6a}$$

$$\boldsymbol{e}_M = \frac{1}{Y_0^1} \sum_{L=|l-1|}^{l+1} \langle l10M \mid l1LM \rangle \boldsymbol{Y}^L_{Ml1}. \tag{6b}$$

Iere l is the l'th multipole moment, j_1 the l'th modified Bessel function, P_1 the l'th egendre polynomial. The basis vector $\boldsymbol{e}_M (M = -1, 0, +1)$ is expanded in terms of the ilebsch-Gordan coefficients $\langle l10M \mid l1LM \rangle$ and the vector spherical harmonics $\boldsymbol{Y}^L_{Ml1}$, nd normalized with the scalar spherical harmonic $\boldsymbol{Y}^L_{Ml1}$.

In deriving Eq. (4) we have used on the left hand side the conjugate product of hase factors,

$$e^{i\phi} e^{-i\phi} = 1. \tag{7}$$

Jsing Eqs. (6a) and (7) it is seen that the product is unity if and only if we sum over all nultipoles with $l \to \infty$ in Eq. (6). In all other cases, the B cyclic theorem is

$$\boldsymbol{B}_{-1} \times \boldsymbol{B}_{1} = -ixB^{(0)}\boldsymbol{B}_{0}, \tag{8}$$

vhere x is different from unity. It is given in Table One for the first four multipoles.

Table 1. Factor x for Various l

Multipole	x
Dipole (l = 1)	$9j_1^2P_1^2$
Quadrupole (l = 2)	$25j_2^2P_2^2$
Octopole (l = 3)	$49j_3^2P_3^2$
Hexadecapole (l = 4)	$81j_4^2P_4^2$

In Table 1, [6],

$$P_l(\cos\theta) = (2\pi(2l+1))^{1/2} Y_0^l(\theta), \tag{9}$$

$$j_l(kr) = \left(-\frac{r}{k}\right)^l \left(\frac{1}{r}\frac{d}{dr}\right) j_0(kr). \tag{10}$$

It is important to note that $\boldsymbol{B}_0$ in Eq. (8) is the same as $\boldsymbol{B}_0$ in Eq. (4), i.e. phaseless, irrotational and divergentless. The factor x arises purely from the truncation of the infinite series (6a) to individual multipole components. It is incorrect and arbitrary to claim that [7]

$$\boldsymbol{B}_0 :=? \; (x^{1/2}B^{(0)})\boldsymbol{e}_0, \tag{11}$$

because this results in a finite divergence for $\boldsymbol{B}_0$. It is also incorrect to claim [8,9] that $\boldsymbol{B}_0$ has a non-zero curl. As discussed by Silver [6], the $\boldsymbol{e}_M$ vectors are polarization vectors for the electromagnetic wave, but are also *spin angular momentum eigenfunctions*. Tautologically therefore, Eq. (4) is a spin angular momentum equation for the photon, with $M = -1, 0, 1$. The photon wavefunction therefore has components $e^{ikz}\boldsymbol{e}_1$, $e^{-ikz}\boldsymbol{e}_{-1}$, and $\boldsymbol{e}_0$. The observables in this theory are therefore energy and $\boldsymbol{B}_0$ [6].

The complete vector fields $\boldsymbol{B}_1$, $\boldsymbol{B}_{-1}$ and $\boldsymbol{B}_0$ are described in terms of the vector spherical harmonics [6] and the B cyclic theorem indicates the existence of an intrinsic magnetic field, $\boldsymbol{B}_0$, which is described by the transformation of the frame under rotation and by the existence of the elementary charge e. As is well known in classical angular momentum theory [6] only the $\boldsymbol{B}_0$ component remains sharply defined under rotation.

The components B_1 and B_{-1} are defined only within an arbitrary phase factor. Within $\hbar$, his is the quantum theory of angular momentum [6].

Discussion

The B_0 field is observed empirically in the inverse Faraday effect [10] and acts on natter through its definition (Eq. (4)), i.e., through $B_1 \times B_{-1}$. In general the interaction s relativistic [1—5], giving rise to useful new effects in ESR and NMR [3]. Recent laims that B_0 is unobservable [11,12] are counter-indicated experimentally by several uccessful verifications [1—4, 13, 14] of the original inverse Faraday effect experiment 10]. Arp [15] has demonstrated empirically that there exist in nature giant, inter-galactic, nagnetic fields, which may originate [2] in a seed field such as the B_0 field of photons. Without some kind of a seed field, the dynamo mechanism cannot be initiated and the seed ield is necessary for the consequent evolution of dynamo generation due perhaps to ermions. Recently, it has been claimed [16] that the universe may exhibit macroscopic nisotropy, and this claim is associated with a novel Faraday effect that may be due to a nagnetic field. This idea is however very recent and has been questioned in the literature 17]. Nevertheless, it is interesting to speculate on the possibility that a seed field such as he intrinsic B_0 may be the cause of these mysterious inter-galactic magnetic phenomena. This speculation is developed in Vol. 3, Chap. 8 of Ref. 2, which is the work of Roy. The nverse Faraday effect is a well known non-linear optical effect which is reproducible and epeatable [1].

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References

[1] M. W. Evans and S. Kielich, eds., *Modern Nonlinear Optics*, Vol. 85 of *Advances in Chemical Physics*, I. Prigogine and S.A. Rice, eds., (Wiley, New York, 1993, paperback edition 1997).

[2] M. W. Evans, J.-P. Vigier, S. Roy, and G. Hunter, eds., *The Enigmatic Photon, Volume 4: New Directions* (Kluwer Academic, Dordrecht, 1997), a collection of contributed articles, in press; M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Volume 1: The Field* $B^{(3)}$ (Kluwer Academic, Dordrecht, 1994); M. W. Evans and

J.-P. Vigier, *The Enigmatic Photon, Volume 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995); M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Volume 3: Theory and Practice of the* $B^{(3)}$ *Field* (Kluwer, Dordrecht, 1996).

[3] M. W. Evans, *Physica B* **182**, 227, 237 (1992); *Physica A* **214**, 605 (1995); *Found. Phys.* **24**, 1671 (1994).

[4] V. V. Dvoeglazov, *Weinberg Formalism and New Looks at the Electromagnetic Theory*, a review in Ref. 2.

[5] B. Lehnert and S. Roy, *Electromagnetic Theory with Space Charges in Vacuo and Nonzero Rest Mass of the Photon* (World Scientific, Singapore, 1997, in press).

[6] B. L. Silver, *Irreducible Tensor Theory* (Academic, New York, 1976).

[7] E. Comay, *Physica B* **222**, 150 (1996).

[8] E. Comay, *Chem. Phys. Lett.* **261**, 601 (1996).

[9] M. W. Evans and S. Jeffers, *Found. Phys. Lett.* **9**, 587 (1996).

[10] J. P. van der Ziel, P. S. Pershan, and L. D. Malmstrom, *Phys. Rev. Lett.* **15**, 190 (1965); *Phys. Rev.* **143**, 574 (1966).

[11] G. L. J. A. Rikken, *Opt. Lett.* **20**, 846 (1995).

[12] M. W. Evans, *Found. Phys. Lett.* **9**, 61 (1996).

[13] G. H. Wagniére, *Linear and Nonlinear Optical Properties of Molecules* (VCH, Basel, 1993).

[14] T. W. Barrett, H. Wohltjen, and A. Snow, *Nature* **301**, 694 (1983).

[15] H. Arp, *Phys. Lett.* **129**, 135 (1988).

[16] B. Nodland and J. Ralston, *Phys. Rev. Lett.* **78**, 3043 (1997).

[17] S. M. Carroll, LANL 9704263, (1997).

Zero Field Helicity of the U(1) Gauge

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In U(1) gauge theory applied to the electromagnetic field the relativistic helicity vanishes as the direct result of using the covariant derivative. This implies that U(1) gauge theory is incompatible with the correspondence principle, and O(3) gauge theory or higher is needed to define the electromagnetic field helicity properly. This can be done if and only if the $\boldsymbol{B}^{(3)}$ field is identically non-zero in vacuo.

Key words: $\boldsymbol{B}^{(3)}$ field; O(3) gauge theory; relativistic field helicity.

1. Introduction

It is well known that general gauge theory [1] defines the field through the commutator of covariant derivatives in any gauge group, including U(1). If covariant derivatives are not used, there is no gauge field. It is shown in this Letter that U(1) gauge theory leads to a zero classical field helicity in vacuo if we are to recover the ordinary U(1) Maxwell equations in vacuo. This result means that U(1) gauge theory is inconsistent with the correspondence principle, because photon helicity is non-zero in relativistic quantum field theory [2—5]. The only way to obtain non-zero field helicity is to use a non-Abelian gauge group such as O(3), which automatically defines the $\boldsymbol{B}^{(3)}$ field in vacuo through the commutator of O(3) covariant derivatives. The internal space of the O(3) gauge group is ((1),(2),(3)). Therefore the $\boldsymbol{B}^{(3)}$ field is the fundamental spin, the classical counterpart of Wigner's photon spin [6]. In order to save the correspondence principle, the $\boldsymbol{B}^{(3)}$ field in vacuo must be identically non-zero and the gauge group of the electrodynamical sector must be rigorously non-Abelian. This conclusion has many fundamental consequences for field theory, including unified field theory.

2. The Classical U(1) Field Helicity

In general gauge theory the field for any gauge group is defined through the commutator of covariant derivatives, giving the result [1],

$$G_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right], \tag{1}$$

where the commutator is non-zero in general. Here A_μ is the connection, or potential, and is defined in general through the gauge group symmetry. The field tensor $G_{\mu\nu}$ is covariant for all gauge groups, and so the general gauge theory is compatible with special relativity for all gauge group symmetries. In the general theory, the homogeneous and inhomogeneous Maxwell equations in vacuo are generalized respectively to

$$D^\nu \tilde{G}_{\mu\nu} = 0, \qquad D^\nu G_{\mu\nu} = 0, \tag{2}$$

where D^ν denotes the covariant derivative pertinent to the gauge group symmetry being used and $\tilde{G}_{\mu\nu}$ is the dual of $G_{\mu\nu}$. It is shown in this section that the electromagnetic helicity vanishes if we use U(1) gauge theory for the field. This is an irretrievable failure of the U(1) gauge theory of electromagnetism. In the next section it is shown that the O(3) gauge theory produces a satisfactorily non-zero field helicity, but only if $\boldsymbol{B}^{(3)}$ is non-zero identically.

In the U(1) gauge theory the commutator in Eq. (1) vanishes, because the U(1) only has one structure constant, a scalar, and the internal space is also a scalar. However, the covariant derivative of U(1) is

$$D^\nu = \partial^\nu + ieA^\nu, \tag{3}$$

where is the identically non-zero elementary charge [1]. Therefore Eqs. (2) reduce to:

$$(\partial^\nu + ieA^\nu)\tilde{F}_{\mu\nu} = 0, \tag{4a}$$

$$(\partial^\nu + ieA^\nu)F_{\mu\nu} = 0. \tag{4b}$$

These become the customary Maxwell equations if and only if

$$A^\nu \tilde{F}_{\mu\nu} = 0, \tag{5a}$$

$$A^\nu F_{\mu\nu} = 0, \tag{5b}$$

or in vector notation,

$$\left.\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{B} &= 0, & \boldsymbol{A}\times\boldsymbol{E} &= \boldsymbol{0}, \\ \boldsymbol{A}\cdot\boldsymbol{E} &= 0, & \boldsymbol{A}\times\boldsymbol{B} &= \boldsymbol{0}, \end{aligned}\right\} \tag{6}$$

where $\boldsymbol{E}$ is the electric field strength and $\boldsymbol{B}$ the magnetic flux density in vacuo. In the U(1) gauge there are only transverse fields in vacuo, usually represented by the well known plane waves,

$$\left.\begin{aligned} \boldsymbol{A} &= \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}, \\ \boldsymbol{B} &= \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i\phi}, \\ \boldsymbol{E} &= \frac{E^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi}, \end{aligned}\right\} \tag{7}$$

where ϕ is the electromagnetic phase, and where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in X and Y, perpendicular to the propagation axis, Z. Using the implied relation,

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{8}$$

it is easily verified that Eqs. (7) are compatible with Eqs. (6). This result shows that plane waves in U(1) produce a *zero* field helicity, and this is the direct and unavoidable result of standard gauge theory [1].

