Part I

Electrodynamics as a Gauge Field Theory

Chapter 1

General Gauge Field Theory Applied to Electrodynamics

In the first part of this fifth volume it is argued that electrodynamics can be developed self consistently as an example of contemporary general gauge field theory. The basic assumption in this development is that the left and right circular polarization discovered by Arago in 1811 can be supplemented by a longitudinal component, (3), forming a complex circular basis ((1), (2), (3)) of *O(3)* symmetry - the symmetry of the rotation group. The nature of the basis has been elaborated in Vols. 1 to 4 [1—4] of this series so we take advantage of this groundwork in this volume to try to establish the complete self consistency of *O(3)* electrodynamics on the classical level. This means that the fields are described classically in terms of physical potentials within general gauge field theory [5]. The latter borrows concepts from general relativity, the most important of which is the covariant derivative [1—5], used extensively in the first four volumes. In *O(3)* symmetry, the field tensor in classical electrodynamics is made up of terms both linear and non-linear in the potential, which is a vector in an internal gauge space ((1), (2), (3)). This space is superimposed on space-time in such a way that indices are matched self consistently, forming an extended Lie algebra in which the spaces are not independent.

The rules of general gauge field theory, rules that have led to the discovery of quarks, for example, are then applied to electrodynamics in this *O(3)* group symmetry and several results obtained which are absent from, or ill defined in, the received *U(1)* electrodynamics. Under certain well defined conditions, non linear *O(3)* electrodynamics are well approximated by the linear *U(1)* electrodynamics. For the free field, however, the *O(3)* gauge field symmetry leads to a novel, always non-zero, fundamental field $\boldsymbol{B}^{(3)}$ [1—4], and to novel concepts such as classical vacuum polarization and magnetization which are missing from *U(1)* electrodynamics entirely. The vector potential in *O(3)* symmetry is a classical object, and the rules of gauge transformation are different from those in the older view. This conclusion leads to many ramifications and concepts which are also missing from *U(1)*. *These concepts are due to the non Abelian nature of the O(3)* theory, for example inherent non linearities such as the well observed conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ of complex vector potentials discussed throughout Vols. 1—4 and elsewhere [5] in the literature.

In the presence of field matter interaction it is shown that the *U(1)* theory can be recovered as an excellent approximation to the *O(3)* theory, because when there is field matter interaction the non linear terms are very small, empirically and theoretically. This correct recovery of the linear *U(1)* field equations from those of the non-linear *O(3)* theory means that the latter can do everything that the former can do plus a lot more. The Coulomb, Ampère, Faraday and Gauss laws can be recovered from the *O(3)* theory when the latter's non linearities can be neglected.

For the free field, however, the non-linearities of the *O(3)* theory are intrinsically important and cannot be approximated or gauged away. For example, the $\boldsymbol{B}^{(3)}$ field of the *O(3)* theory does not exist in the *U(1)* theory. The inverse Faraday effect can be accounted for from first principles in the *O(3)* theory, but it leads to a paradox in the *U(1)* theory. In the latter, the potentials can be regarded as mathematical subsidiary variables [7—9], but in the *O(3)* theory they are physically meaningful, for example, there is a light like $\left(cA^{(0)}, \boldsymbol{A}^{(3)}\right)$, a polar four-vector that quantizes directly to photon momentum and which is missing entirely from the *U(1)* theory. Gauge transformation in the *O(3)* theory is a geometrical process with physical meaning, whereas in the *U(1)* theory it is essentially a mathematical process using the gradient of an arbitrary variable. One consequence is that in the *O(3)* theory the Lorentz transformation has a different meaning; if it applies at all it is a special restriction on the physical vector potential. In the *U(1)* theory it is a key choice of gauge that is ultimately a mathematical convenience, leading as it does to the d'Alembert wave equation and to the gauge fixing term used in *U(1)* quantization.

The empirical evidence for the need for an *O(3)* or *SU(2)* symmetry for classical electrodynamics has been reviewed recently by Barrett [8,9], who argues that the classical Maxwellian view of electrodynamics is a linear theory in which the scalar and vector potentials are arbitrary, and defined only through applied boundary conditions and a subjective choice of gauge such as the Lorentz condition. Barrett [8,9] then exposes several flaws in the received view by arguing that there exist several phenomena of nature that require a physical potential four-vector on the classical as well as the quantum levels. One of these is the Aharonov-Bohm effect, but there are several others. The examples thus far identified include the following: 1) Aharonov-Bohm; 2) Altshuler-Aronov-Spivak; 3) topological phase; 4) Josephson; 5) quantized Hall; 6) Sagnac; 7) de Haas van Alphen; 8) Ehrenberg-Siday; 9) non-linear magneto optical. It is also significant that quantum electrodynamics leads to vacuum polarization, or photon self energy, which is missing from classical *U(1)* theory but is present in classical *O(3)* theory as shown in this chapter. The *O(3)* theory also gives classical vacuum magnetization, also missing from *U(1)* theory.