This is an irretrievable failure of the U(1) gauge theory because the result breaks the correspondence principle. The helicity of the photon is well known to be non-zero. It is not a surprising result however, because without longitudinal structure, helicity is not defined. In other words, the U(1) gauge does not give the correct spin relations [7] for the classical electromagnetic field.

3. The Field Helicity in the O(3) Gauge

If we represent the O(3) gauge group by the internal indices ((1),(2),(3)) formed [8—12] from the unit vector definitions,

$$\left.\begin{aligned} \boldsymbol{e}^{(1)} &:= \frac{1}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j}) = \boldsymbol{e}^{(2)*}, \\ \boldsymbol{e}^{(3)} &:= \boldsymbol{k}, \end{aligned}\right\} \tag{9}$$

we find that the transverse field components are

$$G_{\mu\nu}^{(1)} = \partial_\mu A_\nu^{(1)} - \partial_\nu A_\mu^{(1)}, \qquad (\mu,\nu) \neq (1,2), \tag{10a}$$

and

$$G_{\mu\nu}^{(2)} = \partial_\mu A_\nu^{(2)} - \partial_\nu A_\mu^{(2)}, \qquad (\mu,\nu) \neq (1,2), \tag{10b}$$

but that there is in addition a well defined $\boldsymbol{B}^{(3)}$ field,

$$B^{(3)*} := G_{\mu\nu}^{(3)*} := -ig(A_\mu^{(1)} A_\nu^{(2)} - A_\mu^{(2)} A_\nu^{(1)}), \tag{10c}$$

for $(\mu,\nu) = (1,2)$. The $\boldsymbol{B}^{(3)}$ field [8—12] is therefore a direct consequence of the definition of the gauge field. It can be represented as a vector cross product,

$$\boldsymbol{B}^{(3)*} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{11}$$

and is part of the covariant O(3) gauge field. Therefore $\boldsymbol{B}^{(3)}$ is compatible with special relativity. From Eq. (2), the O(3) Maxwell equations become

$$\left.\begin{aligned} \partial^\nu \tilde{G}_{\mu\nu}^{(1)} &= 0, \\ \partial^\nu \tilde{G}_{\mu\nu}^{(2)} &= 0, \\ \partial^\nu \tilde{G}_{\mu\nu}^{(3)} &= 0, \end{aligned}\right\} \tag{12a}$$

$$\left.\begin{aligned} \partial^\nu G_{\mu\nu}^{(1)} &= 0, \\ \partial^\nu G_{\mu\nu}^{(2)} &= 0, \\ \partial^\nu G_{\mu\nu}^{(3)} &= 0, \end{aligned}\right\} \tag{12b}$$

and happen to be the same in form as the ordinary Maxwell equations, but with superimposed indices (1),(2),(3). In vector form, therefore, the Maxwell equations for $\boldsymbol{B}^{(3)}$ are,

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \boldsymbol{0}, \qquad \nabla \times \boldsymbol{B}^{(3)} = \boldsymbol{0}, \tag{13}$$

and the helicity as defined in Eq. (6) is no longer necessarily zero. The scalar helicity in the O(3) gauge can be shown to be [12] proportional to

$$h := \boldsymbol{A}^{(3)} \cdot \boldsymbol{B}^{(3)}, \tag{14}$$

and is non-zero therefore because the polar $\boldsymbol{A}^{(3)}$ and the axial $\boldsymbol{B}^{(3)}$ are non-zero. The complete relativistic field helicity in O(3) becomes the Pauli-Lubanski vector,

$$h^\mu := A^{(0)}(B^{(0)}, \boldsymbol{B}^{(3)}), \tag{15}$$

which transforms under parity as a pseudo four-vector. It is the precise classical equivalent of the Pauli-Lubanski operator in the quantum field theory developed originally by Wigner [6]. The $\boldsymbol{B}^{(3)}$ field is therefore well defined, compatible with Maxwell's equations, and covariant. It is also a physical observable in magneto-optics [8—12]. Self-consistently, the Maxwell equations (12) can be obtained from the general equations (2) if and only if $\boldsymbol{A}^{(3)}$ is mutually perpendicular to $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ and $\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*}$.

4. Discussion

In general gauge theory, the homogeneous Maxwell equations in vacuo become a Jacobi identity of the underlying gauge group, as first shown by Feynman [13]. This means that unified field theory, for example, can be pursued within gauge theory. The reason why $\boldsymbol{B}^{(3)}$ is critical to this development is that without it, the U(1) gauge is used for the electromagnetic sector, with consequent problems as discussed already. It is significant that the theory of the strong field (chromodynamics) relies as is well known on a non-Abelian SU(3) gauge group in which the interaction between quarks is mediated by gluons with color taking the role of charge. The SU(3) group's basic structure is the same as Eq. (1), but with nine structure constants [1]. Recami *et al.* [14] have already indicated that an SU(3) structure is present in the Majorana form of Maxwell's equations. Formally, the $\boldsymbol{B}^{(3)}$ field in SU(3) takes the form,

$$B^{(3)} = \frac{g'}{2}(\alpha_1^4 \alpha_2^5 - \alpha_1^6 \alpha_2^7), \tag{16}$$

where the α coefficients are appropriate connection coefficients in the group space. Therefore it is not inappropriate to consider the development of $\boldsymbol{B}^{(3)}$ electrodynamics within an overall SU(3) structure with the intent of understanding photon structure in terms of gluons; charge in terms of color structure; and electrons in terms of quarks.

'his process can be extended, theoretically, to flavor dynamics (electroweak theory) and ravitational theory within a non-Abelian, Riemannian framework. In this way, unification f all four fields may be achieved within non-Abelian gauge theory. The $B^{(3)}$ field allows he electrodynamical sector to become non-Abelian, and consistently defines some ındamental quantities such as the field spin [6]. Non-Abelian electrodynamics is eveloped in detail elsewhere [15].

Equation (1) shows that the fundamental origin of $B^{(3)}$ resides in curvilinear opology, i.e., in the commutator of covariant derivatives within a non-Abelian gauge roup. This may be O(3) or a higher non-Abelian group (Poincaré group, Einstein group, U(3), SU(4), etc.). We have shown that it is possible to devise a non-Abelian structure or the electromagnetic sector, and to write the free space Maxwell equations as in (12). 'hese are non-Abelian equations but for each index are the original Maxwell equations. 'hus $B^{(3)}$ can be built up [8--12] from plane wave solutions of the vacuum Maxwell quations for indices (1) and (2). The fact that the gauge structure of electrodynamics ecomes non-Abelian appears to be of key importance for the unification of fields as iscussed.

It is concluded that the empirically verified existence of the conjugate product $A^{(1)} \times A^{(2)}$ in magneto-optics [8—12] provides conclusive evidence for the fact that the auge group of electrodynamics is O(3) or higher. The existence of $A^{(1)} \times A^{(2)}$ is ıcompatible with a U(1) gauge.

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eferences

1] L. H. Ryder, *Quantum Field Theory*, 2nd. edn., (Cambridge University Press, Cambridge, 1987).

2] V. V. Dvoeglazov, *Int. J. Mod. Phys.* **34**, 2467 (1995); *Rev. Mex. Fis.* **4**, 159 (1995); *Nuovo Cim. A* **108**, 1467 (1995); *Helv. Phys. Acta.* in press.

3] M. W. Evans and S. Kielich, eds., *Modern Nonlinear Optics*, Vols. 85(1), 85(2), 85(3) of *Advances in Chemical Physics*, I. Prigogine and S.A. Rice, eds., paperback edn., (Wiley, New York, 1997).

4] L. H. Ryder, *Elementary Particles and Symmetries* (Gordon and Breach, London, 1986).

5] P. W. Atkins, *Molecular Quantum Mechanics*, 2nd edn., (Oxford University Press, Oxford, 1983).

[6] E. P. Wigner, *Ann. Math.* **140**, 149 (1939).

[7] V. V. Dvoeglazov, a review in M. W. Evans, J.-P. Vigier, S. Roy, and G. Hunter, eds., *The Enigmatic Photon, Volume 4: New Directions* (Kluwer Academic, Dordrecht, 1997) in press.

[8] M. W. Evans, *Physica B* **182**, 227, 237 (1992); *Found. Phys.* **24**, 1671 (1994); *Physica A* **214**, 605 (1995).

[9] L. D. Barron, *Physica B* **190**, 307 (1993); M. W. Evans, ibid., p. 310; A. D. Buckingham, and L. Parlett, *Science* **264**, 1748 (1994); A. D. Buckingham, *Science* **266**, 665 (1994); A. Lakhtakia, *Physica B* **191**, 362 (1993); D. M. Grimes, *Physica B* **191**, 367 (1993); M. W. Evans, *Found. Phys. Lett.* **8**, 563 (1995); A. Lakhtakia, *Found. Phys. Lett.* **8**, 183 (1995); M. W. Evans, *Found. Phys. Lett.* **8**, 187 (1995); S. J. van Enk, *Found. Phys. Lett.* **9**, 183 (1996); M. W. Evans, ibid., 191; G. L. J. A. Rikken, *Opt. Lett.* **20**, 846 (1995); M. W. Evans, *Found. Phys. Lett.* **9**, 61 (1996); E. Comay, *Chem. Phys. Lett.* **261**, 601 (1996); M. W. Evans and S. Jeffers, *Found. Phys. Lett.* **9**, 587 (1996); E. Comay, *Found. Phys. Lett.*, in press (1997); M. W. Evans, *Found. Phys. Lett.*, in press (1997); V. V. Dvoeglazov, *Found. Phys. Lett.*, in press (1997).

[10] M. W. Evans, *The Photon's Magnetic Field* (World Scientific, Singapore, 1992); M. W. Evans and A. A. Hasanein, *The Photomagneton in Quantum Field Theory* (World Scientific, Singapore, 1994).

[11] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Volume 1: The Field* $B^{(3)}$ (Kluwer Academic, Dordrecht, 1994); M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Volume 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995); M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Volume 3: Theory and Practice of the* $B^{(3)}$ *Field* (Kluwer, Dordrecht, 1996); M. W. Evans, J.-P. Vigier, S. Roy, and G. Hunter, eds., *The Enigmatic Photon, Volume 4: New Directions* (Kluwer Academic, Dordrecht, 1997), in press.

[12] A. E. Chubykalo and R. Smirnov-Rueda, *Phys. Rev. E* **53**, 5373 (1996); ibid., *Int. J. Mod. Phys.*, in press; *Mod. Phys. Lett.*, in press; A. E. Chubykalo, R. Smirnov-Rueda, and M. W. Evans, *Found. Phys. Lett.*, in press (1997); A. E. Chubykalo and R. Smirnov-Rueda, a review in Ref. (7); H. A. Múnera, D. Buritica, O. Guzmán, and J. I. Vallejo, *Rev. Colomb. Fis.* **27**, 215 (1995); H. A. Múnera and O. Guzmán *Found. Phys. Lett.*, in press (1997); ibid., submitted (1997); ibid., *Mod. Phys. Lett.*, submitted (1997); B. Lehnert, review in Ref. 7; B. Lehnert and S. Roy, monograph on non-Abelian electrodynamics (World Scientific, Singapore, 1998), in preparation; C. R. Keys and M. W. Evans, eds., *Apeiron* **4**, no. 2—3, (1997), special double issue on the $B^{(3)}$ Field; M. Mészáros, P. Molnár and T. Borbely, in Ref. 7.

[13] R. P. Feynman, *Gauge Theories*, in R. Balian and C. H. Llewelyn Smith, eds., Les Houches Session XXIX, 1976, *Weak and Electromagnetic Interactions at High Energy* (North Holland, Amsterdam, 1977).

[14] E. Recami and M. W. Evans, in Ref. 7.

15] For example, the papers, reviews and volumes cited in Refs. 8 to 14.

Link between Yang-Mills Theory of Electrodynamics and Relativistic Helicity

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Using a three dimensional gauge for electromagnetism in the Yang-Mills formalism produces a non-Abelian gauge theory of the $\boldsymbol{B}^{(3)}$ field. The structure of this theory is used to represent the fundamental relativistic helicity of vacuum electromagnetism in terms of the $\boldsymbol{B}^{(3)}$ component.

1. Introduction

Recently, several theories have arrived at the conclusion that the structure of electromagnetism in the vacuum is non-Abelian [1—10]. In this Letter it is shown (Section 2) that a gauge representation of the electromagnetic field can be built up using the Yang-Mills formalism in an isospace which corresponds with the physical space ((1), (2), (3)) — the complex circular representation [1—5]. In Sec. 3 this gauge theory is used as a model for the relativistic helicity [11—15] of the electromagnetic field by extending ((1), (2), (3)) to the four dimensional space-time of the Poincaré group. It is shown that the novel $\boldsymbol{B}^{(3)}$ component recently identified in the literature [1—10] must be non-zero if the helicity of the electromagnetic field is non-zero as measured experimentally in the well known Lebedev and Beth experiments of optics.

2. The Yang-Mills Theory as a non-Abelian Electromagnetic Gauge.

The Yang-Mills gauge theory [1—5] uses an isospace which is superimposed on the four dimensional space of the usual field tensor. This device leads to useful gauge theories [16] such as non-Abelian quantum electrodynamics and the 't Hooft-Polyakov theory of the magnetic monopole. Until recently [1—10] however, the gauge theory of electrodynamics was thought to be based on a U(1) (Abelian) group symmetry, both classically and quantum mechanically. However, the interpretation of magneto optics in terms of the longitudinal magnetic component, $\boldsymbol{B}^{(3)}$, of vacuum electromagnetism led to a non-Abelian cyclic relation between field components,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \text{ et cyclicum} \qquad \textbf{(1)}$$

which can be referred to as the B cyclic theorem [1—10]. Here $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ is the usual transverse magnetic component, for example a plane wave. Further work has shown [17,18] that the relativistic helicity of the field can be represented in terms of the four-vector $(B^{(0)}, \boldsymbol{B}^{(3)})$, where $B^{(0)}$ is the magnitude of $\boldsymbol{B}^{(3)}$. Therefore we lose the field completely if $\boldsymbol{B}^{(3)}$ is non zero. Refs. 1—10, furthermore, give many arguments for the existence of a non-zero $\boldsymbol{B}^{(3)}$. One of the major implications of these findings is that the gauge theory of classical and quantum electrodynamics is non Abelian. The Yang-Mills theory is the obvious framework on which to develop the non Abelian theory, and this work is in fact already well developed in the subject of non Abelian quantum electrodynamics [16]. This intricate development took place however without ever realizing the existence of the fundamental field of helicity, $\boldsymbol{B}^{(3)}$. The existence of the latter is signaled through conjugate products such as $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}$ of Eq. (1), or equivalently $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, the conjugate product of vector potentials.

In the well developed Yang-Mills theory [16] the existence of

$$\boldsymbol{B}^{(3)*} = -i\frac{\kappa}{A^{(0)}}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{2}$$

means that the isospace of the non-Abelian Yang-Mills equations is ((1), (2), (3)). In Eq. (2), κ is the scalar magnitude of the wavevector and $A^{(0)}$ the magnitude of $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$. The structure of the Yang-Mills equations then demands $\left(g := \kappa/A^{(0)}\right)$

$$\begin{aligned} \partial^\nu \boldsymbol{G}_{\mu\nu}^{(3)*} &= ig\boldsymbol{A}^{\nu(1)} \times \boldsymbol{G}_{\mu\nu}^{(2)}, \\ \partial^\nu \boldsymbol{G}_{\mu\nu}^{(1)*} &= ig\boldsymbol{A}^{\nu(2)} \times \boldsymbol{G}_{\mu\nu}^{(3)}, \\ \partial^\nu \boldsymbol{G}_{\mu\nu}^{(2)*} &= ig\boldsymbol{A}^{\nu(3)} \times \boldsymbol{G}_{\mu\nu}^{(1)}, \end{aligned} \tag{3}$$

where the vector symbols must be interpreted to mean that the components of the field tensor can be superscripted (1), (2) or (3). If we align (3) with Z of the laboratory frame, then by this definition, the only non-zero component of $\boldsymbol{B}^{(3)}$ is the Z component, and $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ are mutually perpendicular to $\boldsymbol{B}^{(3)}$. This geometry gives the B cyclics (1).

The non-Abelian structure (2) contains intrinsically the vector $\boldsymbol{A}^\nu$, and is intrinsically non linear. When there is no (3) component it reduces to the familiar Maxwellian structure,

$$\partial^\nu \boldsymbol{G}_{\mu\nu}^{(1)} = \partial^\nu \boldsymbol{G}_{\mu\nu}^{(2)*} = 0. \tag{4}$$

The Yang-Mills gauge theory is covariant by definition, and so are the B cyclics (1). As they stand, Eqs. (3) are linked non-linear differential equations which are classical in nature. There is no hint yet of the quantum theory.

The latter can be introduced, however, through the vacuum relation [1—5],

$$\hbar\kappa = eA^{(0)}, \tag{5}$$

which is consistent with the minimal prescription and with the definition of the fine structure constant. Therefore Eq. (5) is a self-consistent quantum hypothesis and correspondence principle which expresses the ratio $\kappa/A^{(0)}$ as $e/\hbar$. In the Yang-Mills theory, this is also the result of non-Abelian topology, because if g were to be zero, the structure of the theory becomes Abelian irrespective of the existence of cross products on the right hand side of Eqs. (3. The quantum theory is therefore rigorously non Abelian, but in the usual approach to quantum electrodynamics, the gauge symmetry used is U(1). The latter reduces Eqs. (3) to (4), irrespective of the constancy of $e/\hbar$, because in the U(1) gauge, the cross products on the right hand side vanish. Thus, Abelian quantum electrodynamics is based on the assertion that there exists nothing physical in the (3) (= Z) axis for vacuum electrodynamics.

If we enlarge the isospace in Eq. (3) to four dimensions, its gauge group becomes the Poincaré group. By making the isospace identical with that of space-time, the Yang-Mills theory becomes the relativistic theory of electromagnetic helicity [11—15] as demonstrated in Sec. 3. Using the quantum hypothesis,

$$\partial^\mu \to -\frac{i}{\hbar}p^\mu = -i\frac{e}{\hbar}A^\mu, \tag{6}$$

transforms Eqs. (3) into a cyclic structure similar to Eq. (1),

$$\begin{aligned} A^\nu \boldsymbol{G}_{\mu\nu}^{(3)*} &= \boldsymbol{G}_{\mu\nu}^{(2)} \times \boldsymbol{A}^{\nu(1)}, \\ A^\nu \boldsymbol{G}_{\mu\nu}^{(1)*} &= \boldsymbol{G}_{\mu\nu}^{(3)} \times \boldsymbol{A}^{\nu(2)}, \\ A^\nu \boldsymbol{G}_{\mu\nu}^{(2)*} &= \boldsymbol{G}_{\mu\nu}^{(1)} \times \boldsymbol{A}^{\nu(3)}, \end{aligned} \tag{7}$$

with

$$A^\nu = \frac{1}{3}\left(\boldsymbol{A}^{\nu(1)} \cdot \boldsymbol{A}^{\nu(1)*} + \boldsymbol{A}^{\nu(2)} \cdot \boldsymbol{A}^{\nu(2)*} + \boldsymbol{A}^{\nu(3)} \cdot \boldsymbol{A}^{\nu(3)*}\right)^{1/2}. \tag{8}$$

Equations (7) already has the outline of relativistic helicity theory [11—15] but still works in two spaces, physical space-time and the isospace ((1), (2), (3)).

3. Theory of Relativistic Helicity from Eq. (7)

The isospace in Eq. (7) is enlarged firstly from the three dimensional ((1), (2), (3)) to the four dimensions of space-time. The enlarged isospace is identified secondly with the physical space-time itself, so that the two spaces of the Yang-Mills theory become the same. The structure of Eq. (7) is obtained if and only if g is non-zero, and this puts a minimum value on $e/\hbar$ topologically. The procedure replaces the cyclic equations (7) with

$$A^0\tilde{G}_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\nu\rho}A^\sigma, \tag{9}$$

in which appears the vector dual $\tilde{G}_\mu$, representing relativistic helicity [5,6]. The cross products in the three dimensional isospace of Eqs. (7) are replaced by the cross product in four dimensions of Eq. (9). The latter is expressed using the four dimensional antisymmetric unit tensor $\epsilon_{\mu\nu\rho\sigma}$.

The Maxwell equations and gauge conditions of the usual electrodynamics are then obtained from [5,11]

$$\partial_\mu\tilde{G}^\mu = 0, \tag{10}$$

which is the orthogonality property of the Poincaré group [5]. Therefore Eq. (10) is consistent with the Maxwell equations and with the cyclic structure (1), suggesting strongly that the former are equivalent to a linear equation of conservation of angular momentum, whose commutator structure is non linear.

3. Discussion

This result has been obtained by identifying the isospace of the Yang-Mills theory with the four dimensional space of special relativity. More generally, these two mathematical spaces are not identical, giving rise to the possibility of super-symmetry and the superimposition of other field sectors on to the electromagnetic sector and vice versa. The general structure for such a theory is already available in general n dimensional gauge-field theory, which is automatically covariant and which is a local interaction theory of relativity. The correct description of classical relativistic helicity is obtained from the Yang-Mills theory by using a four dimensional isospace which is identified with physical

space-time. This is a classical procedure, but one which is consistent with the basic hypothesis of quantum mechanics, that there is a non-zero $e/\hbar = \kappa/A^{(0)}$. In relativistic helicity theory, covariant derivatives are worked out within the Poincaré group and are inherent in the structure of the gauge theory. The Maxwell equations in standard form, and the standard gauge relations, are given by Eq. (9), whose left hand side is experimentally observable but which vanishes if there is no $\boldsymbol{B}^{(3)}$. In general, therefore, the field matrix is a commutator of covariant derivatives worked out in the Poincaré group, and whose structure is determined in a well known way by n dimensional gauge theory. The latter gives the most rigorous form of $\boldsymbol{B}^{(3)}$ in classical and quantum electrodynamics.

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References

[1] M. W. Evans and S. Kielich, eds., *Modern Nonlinear Optics*, Vols. 85(1), 85(2), 85(3) of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, eds. (Wiley-Interscience, New York, 1993).

[2] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $\boldsymbol{B}^{(3)}$ (Kluwer Academic, Dordrecht, 1994).

[3] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).

[4] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).

[5] M. W. Evans, J.-P. Vigier, S. Roy, and G. Hunter, eds., *The Enigmatic Photon, Vol. 4: New Directions* (Kluwer, Dordrecht, in prep), a collection of contributed papers.

[6] M. W. Evans, *Physica A* **214**, 605 (1995).

[7] V. V. Dvoeglazov, *Hadronic J. Suppl.* **10**, 349 (1995); *Nuovo Cim.* **111B**, 483 (1996); *Int. J. Theor. Phys.* **34**, 2467 (1995); *Rev. Mexicana Fis.* **41(1)**, 159 (1995); *Nuovo Cim.* **108A,** 1467 (1995); *Int. J. Theor. Phys.* **35,** 115 (1996); and in Ref. 5, *The Weinberg Formalism and New Looks at Electromagnetic Theory.* a review.

[8] A. Chubykalo and R. Smirnov-Rueda, *Phys. Rev. E* **53**, 5373 (1996).

[9] N. Schnerb and L. P. Horwitz, *Phys. Rev. A* **48**, 4068 (1993).

[10] V. V. Dvoeglazov, M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 5: Unified Field Theory* (Kluwer, Dordrecht, planned).