So to accept the suggestion of *O(3)* electrodynamics it is necessary to consider the empirical data given by Barrett, and to accept the hypothesis that gauge field theory can be developed with *O(3)* covariant derivatives which can be classified with group theory and which can be applied to classical electrodynamics [1—4]. Once this hypothesis is accepted and tested for self-consistency, several advantages follow which are described in the first part of this volume. Resistance to the hypothesis based on the standard model is counter-argued in Refs.1 through 4 and on the key empirical observations listed above, for example those by Barrett [8,9] and the empirical observation [1—4] of the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. This is rigorously zero in *U(1)* electrodynamics by definition [1—6], but is non-zero in *O(3)* electrodynamics. There is no difficulty in principle in extending quantum electrodynamics to a non-Abelian theory, which becomes akin to quantum chromodynamics. The latter is well known to be renormalizable at all orders. The mathematical structure of non Abelian *qed* is that of *qcd*, but with an internal gauge space ((1), (2), (3)). As shown in Chap. 2, the gauge space and space-time form an extended Lie algebra in electrodynamics, even in the *U(1)* theory. The two spaces are not independent of each other, even in the standard model.

Therefore the first part of this volume develops the ideas of *O(3)* electrodynamics, giving an unusual amount of technical detail because the hypothesis and concomitant ideas may be new to the classical electrodynamicist versed in the standard model, which allows only *U(1)* theory for the electromagnetic sector. The contemporary gauge field theorist will be unfamiliar with the fact that the internal gauge space and the space-time of both the *U(1)* and the *O(3)* theory are not independent (Chap. 2), in the sense that they form an extended Lie algebra as discussed elegantly by Aldrovandi [10]. The concept of *O(3)* electrodynamics must not be confused as an abstract analogy of the *U(1)* electrodynamics. The former produces physical equations of classical electrodynamics which reduce to the form of the Maxwell equations for polarizations (1) and (2), so in this sense the *O(3)* (non-linear) theory, reduces to the *U(1)* theory when non linearities are small. This occurs in field matter interaction. For example, the non-linear inverse Faraday effect is in magnitude a very small effect of magneto optics which was finally observed with considerable difficulty in 1965 [1—4]. The linear Maxwell equations describe the much more accessible and more easily observable optical effects of nature that go back to the discovery of circular polarization by Arago in 1811, and to the work of Coulomb in the late eighteenth century. The Maxwell equations work well because the optical non linearities in nature are so small in field matter interaction.

At the risk of boring the initiated therefore, we provide throughout the opening chapters of this volume copious details of the new theory, to try to minimize confusion and obscurity, and to help the student. The first section of this chapter deals with the fundamental vector algebra of the complex circular basis ((1), (2), (3)), showing that it is, indeed, a basis that can be used as a representation of *O(3)* space. As intimated, the use of this basis is suggested by the empirical existence of right and left circular polarization, which must be described in a complex representation by at least two basis vectors, $\boldsymbol{i}$ and $\boldsymbol{j}$ in the Cartesian representation, $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ in the complex circular representation. As argued elegantly by Barrett [8,9] this basic fact about light leads to the need for an *SU(2)* electrodynamics.

In our view, the $\boldsymbol{B}^{(3)}$ field emerges once we accept an *SU(2)* or *O(3)* electrodynamics for the vacuum as well as for field matter interaction. It turns out that the hypothesis of *O(3)* electrodynamics is self consistent and is as valid in this sense as *U(1)* electrodynamics. It is recognized however that all Maxwellian type theories have serious flaws inherent in them, and the extension from *U(1)* to *O(3)* does not cure all of these. The standard model is rigidly cemented in *U(1)* theory and carries with it all these serious flaws listed, for example by Bearden [11], and recently discussed by Fritzius' translation [12] of Ritz [13]. These have each argued elegantly against the *U(1)* electrodynamics for a number of years.