[11] G. N. Afanasiev and Yu. P. Stepanofsky, *The Helicity of the Free Electromagnetic Field and its Physical Meaning* Dubna **E2-95-413** (1995); *Nuovo Cim.*, submitted for publication.
[12] H. Bacry, Helv. *Phys. Acta* **67**, 632 (1994).
[13] A. F. Ranada, *Eur. J. Phys.* **13**, 70 (1992); *J. Phys. A* **25**, 1621 (1992).
[14] H. K. Moffat, *Nature* **147**, 367 (1990).
[15] M. W. Evans, *Found. Phys.*, submitted for publication.
[16] L. H. Ryder, *Quantum Field Theory*, 2nd edn., paperback (Cambridge University Press, 1987).
[17] V. V. Dvoeglazov, Ref. 5.
[18] M. W. Evans and J.-P. Vigier, Ref. 2, Chap. 11.

Description of the Inverse Faraday Effect in Terms of Rotational Energy Transferred in a Photon-Electron Collision

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The description of the inverse Faraday effect is simplified to its essentials by calculating the rotational energy and angular momentum transferred from a photon to an electron in a collision. At visible frequencies, the angular momentum transferred is only about $10^{-5}\hbar$ of the photon. This process is proportional to the power density, I, of the radiation. For high intensity microwave pulses, however, nearly all $\hbar$ is transferred elastically to the electron in a process proportional to $I^{1/2}$ and to the Evans-Vigier field, $\boldsymbol{B}^{(3)}$.

Key Words: photon-electron collision, Evans-Vigier field.

1. Introduction

The inverse Faraday effect has been observed experimentally [1-3] and is the phenomenon of magnetization by circularly polarized electromagnetic radiation. A considerable amount of theoretical work [4—8] has been carried out on the effect, including a non-equilibrium statistical mechanical development by Talin *et al.* [4] which contains a simple description in terms of one electron in a circularly polarized electromagnetic field. This was based on the Hamilton-Jacobi equation as in the classic text by Landau and Lifshitz [9], and shows that the effect is in general relativistic in nature. In this Letter the inverse Faraday effect is reduced to its essence by expressing it in terms of the rotational energy and angular momentum transferred during a photon-electron collision for visible frequency radiation and microwave pulses of high intensity. It is shown in Sec. 2 that the angular momentum transferred in the non-relativistic limit (visible frequencies) is only about $10^{-5}\hbar$, whereas in the relativistic limit (high power density microwave pulses) almost the whole of the angular momentum, $\hbar$, of the photon is transferred to the electron. At visible frequencies the inverse Faraday effect is proportional to electromagnetic power density (I), whereas for intense microwave pulses it is proportional to the square root of the power density.

An experimental arrangement is suggested whereby the expected $I^{1/2}$ dependence can be measured, thus showing empirically the presence of the recently inferred $\boldsymbol{B}^{(3)}$ field [10—15] of vacuum electromagnetism.

. Transfer of Rotational Energy and Angular Momentum

The rotational energy acquired by the electron is described by Talin *et al.* [4] as

$$En = \omega |\boldsymbol{J}|, \tag{1}$$

here ω is the angular frequency of the electromagnetic radiation and where $\boldsymbol{J}$ is the ngular momentum acquired by the electron. In classical terms [10] the transferred rotaonal energy is given by the relativistic Hamilton-Jacobi equation,

$$En = \frac{e^2c^2}{\omega}\left(\frac{B^{(0)}}{\left(m_0^2\omega^2 + e^2B^{(0)2}\right)^{1/2}}\right) |\boldsymbol{B}^{(3)}|, \tag{2}$$

here $\boldsymbol{B}^{(3)}$ is the Evans-Vigier field [10—15], $B^{(0)}$ the scalar magnitude of the magnetic ux density, e the charge and m_0 the mass of the electron. The speed of light in vacuo ; c. The propagating source of the $\boldsymbol{B}^{(3)}$ field is as described in the following cyclically ymmetric relations between field components in vacuo,

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \text{ et cyclicum} \tag{3}$$

vhere $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ is the plane wave of magnetic flux density in tesla [16], and vhere $\boldsymbol{B}^{(2)}$ is its complex conjugate. Note that the result (2) is in S.I. units, not the Gaussian units used by Landau and Lifshitz [9] and Talin *et al.* [4]. It is clear from Eq. 2) that if there were no $\boldsymbol{B}^{(3)}$ field, there would be no inverse Faraday effect. Also, if here were no matter in the beam, i.e., no charge, e, there would be no effect, showing hat there is no Faraday induction due to $\boldsymbol{B}^{(3)}$ in the vacuum in the absence of matter 15]. This has been confirmed experimentally by Deschamps *et al.* [3], who showed the nverse Faraday effect in an electron plasma using an induction coil. When the plasma was emoved [3] the induction voltage disappeared. Symmetry and relativity [10] both forbid he existence of a phase free electric field propagating with the plane wave in vacuo, but llow the existence of $\boldsymbol{B}^{(3)}$ [10]. It is also clear from Eq. (2) that the inverse Faraday effect at any frequency proves the existence of $\boldsymbol{B}^{(3)}$. It is of great importance however o obtain empirical evidence for the $I^{1/2}$ dependence expected of magnetization due to $\boldsymbol{B}^{(3)}$, because such a dependence cannot be interpreted in terms of oscillatory plane waves uch as $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$.

The visible frequency inverse Faraday effect has been shown experimentally [1,2] o be proportional to I, and this effect is obtained from the correctly relativistic Eq. (2) n the limit,

$$\omega \gg \frac{e}{m_0}B^{(0)}, \tag{4}$$

i.e., when the frequency is high and the intensity relatively low. This is the limit assumed implicitly by Talin *et al.* [4], in which we obtain,

$$En \rightarrow \frac{e^2c^2}{m_0\omega^2}B^{(0)}|\boldsymbol{B}^{(3)}| = \chi' B^{(0)2}, \tag{5}$$

where χ' is a susceptibility in S.I. units [10]. It can be considered as a *one electron* susceptibility generated by the orbital motion of the electron in the field. This classical description can be reduced to its essentials as follows.

Since the inverse Faraday effect occurs in general without the necessity for spectral absorption [1—3], a fraction of the rotational energy, $\hbar\omega$, of one photon can be transferred to the electron. So there would be a small frequency shift in a photon reflected off an electron due to the visible frequency inverse Faraday effect because the photon would have lost part of its energy, and therefore ω would be decreased. The shift can be calculated using the standard minimal prescription [17] for the free photon, referred to elsewhere [10] as the charge quantization condition that emerges self-consistently from $\boldsymbol{B}^{(3)}$ theory in several ways,

$$p = eA^{(0)} = \hbar\kappa, \tag{6}$$

where p is the vacuum linear momentum magnitude of one photon, the magnitude of its wave-vector, and $A^{(0)}$ the scalar magnitude of the classical vector potential, a plane wave [10]. Using the relations [10—15] between magnitudes in vacuo,

$$A^{(0)} = \frac{B^{(0)}}{\kappa} = \frac{cB^{(0)}}{\omega}, \tag{7}$$

we obtain,

$$B^{(0)} = \frac{\omega}{ec^2}\hbar\omega, \tag{8}$$

which is an interesting indication of the nature of the quantized field, which carries the charge, e, on the electron. This inference is an immediate consequence of the standard minimal prescription [17] in the absence of initial electron momentum. Equation (8) translates the classical Eq. (5) into a physically transparent result in the quantum field theory,

$$En = \left(\frac{\hbar\omega}{m_0 c^2} \right) \hbar\omega. \tag{9}$$

he rotational energy transferred from the photon to the electron in the inverse Faraday fect far from optical resonance (i.e., in spectral regions where there is no absorption) is fraction of the rotational energy of one photon, $\hbar\omega$, a fraction given by,

$$f := \frac{\hbar\omega}{m_0 c^2}. \tag{10}$$

his is a two photon process, because of the occurrence of $\hbar\omega$ squared in Eq. (9), and assically it is a second order process described by $iB^{(0)}\boldsymbol{B}^{(3)}$.

For visible frequency radiation, $\omega \sim 10^{14}\ rads^{-1}$, and so,

$$f \sim 10^{-5}, \tag{11}$$

nd the angular momentum magnitude transferred to the electron is

$$|\boldsymbol{J}| \sim 10^{-5}\hbar. \tag{12}$$

his translates into a Zeeman shift of order of magnitude in the gigahertz range, a shift hich is easily observable in principle in atomic samples using an ESR type probe tuned resonance in the same gigahertz range.

At visible frequencies therefore, the angular momentum transferred to the electron a small fraction of $\hbar$. Equation (9) reduces the non-relativistic inverse Faraday effect its essence, it is an exchange of angular momentum between two photons and one lectron. At visible frequencies the angular momentum given up to the electron is very nall compared to $\hbar$. At X ray frequencies however, the fraction f can approach and then xceed unity, so the angular momentum transferred to the electron in this non-relativistic mit can exceed $\hbar$.

At microwave frequencies, if sufficient power density is used, it is possible to xperimentally attain the condition,

$$\omega \ll \frac{e}{m_0} B^{(0)}, \tag{13}$$

hich reduces Eq. (2) to its relativistic limit,

$$En \rightarrow m_d |\boldsymbol{B}^{(3)}|, \qquad m_d = \frac{ec^2}{\omega}. \tag{14}$$

Here, $\boldsymbol{m}_d$ has the units of magnetic dipole moment, and the energy transferred from the classical field to the electron is proportional to $\boldsymbol{B}^{(3)}$, and so to the square root of the beam intensity. Using Eq. (1),

$$|\boldsymbol{J}| = \hbar, \tag{15}$$

showing that in this limit, the angular momentum of the photon is fully transferred to the electron. This limit corresponds to Eq. (13), and requires a low angular frequency (microwave range) and high power density. Since electron rest mass can never vanish, the limit (14) corresponds more accurately to

$$|\boldsymbol{J}| \xrightarrow[m_0 \rightarrow 0]{} \hbar. \tag{16}$$

Therefore the relativistic limit of Eq. (2) can be reached experimentally in an essentially elastic transfer of angular energy and angular momentum from a photon to an electron in a 3 GHz pulse of high powered microwave radiation directed at an electron beam.

3. Discussion

Detection of the square root I dependence created by the $\boldsymbol{B}^{(3)}$ field is a method of proving its existence empirically. In the relativistic limit, a photon colliding with an electron of the beam transfers almost all of its angular momentum, $\hbar$, to the electron and so is left with essentially no energy, $\hbar\omega$. The electron acquires an orbital angular momentum $\hbar$ and so becomes spin polarized, an effect which can be measured in principle by measuring the spin polarization acquired by an initially unpolarized beam of electrons after interaction with a microwave pulse of high power density. The degree of spin polarization is proportional to the angular momentum acquired by the electron, whose magnitude is

$$|\boldsymbol{J}| = \frac{ec^2}{\omega^2} B^{(0)}. \tag{17}$$

For a given electromagnetic angular frequency ω this is proportional to $B^{(0)}$ and therefore to the square root of the beam intensity I through the relation [10],

$$B^{(0)} = \left(\frac{I}{\epsilon_0 c^3} \right)^{1/2} = \left(\frac{\mu_0 I}{c} \right)^{1/2}, \qquad \mu_0 \epsilon_0 = \frac{1}{c^2}. \tag{18}$$

Therefore the measurement of the acquired electron spin polarization should be linear in $I^{1/2}$. Such a plot provides empirical evidence for the existence of $\boldsymbol{B}^{(3)}$ in the vacuum, ith many consequences for field theory [10].

eferences

[1] J. P. van der Ziel, P. S. Pershan, and L. D. Malmstrom, *Phys. Rev. Lett.* **15**, 190 (1965).
[2] P. S. Pershan, J. P. van der Ziel, and L. D. Malmstrom, *Phys. Rev.* **143**, 574 (1966).
[3] J. Deschamps, M. Fitaire, and M. Lagoutte, *Phys. Rev. Lett.* **25**, 1330 (1970); ibid., *Rev. Appl. Phys.* 7, 155 (1972).
[4] B. Talin, V. P. Kaftandjan, and L. Klein, *Phys. Rev. A* **11**, 648 (1975).
[5] T. W. Barrett, H. Wohltjen, and A. Snow, *Nature* **301**, 694 (1983).
[6] N. Sanford, R. W. Davies, A. Lempicki, W. J. Miniscalco, and S. J. Nettel, *Phys. Rev. Lett.* **50**, 1803 (1983).
[7] S. Woźniak, M. W. Evans, and G. Wagnière, *Mol. Phys.* **75**, 81 (1992).
[8] R. Zawodny in M. W. Evans and S. Kielich, eds., *Modern Nonlinear Optics*, Vols. 85(1), 85(2), 85(3) of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, eds., (Wiley Interscience, New York, 1993).
[9] *L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields*, 4th edn. (Pergamon, Oxford, 1975).
[10] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Volume 1, The Field* $\boldsymbol{B}^{(3)}$ (Kluwer Academic Publishers, Dordrecht, 1994), *The Enigmatic Photon, Volume 2, Non-Abelian Electrodynamics* (Kluwer Academic Publishers, Dordrecht, 1995).
[11] A. A. Hasanein and M. W. Evans, *The Photomagneton in Quantum Field Theory* (World Scientific, Singapore, 1994).
[12] M. W. Evans in A. van der Merwe and A. Garuccio eds., *Waves and Particles in Light and Matter.* (Plenum, New York, 1994).
[13] M. W. Evans, Vol. 85(2) of Ref. 8.
[14] M. W. Evans, *Found. Phys.* **24**, 892, 1519, 1671 (1994).
[15] M. W. Evans, *Found. Phys. Lett.* 7, 67, 209 and in press (1995).
[16] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).
[17] L. H. Ryder, *Quantum Field Theory*, 2nd. edn. (Cambridge University Press, Cambridge, 1987).

Molecular Dynamics Computer Simulation of the Inverse Faraday and Beth Effects in Liquid Water

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A novel computer simulation method is used to demonstrate the existence of Langevin-Kielich (LK) functions of the inverse Faraday and Beth effects (the induction by circularly polarized electromagnetic radiation of a molecular magnetic dipole moment and angular momentum respectively). The LK functions are constructed as a function of field strength in the relativistic and non-relativistic limits, where the effects are proportional respectively to $B^{(0)}$ and $B^{(0)2}$, where $B^{(0)}$ is the scalar magnitude of the magnetic flux density of the radiation field. The LK function of molecular torque goes through a characteristic maximum and returns to zero, revealing a non-Markovian and non-linear response to the applied field. In the relativistic limit the simulation shows that the electromagnetic field-ensemble interaction is describable in terms of the recently inferred $\boldsymbol{B}^{(3)}$ field in vacuo, and shows that the gauge symmetry of electromagnetism is that of the Poincaré group rather than the U(1) group.

1. Introduction

The interaction of electric, magnetic and electromagnetic fields with molecular matter is fundamentally important. Without this interaction, the existence of fields would be by tautology, hypothetical. The method of molecular dynamics computer simulation was first applied to this problem in 1982 [1—5] and rapidly developed thereafter [6—14]. The original method was based on coding in to the forces loop a torque set up between a molecular property tensor and a component of the electromagnetic, electric or magnetic field, and has produced numerous results which are still inaccessible to experiment or simple theory. It produces, for example, the set of Langevin-Kielich functions over the whole range of response, from linear to saturation [1—5]; it has demonstrated the existence of non-linear response through fall transient acceleration [6]; and of several novel response phenomena unknown either analytically or experimentally, for example, a novel type of Rosenfeld birefringence [9]. It has also been used to confirm computationally the inverse Faraday effect, magnetization by circularly polarized electromagnetic radiation [10], and related phenomena such as a frequency doubled optical Stark effect [11] and optical Faraday effect [12]. These simulations [13,14] confirm the analytical redictions of non-linear optical theory [15], even in cases where the predicted xperimental effects have yet to be observed. (The experimental data on the inverse araday effect, for example, are sparse [16—18], but reproducible and repeatable.) From nese simulation data the principles of group theoretical statistical mechanics have been eveloped and summarized in review [8]. The basic methods of computer simulation have eveloped rapidly, and are summarized, for example, in Ref. (14), with over 2,000 eferences.

In Sec. 2, the original method [1—5] is developed by coding indirectly to the forces oop an induced molecular *angular momentum*. The code responds to this external nfluence in a manner that can be understood only with a non-Markovian theory using nemory functions [19]: the effect is to produce Langevin-Kielich (LK) functions [1—14] f molecular angular momentum, angular velocity and torque, together with a novel ransient orientational anisotropy. The LK function of molecular torque goes through a haracteristic maximum dependent only on molecular structure, and returns to zero in the mit of intense applied field strength, a limit in which the induced angular momentum of ach molecule becomes a constant spinning motion about the axis of applied field angular nomentum. In other words the mean molecular angular velocity and angular momentum ecome parallel and constant in magnitude and direction. In this limit, inaccessible to resent day experiments, the spins are completely aligned. The inverse Faraday and Beth ffects (induced molecular magnetic dipole and molecular angular momentum respectively) re revealed clearly and unequivocally. The method is an improvement over the first ttempts [10] at simulating these phenomena because those depended on a highly scillatory torque function. In these new results, the interfering high frequency oscillations re removed completely, revealing the underlying phenomena in all detail.

In Sec. 3 the relativistic nature of these effects is considered by adopting the precise lescription of the interaction of one electron with the classical electromagnetic field given y the classical, relativistic, Hamilton-Jacobi equation [20]. The latter shows that in the elativistic limit, the interaction of field with electron is governed by the $\boldsymbol{B}^{(3)}$ component of the former in vacuo. This exact result of the relativistic field theory is adopted to the ase of molecular ensembles in this section. The result is that the induced molecular nagnetic dipole moment and molecular angular momentum can be described in the elativistic limit through the interaction with $\boldsymbol{B}^{(3)}$ of a symmetric susceptibility, a nolecular property tensor. In the other non-relativistic limit (the one in which all experimental data [16—18] have been gathered to date) the relevant molecular property tensor is a fully antisymmetric rank three hypersusceptibility, and the field property with which it interacts is the conjugate product, related to $\boldsymbol{B}^{(3)}$ through the B Cyclic theorem [20—25],

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}, \qquad et\ cyclicum. \tag{1}$$

Here $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ is a plane wave in vacuo of magnetic flux density (tesla), and $\boldsymbol{B}^{(3)}$ is aligned in the axis of propagation, conveniently Z. The classical relation (1) is

tautologically an angular momentum relation in the complex basis ((1), (2), (3)) [23], and obeys the $\hat{C}\hat{P}\hat{T}$ theorem in the quantum field theory. Its underlying symmetry is that of the Poincaré group, i.e., of special relativity [23].

Therefore these simulations show the effect on a molecular ensemble of the fundamental $\boldsymbol{B}^{(3)}$ field over the whole range of the LK functions, from linear response to saturation.

2. The Molecular Dynamics Method

The essential features of respectively the inverse Faraday and Beth effects are the induction of a magnetic dipole moment and angular momentum by a circularly polarized electromagnetic field. Therefore it is an advantage to simulate this feature directly by coding in to the forces loop of a standard molecular dynamics [8,14] code an external angular momentum, the induced molecular magnetic dipole moment divided by the gyromagnetic ratio, a scalar coefficient. The code responds to this external influence through the forces loop, creating LK functions. In the non-relativistic limit the induced molecular magnetic dipole moment of the inverse Faraday effect is [15],

$$m_i = \chi''_{ijk} B_j B_k^*, \tag{2}$$

where χ''_{ijk} is a fully antisymmetric imaginary part of the hypersusceptibility, a molecular property tensor. The tensor product $B_j B_k^*$ represents the imaginary antisymmetric conjugate product [15] of the radiation field, and is phase free. The latter property implies that m_i is also phase free. It is observable [16—18] experimentally in an induction coil wound around a sample subjected to a pulse of laser radiation. The Beth effect was observed originally [26] in a crystal mounted on a torsion wire, and measures the angular momentum imparted to the sample by the circularly polarized field. In this simulation, the sample is 108 water molecules, interacting with a modified ST2 [8,14]. As with all potentials, this one has advantages and shortcomings discussed in detail in Ref. (14): it is basically a Lennard-Jones atom-atom potential with partial charges to simulate the water structure.

The induced molecular angular momentum proportional to m_i of Eq. (2) is coded into the molecule fixed frame labeled (1,2,3) (not to be confused with the complex basis ((1), (2), (3)) of Eq. (1)), a frame defined by the principal molecular moments of inertia. In this frame, each component of the induced magnetic dipole is given by [10],

$$m_1 = 2\chi''_{123} e_{1Z} B^{(0)2}, \quad m_2 = 2\chi''_{231} e_{2Z} B^{(0)2}, \quad m_3 = 2\chi''_{312} e_{3Z} B^{(0)2}, \tag{3}$$

where e_{1Z}, e_{2Z}, and e_{3Z} are the Z components of the unit vector e_1 in axis 1. The derivation of Eq. (3) uses the C_{2V} symmetry of the water molecule [10]. The anisotropy of the antisymmetric hypersusceptibility is unknown experimentally, so we have used the same procedure as in the original simulations [10] of the inverse Faraday effect and have coded in

$$\chi''_{123} : \chi''_{231} : \chi''_{312} = 1 : 2 : 3 \, . \tag{4}$$

As in the original simulations [10] using the torque method [1—5] this procedure does not result in loss of generality. Although non-linear optics is a developed subject area, very little is known about the detailed properties of molecular property tensors. For example, the anisotropy of polarizability of water is for all practical purposes unknown, because [10b] some experiments give a different sign from others.

As detailed elsewhere [10a] the molecule frame components in Eq. (3) are back transformed into the laboratory frame (X, Y, Z) using a rotation matrix. This procedure was coded into the forces loop of the molecular dynamics program, which is given in full in an Appendix to Ref. (8). A series of auto and cross correlation functions were then evaluated with code also given in Ref. (8). The auto correlation functions (a.c.f.'s) were normalized to unity at the origin as follows:

$$C_x(t) = \frac{<x(t)\,x(0)>}{<x(0)^2>} \, , \tag{5}$$

and were evaluated for molecular center of mass velocity; angular momentum; orientation (e_1) vector; net molecular force; net molecular torque; rotational velocity ($\dot{e}_1$); and angular velocity. The three individual components a.c.f.'s were evaluated over 6,000 time steps of 0.005 ps each for each vector as a function of $B^{(0)2}$ in the non-relativistic limit represented by Eq. (2). Langevin-Kielich functions were evaluated in a series of simulations from the long time limit of the Z component a.c.f. for molecular angular velocity; angular momentum; and torque. Some individual a.c.f. results are given in Figure 1 and the LK functions in Figure 2. The main features of the results (to be discussed in more detail later) are that the angular velocity and momentum LK functions saturate at high field strength, but the torque LK function goes through a novel, characteristic maximum and returns to zero. A transient anisotropy develops in the orientational and rotational velocity a.c.f.'s. There is no development of c.c.f.'s in the laboratory frame.

The simulations, carried out on an IBM Power P.C., were repeated in the fully relativistic limit represented (Sec. 3) by the induced molecular magnetic dipole moment,

$$\boldsymbol{m}_1 = \chi'_{11} e_{1Z} B^{(0)} \, , \qquad \boldsymbol{m}_2 = \chi'_{22} e_{2Z} B^{(0)} \, ,$$
$$\boldsymbol{m}_3 = \chi'_{33} e_{3Z} B^{(0)} \, , \tag{6}$$

where χ'_{ii} are molecule fixed frame components of the symmetric, real, molecular susceptibility. Langevin-Kielich functions were computed as a function of $B^{(0)}$, the magnitude of the vacuum $\boldsymbol{B}^{(3)}$ field [20—25] and the same pattern of results obtained, i.e., a torque LK function that goes through a maximum, but this time as a function of $B^{(0)}$ and not of $B^{(0)2}$.

The simulation shows directly the existence of the inverse Faraday and Beth effects over the complete range of the LK functions, from linear response to saturation. This cannot yet be repeated experimentally or with analytical non-linear response theory without great technical difficulty and over-parameterization.

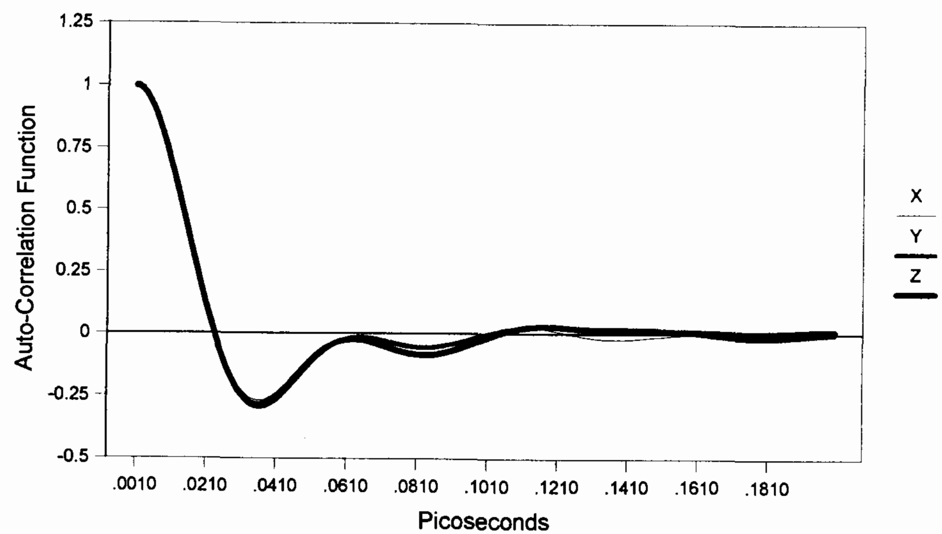


Fig. 1a-1. Plot of autocorrelation function for normalized angular velocity. Field off condition.

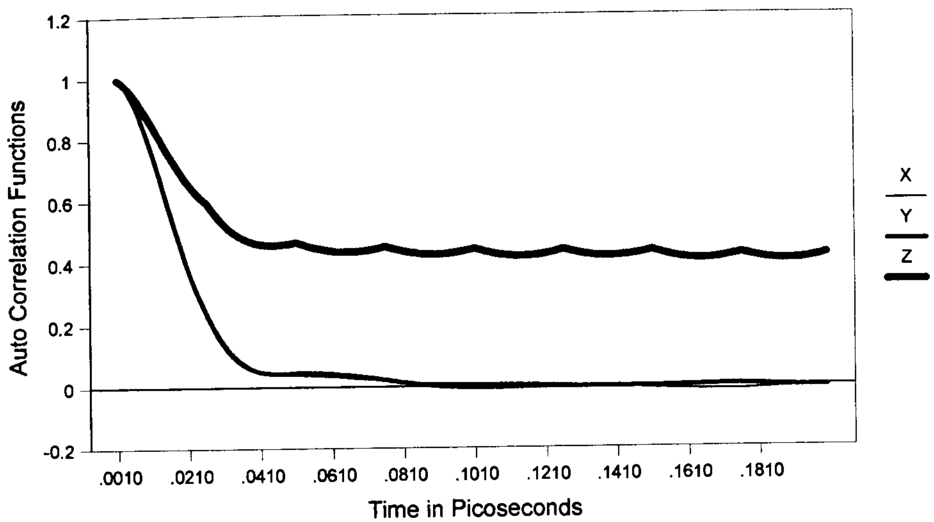


Fig. 1a-2. Plot of autocorrelation function for normalized angular velocity. Series 2 level (intermediate field strength).

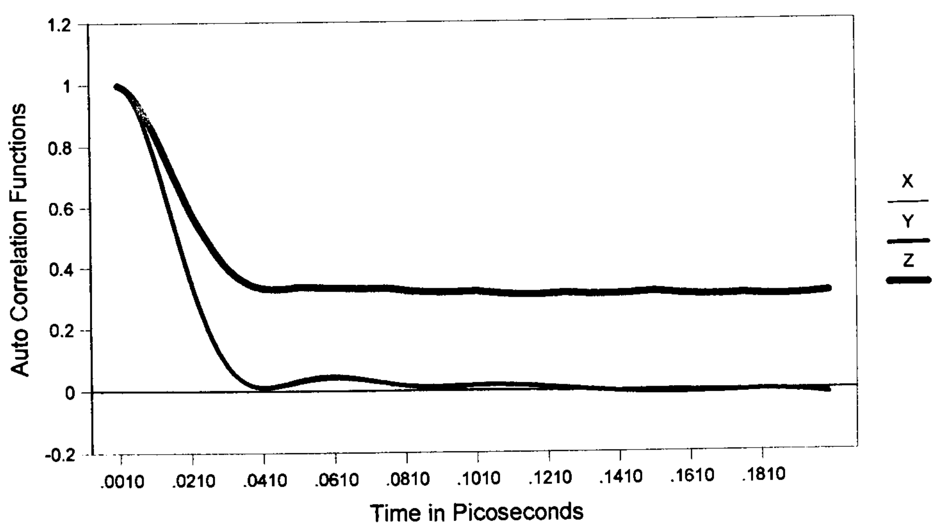


Fig. 1a-3. Plot of autocorrelation function for normalized angular velocity. Series 4 level (high field strength).

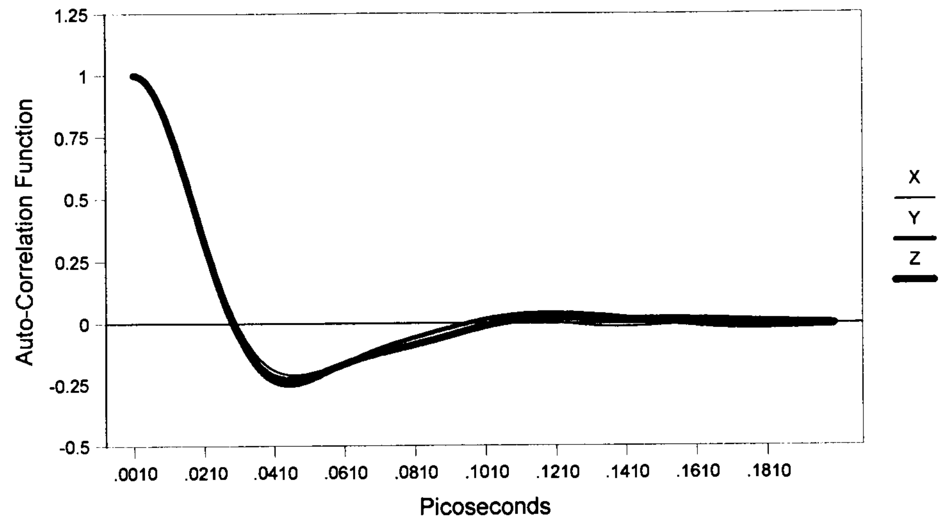


Fig. 1b-1. Plot of autocorrelation function for normalized angular momentum. Field off condition.

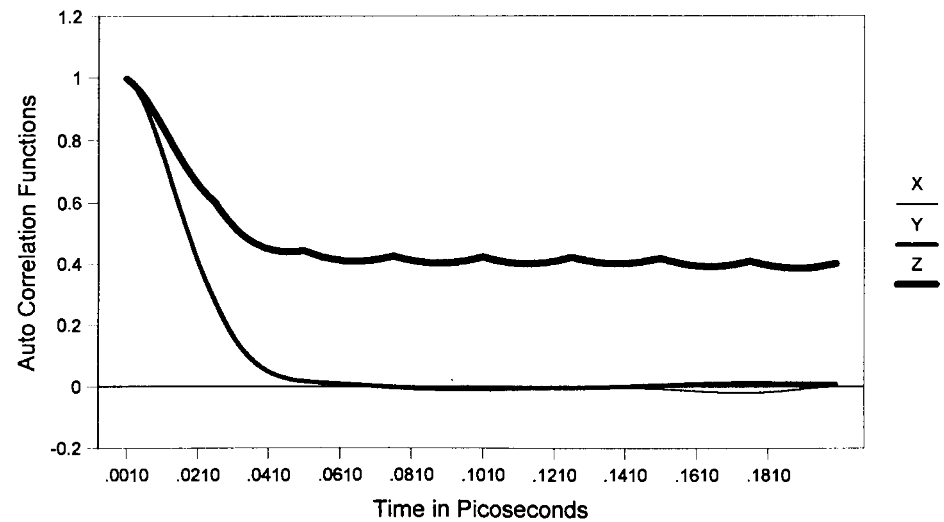


Fig. 1b-2. Plot of autocorrelation function for normalized angular momentum. Series 2 level (intermediate field strength).

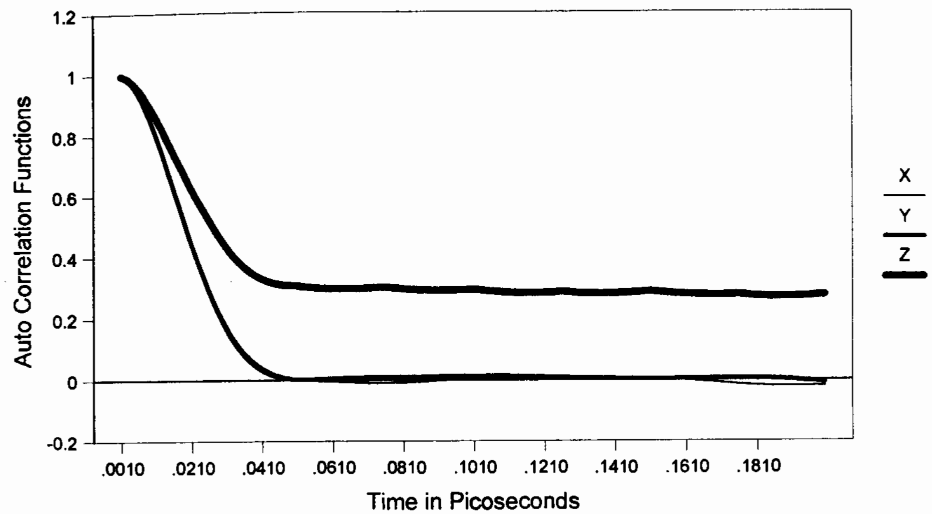


Fig. 1b-3. Plot of autocorrelation function for normalized angular momentum. Series 4 level (high field strength).

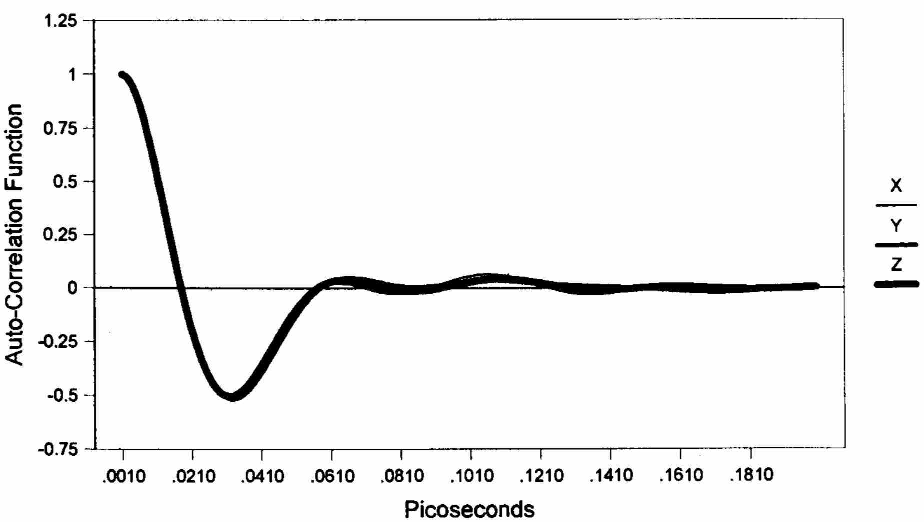


Fig. 1c-1. Plot of autocorrelation function for normalized torque. Field off condition.

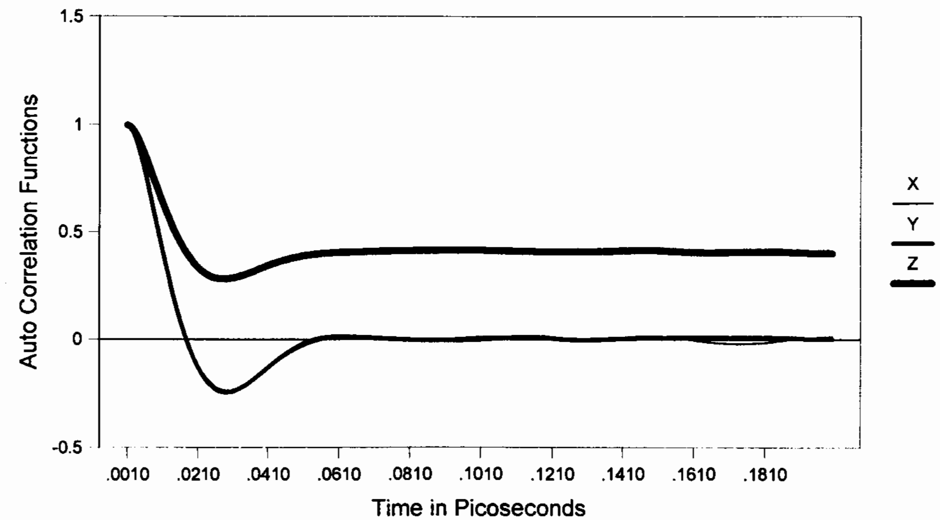


Fig. 1c-2. Plot of autocorrelation function for normalized torque. Series 2 level (intermediate field strength).

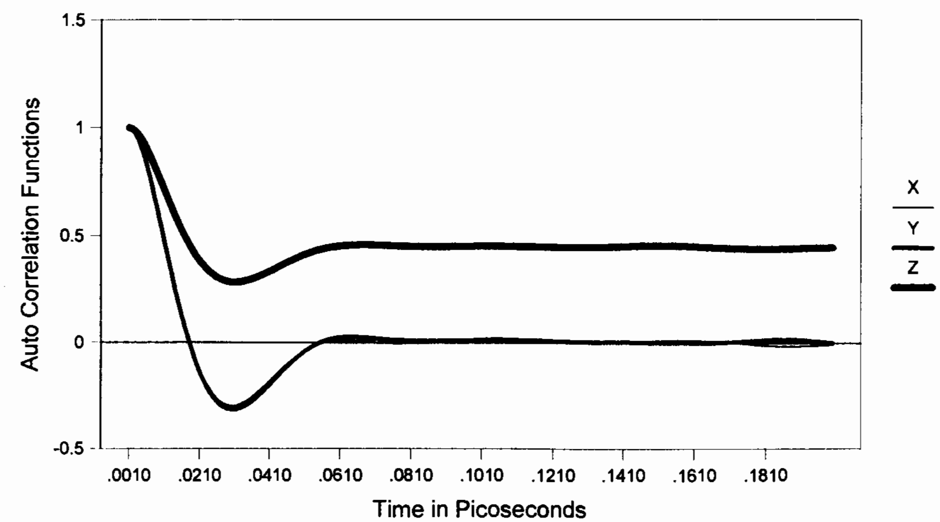
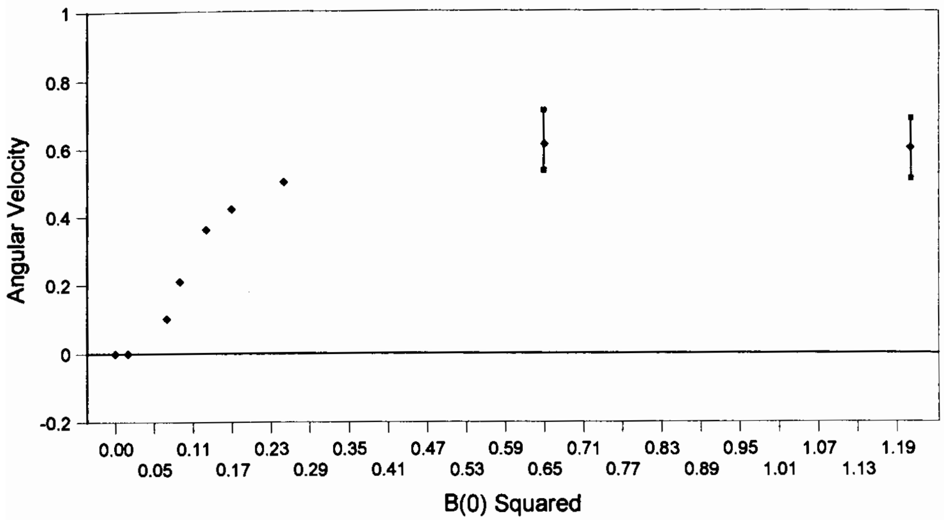
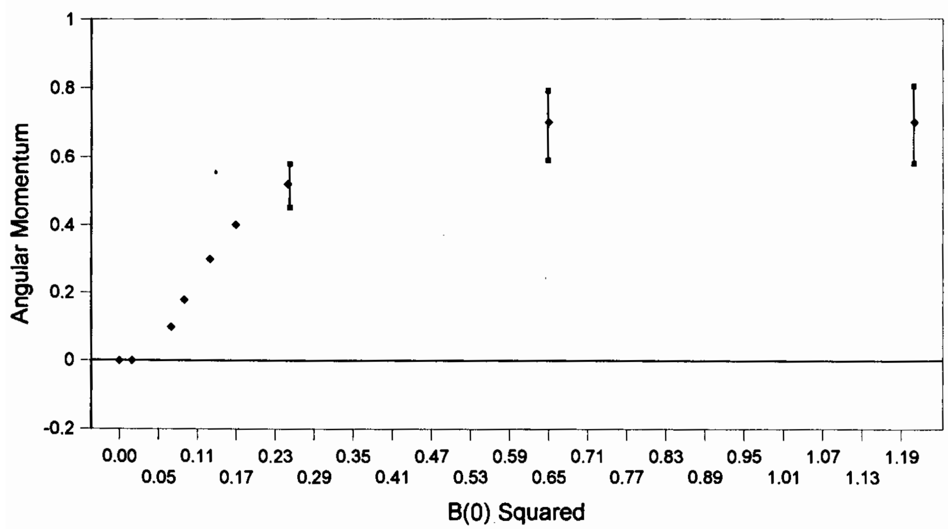


Fig. 1c-3. Plot of autocorrelation function for normalized torque. Series 4 condition (high field strength).



ig. 2a. Langevin-Kielich function of angular velocity plotted against $B^{(0)2}$.



ig. 2b. Langevin-Kielich function of angular momentum plotted against $B^{(0)2}$.

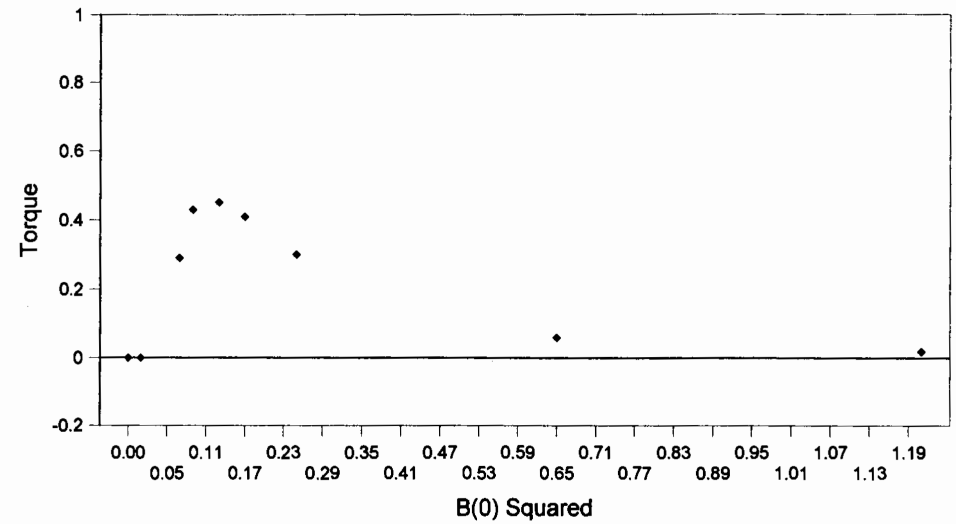


Fig. 2c. Langevin-Kielich function of torque plotted against $B^{(0)2}$.

3. Relativistic Considerations

As first shown in Ref. (27), the inverse Faraday (and by implication the Beth) effect must be corrected relativistically under well defined conditions considered in detail by Evans and Vigier in Ref. (20) using a method originally due to Landau and Lifshitz [28] based on the classical relativistic Hamilton-Jacobi equation of motion. This method is exact for one electron in a classical electromagnetic field. Classically there is no intrinsic electron spinor, and so the Beth effect for one electron is given by [20]

$$\boldsymbol{J}^{(3)} = \frac{e^2c^2}{\omega^2}\left(\frac{B^{(0)}}{\left(m_0^2\omega^2 + e^2B^{(0)2}\right)^{1/2}}\right)\boldsymbol{B}^{(3)}, \tag{7}$$

and the inverse Faraday effect by

$$\boldsymbol{m}^{(3)} = -\frac{e}{m_0}\boldsymbol{J}^{(3)}. \tag{8}$$

Here $\boldsymbol{J}^{(3)}$ is the induced electronic angular momentum and $\boldsymbol{m}^{(3)}$ the induced electronic magnetic dipole moment. The gyromagnetic ratio is taken to be the electronic charge to mass ratio e/m_0. In Eq. (7), ω is the angular frequency of the radiation field, c the speed of light in vacuo. In an electron gas of volume V containing N electrons, the magnetic field strength set up in the sample is given in Appendix F of Ref. (22), and is

$$B^{(3)}_{\text{in sample}} = \frac{N}{V}\frac{\mu_0 e^3 c^2}{m\omega^2} \times \left(\frac{B^{(0)}}{(m^2\omega^2 + e^2 B^{(0)2})^{1/2}}\right) B^{(3)}_{\text{free space}} , \tag{9}$$

where μ_0 is the permeability in vacuo. Here $B^{(3)}_{\text{in sample}}$ is the free space value of $B^{(0)}$ in tesla. In the low field, non-relativistic (visible frequency) limit, $eB^{(0)} \ll m\omega$ [22] and

$$B^{(3)}_{\text{in sample}} \rightarrow \frac{N}{V}\left(\frac{\mu_0 e^3 c^2}{m^2\omega^3}\right) B^{(0)} B^{(3)}_{\text{free space}} . \tag{10}$$

In the high field, relativistic (radio frequency) limit, $m\omega \ll eB^{(0)}$ [22] and

$$B^{(3)}_{\text{in sample}} \rightarrow \frac{N}{V}\left(\frac{\mu_0 e^2 c^2}{m\omega^2}\right) B^{(3)}_{\text{free space}} . \tag{11}$$

The free space value of $B^{(0)}$ is

$$B^{(0)} = \left(\frac{\mu_0}{c} I\right)^{1/2} , \tag{12}$$

where I is the field intensity in watts m^{-2}. To date, experimental data on the inverse Faraday effect are available only in the non-relativistic limit, (low field limit) [22], in which

$$B^{(3)}_{\text{in sample}} \rightarrow \frac{N}{V}\left(\frac{\mu_0 e^3 c}{m^2}\right) \frac{I}{\omega^3} e^{(3)} . \tag{13}$$

Under these conditions, (for example those used by Rikken [29], i.e., $I = 5.5 \times 10^{12}\ Wm^{-2}$; $\omega = 1.77 \times 10^{16}\ rad\ \sec^{-1}$) we obtain from Eq. (13)

$$B^{(3)}_{\text{in sample}} = 1.06 \times 10^{-35} \frac{N}{V} e^{(3)} \sim 10^{-9} \text{ tesla} , \tag{14}$$

which, for $N/V = 10^{26}\ m^{-3}$ is about the same order of magnitude as reported by van der Ziel *et al.* [16—18] in liquids and glasses.

The N electron magnetization from Eq. (8) is

$$\boldsymbol{M}^{(3)} := N\boldsymbol{m}^{(3)} = N\left(a\chi' + bB^{(0)}\chi''\right)\boldsymbol{B}^{(3)} \tag{15}$$

where χ' and χ'' are respectively the electronic susceptibility and hypersusceptibility:

$$\chi' := \frac{e^2 c^2}{m_0 \omega^2} , \qquad \chi'' := \frac{e^3 c^2}{m_0^2 \omega^3} , \tag{16}$$

and where the factors a and b are given by

$$a := \left(1 + x^2\right)^{-1/2} , \qquad b = \left(1 + \frac{1}{x^2}\right)^{-1/2} , \tag{17}$$

with

$$x = \frac{m\omega}{eB^{(0)}} , \tag{18}$$

in dimensionless S.I. units.

Therefore both the inverse Faraday and Beth effects for one electron are in general described by appropriate combinations of χ' and χ''. These combinations are simplified to

$$\boldsymbol{M}^{(3)} \rightarrow N\chi'\boldsymbol{B}^{(3)} , \tag{19}$$

and

$$\boldsymbol{M}^{(3)} \rightarrow N\chi'' B^{(0)}\boldsymbol{B}^{(3)} , \tag{20}$$

respectively in the relativistic and non-relativistic limits.

Therefore the problem at hand is defined as that of extending the classical one electron results to molecular property tensors. This problem has a solution only for one electron, as just described, and even that has inherent approximations [20]. In the relativistic quantum field theory, the Dirac equation has been used [22] for the same problem, yielding a direct interaction between

$$B^{(3)*} := -i\frac{e}{\hbar}A^{(1)} \times A^{(2)} , \tag{21}$$

and the spinor. For atoms and molecules the problem must be approached by approximation and phenomenology, or otherwise through ab initio computations [30]. The molecular dynamics computer simulation method used in this work is however classical, and we prefer to make a first approximation at a solution through a simple relativistic correction suggested by the one electron results. This is,

$$m_i = \frac{\chi^{//}_{ijk} B_j B^*_k}{\left(1 + y^2 B^{(0)2}\right)^{1/2}} . \tag{22}$$

In the non-relativistic limit, $y^2 B^{(0)2} \ll 1$, and

$$m_i \rightarrow \chi^{//}_{ijk} B_j B^*_k . \tag{22a}$$

In the opposite relativistic limit, $y^2 B^{(0)2} \gg 1$, then,

$$m_i \rightarrow \frac{\chi^{//}_{ijk}}{y} \frac{B_j B^*_k}{B^{(0)}} = \chi^{/}_{il} B_l , \tag{22b}$$

using the antisymmetric unit tensor ϵ_{ijk} as follows:

$$B_j B^*_k = \epsilon_{jkl} B^{(0)} B_l , \tag{23a}$$

$$\chi^{//}_{jki} = y \epsilon_{jkl} \chi^{/}_{il} . \tag{23b}$$

Therefore the conjugate product is expressed as $iB^{(0)}B_l$ [20] and the antisymmetric rank three-tensor $\chi^{//}_{ijk}$ is reduced to the symmetric rank two-tensor using (23b). This procedure is the same, mathematically, as expressing an axial vector as an antisymmetric tensor [31], and vice-versa.

4. Discussion

The novel molecular dynamics method developed here makes dynamical sense as can be seen through the LK function of torque in Figure 2. The net molecular torque in the sample is initially zero, and the long time limit of the Z component of the normalized torque a.c.f. (Figure 1) is zero because the sample is isotropic. As the electromagnetic field begins to induce an angular momentum the code shows the presence of a non zero net molecular angular momentum, angular velocity and torque, and an anisotropy (Figure

1) in the orientational a.c.f.'s (the Z component begins to split away from the X and Y components which are the same within the noise). As the field intensity is increased the torque LK function goes through a maximum (Figure 2) and returns to zero in the region where the LK functions of angular momentum and angular velocity are saturated.

In the relativistic limit this occurs as a function of $B^{(0)2}$. In the non-relativistic limit as a function of $B^{(0)}$, the magnitude of $\boldsymbol{B}^{(3)}$.

A qualitative analytical explanation of this behavior can be found through the precessional dynamics of the water molecules subjected to an external angular momentum which tends to align them against the Brownian motion [32] into a collection of molecules spinning each about the Z axis. The latter condition is reached in the saturation limit of the angular momentum and velocity LK functions in Figure 2. In this limit each spin is aligned in Z, the angular momentum is a constant in magnitude and direction and its time derivative, the torque, is zero. The Langevin equation gives [33], to a rough but adequate approximation

$$Tq_Z := \dot{J}_Z \doteq -\beta J_Z + \lambda_Z , \tag{24}$$

where $<\lambda_Z>$ is the internal random torque to Brownian motion and β is the friction coefficient. Linear response theory gives [33,34] from Eq. (24),

$$\frac{<J_Z(t)>}{<J_Z(0)>} = \frac{<J_Z(t)J_Z(0)>}{<J_Z^2(0)>} = e^{-\beta t} . \tag{25}$$

It is clear by comparison of Eq. (25) with Figure 1 that the Langevin equation is not a description of the simulated autocorrelation functions — for reasons which are well known [33,34]. The simulations need for their description a non-linear, non-Markovian theory [33,34] in which the friction coefficient evolves into a chain of memory functions. However, the Langevin equation is useful as a first approximation if we allow for the fact that its applicability is restricted to linear and Markovian processes. It is strictly valid only in the linear response approximation [33,34].

Considering the torque LK function of Figure 2, its qualitative features can be explained as follows. In the linear response region (weak applied field) the induced molecular angular momentum is such as to only slightly augment the friction term βJ_Z with a non-zero term proportional to $\beta<J_Z>_{t\rightarrow\infty}$. This produces a non-zero $<Tq>_{t\rightarrow\infty}$ from Eq. (24). As the applied field strength is increased, so does $\beta<J_Z>_{t\rightarrow\infty}$ and $<Tq>_{t\rightarrow\infty}$; but the latter reaches a maximum while $<J_Z>_{t\rightarrow\infty}$ saturates (Figure 2). This can only be explained by developing the friction coefficient β of the Langevin equation into a chain of memory functions, as for example in Mori theory [33,34], a theory capable of describing at least qualitatively the saturation limit of the LK functions in Figure 2. The computer results show therefore that the response of the ensemble is in general non-Markovian and non-linear. In the saturation limit, the applied field angular momentum is so great that it overwhelms the Brownian motion and the system becomes one of aligned molecular spins of constant magnitude and direction. In this limit $<Tq> = <\dot{J}_Z> = 0$, $<J_Z>$ is a constant; and so β becomes effectively *zero*.

These results therefore make dynamical sense, and illustrate the nature of the Beth and inverse Faraday effects over the complete range of the LK functions and over the range from the non-relativistic to relativistic. The results illustrate the way in which the $\boldsymbol{B}^{(3)}$ field interacts with a molecular ensemble. It has been demonstrated rigorously [23] that the novel $\boldsymbol{B}^{(3)}$ field obeys the $\hat{C}\hat{P}\hat{T}$ theorem in the quantum field theory and Maxwell's equations in the classical theory of fields. The B cyclic theorem (1) is rigorously Lorentz covariant [23], therefore in the present state of knowledge $\boldsymbol{B}^{(3)}$ has the same theoretical validity as $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$. The inverse Faraday and Beth effects show its existence experimentally.

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References

[1] M. W. Evans, *J. Chem. Phys.* **76**, 5473, 5480 (1982); **77**, 4632 (1982); **78**, 925, 5403 (1983).
[2] M. W. Evans, *Phys. Rev. Lett.* **50**, 351 (1983).
[3] M. W. Evans, *Phys. Rev. A* **30**, 2062 (1984).
[4] M. W. Evans, *Phys. Rev. Lett.* **55**, 1551 (1985).
[5] M. W. Evans, *Phys. Rev. A* **34**, 468 (1986); **35**, 2989 (1987).
[6] M. W. Evans, P. Grigolini and G. Pastori-Paravicini, eds., *Memory Function Approaches to Stochastic Problems in Condensed Matter*, Vol. 62 of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, eds. (Wiley Interscience, New York, 1985).
[7] M. W. Evans, ed., *Dynamical Processes in Condensed Matter*, Vol. 63 of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, eds. (Wiley Interscience, New York, 1985).
[8] M. W. Evans, *Simulation and Symmetry in Molecular Diffusion and Spectroscopy*, a review in Vol. 81 of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, eds., (Wiley Interscience, New York, 1992) pp 361-702.
[9] M. W. Evans, S. Woźniak and G. Wagnière, *Physica B* **179**, 133 (1992).
[10] M. W. Evans, S. Woźniak and G. Wagnière, *Physica B* **176**, 33 (1992); M. W. Evans, *Phys. Lett. A* **158**, 216 (1991).
[11] M. W. Evans, *Z. Phys. B* **85**, 135 (1991).

[12] M. W. Evans, *J. Mol. Liq.* **55**, 127 (1993).
[13] M. W. Evans and S. Woźniak, *Physica B* **179**, 6 (1992).
[14] G. W. Robinson, S.-B. Zhu, S. Singh and M. W. Evans, *Water in Biology, Chemistry and Physics Experimental Overview and Computational Methodologies* (World Scientific, Singapore, 1996).
[15] M. W. Evans and S. Kielich, eds., *Modern Nonlinear Optics*, Vols. 85(1), 85(2), 85(3) of *Advances in Chemical Physics*, I. Prigogine and S. A. Rice, eds. (Wiley-Interscience, New York, 1993, paperback edition, 1997).
[16] J. P. van der Ziel, P. Pershan, and L. D. Malmstrom, *Phys. Rev. Lett.* **15**, 190 (1965); *Phys. Rev.* **143**, 574 (1966).
[17] J. Deschamps, M. Fitaire, and M. Lagoutte, *Phys. Rev. Lett.* **25**, 1330 (1970); *Rev. Appl. Phys.* 7, 155 (1972).
[18] T. W. Barrett, H. Wohltjen, and A. Snow, *Nature* **301**, 694 (1983).
[19] L. A. Pozhar, *Transport Theory of Inhomogeneous Fluids* (World Scientific, Singapore, 1994); W. T. Coffey, Yu. P. Kalmykov, and J. T. Waldron, *The Langevin Equation with Applications in Physics, Chemistry and Electrical Engineering* (World Scientific, Singapore, 1996).
[20] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $\boldsymbol{B}^{(3)}$. Chap. 12, (Kluwer Academic, Dordrecht, 1994).
[21] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).
[22] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).
[23] M. W. Evans, J.-P. Vigier, S. Roy, and G. Hunter, eds., *The Enigmatic Photon, Volume Four, New Directions.* (Kluwer, Dordrecht, 1997), in preparation, with contributed review articles.
[24] M. W. Evans, *Physica B* **182**, 227, 237 (1992); *Physica A* **214**, 605 (1995).
[25] M. W. Evans, *Found. Phys.* **24**, 1671 (1994).
[26] R. A. Beth, *Phys. Rev.* **50**, 115 (1936); P. W. Atkins, *Molecular Quantum Mechanics*, 2nd edn. (Oxford University Press, Oxford, 1983), p. 211.
[27] B. Talin, V. P. Kaftandjan, and L. Klein, *Phys. Rev. A* **11**, 648 (1975).
[28] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th edn. (Pergamon, Oxford, 1975).
[29] G. L. J. A. Rikken, *Opt. Lett.* **20**, 846 (1995); M. W. Evans, *Found. Phys. Lett.* **9**, 61 (1996).
[30] A. A. Hasanein and M. W. Evans, *Computational Quantum Chemistry* (World Scientific, Singapore, 1996).
[31] L. D. Ryder, *Quantum Field Theory*, 2nd paperback edition, (Cambridge University Press, Cambridge, 1987).
[32] J.-L. Dejardin, *Dynamic Kerr Effect, The Use and Limits of the Smoluchowski Equation and Nonlinear Inertial Responses* (World Scientific, Singapore, 1995).
[33] M. W. Evans, G. J. Evans, W. T. Coffey and P. Grigolini, *Molecular Dynamics and the Theory of Broad Band Spectroscopy.* (Wiley Interscience, New York, 1982).

[34] W. T. Coffey, M. W. Evans and P. Grigolini, *Molecular Diffusion and Spectroscopy.* (Wiley Interscience, New York, 1984); Russian edition by Yu. P. Kalmykov and V. Gaiduk, Nauka, Moscow, 1987.

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