1.1 Elements of Vector Analysis in the Circular Basis ((1), (2), (3))

The ((1), (2), (3)) basis is hereinafter referred to as the complex circular basis because it is formed from a complex combination of Cartesian unit vectors as they appear in the description of circular polarization. The basis vectors are therefore,

$$\boldsymbol{e}^{(1)} = \frac{1}{\sqrt{2}}(\boldsymbol{i}-i\boldsymbol{j}), \qquad \boldsymbol{i} = \frac{1}{\sqrt{2}}(\boldsymbol{e}^{(1)}+\boldsymbol{e}^{(2)}),$$

$$\boldsymbol{e}^{(2)} = \frac{1}{\sqrt{2}}(\boldsymbol{i}+i\boldsymbol{j}), \qquad \boldsymbol{j} = \frac{i}{\sqrt{2}}(\boldsymbol{e}^{(1)}-\boldsymbol{e}^{(2)}), \tag{1.1.1}$$

$$\boldsymbol{e}^{(3)} = \boldsymbol{k}.$$

If the phase factor $e^{-i\phi}$ of electromagnetic radiation is kept constant, then $\boldsymbol{e}^{(1)} = \boldsymbol{e}^{(2)*}$ is the vectorial part of the circular description of right and left circularly polarized radiation. Note carefully however that in forming the complex conjugate of a plane wave such as $\boldsymbol{B}^{(1)}$ the phase factor also changes from $e^{-i\phi}$ to $e^{i\phi}$. These matters are described at length in Ref. 14.

The vectors $\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}$ and $\boldsymbol{e}^{(3)}$ form the *O(3)* type cyclic permutation relations [1—4],

$$\begin{aligned} \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)} &= i\boldsymbol{e}^{(3)*}, \qquad \boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}, \\ \boldsymbol{e}^{(2)} \times \boldsymbol{e}^{(3)} &= i\boldsymbol{e}^{(1)*}, \qquad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}, \\ \boldsymbol{e}^{(3)} \times \boldsymbol{e}^{(1)} &= i\boldsymbol{e}^{(2)*}, \qquad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}. \end{aligned} \tag{1.1.2}$$

A closely similar complex circular basis has been described for example by Silver [15], and is well known in tensor analysis.

1.1.1 The Unit Vector Dot Product

In the complex circular basis,

$$\begin{aligned} \boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(2)} = \boldsymbol{e}^{(2)} \cdot \boldsymbol{e}^{(1)} = \boldsymbol{e}^{(3)} \cdot \boldsymbol{e}^{(3)} = 1 \\ \boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(1)} = \boldsymbol{e}^{(2)} \cdot \boldsymbol{e}^{(2)} = 0 . \end{aligned} \tag{1.1.3}$$

In the Cartesian basis,

$$\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1, \qquad \boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{0} . \tag{1.1.4}$$

1.1.2 Vectors

In the complex circular basis the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ can be defined as

$$\begin{aligned} \boldsymbol{A} &:= \boldsymbol{A}^{(1)} + \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(3)} = A^{(1)}\boldsymbol{e}^{(1)} + A^{(2)}\boldsymbol{e}^{(2)} + A^{(3)}\boldsymbol{e}^{(3)}, \\ \boldsymbol{B} &:= \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(2)} + \boldsymbol{B}^{(3)} = B^{(1)}\boldsymbol{e}^{(1)} + B^{(2)}\boldsymbol{e}^{(2)} + B^{(3)}\boldsymbol{e}^{(3)}, \end{aligned} \tag{1.1.5}$$

In these definitions, $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ are scalars, linked to their Cartesian counterparts as follows,

$$A^{(1)} = \frac{1}{\sqrt{2}}\left(A_X - iA_Y\right) = A^{(2)*}, \qquad A^{(3)} = A_Z . \tag{1.1.6}$$

1.1.2.1 Unit Vector Premultipliers

In the logic of the complex circular basis scalar unity is expressed as the product of two complex conjugates, referred to here as *complex unity*,

$$1^2 := 1^{(1)}1^{(2)}, \tag{1.1.7}$$

where,

$$1^{(1)} = \frac{1}{\sqrt{1}}(1 - i), \qquad 1^{(2)} = \frac{1}{\sqrt{2}}(1 + i), \tag{1.1.8}$$

so the dot product of $\boldsymbol{e}^{(1)}$ with $\boldsymbol{e}^{(2)}$ or of vectors $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ is

$$\left.\begin{aligned} \boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(2)} &= 1^{(1)}\boldsymbol{e}^{(1)} \cdot 1^{(2)}\boldsymbol{e}^{(2)} = 1^{(1)}1^{(2)} = 1^2 = 1, \\ \boldsymbol{A}^{(1)} \cdot \boldsymbol{A}^{(2)} &= A^{(1)}\boldsymbol{e}^{(1)} \cdot A^{(2)}\boldsymbol{e}^{(2)} = A^{(1)}A^{(2)} = A^{(0)2} . \end{aligned}\right\} \tag{1.1.9}$$

Since the product $1^{(1)}1^{(2)}$ is always unity, it makes no difference to the dot product of unit vectors or of conjugate vectors such as $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$, but the dot product of a vector $\boldsymbol{A}^{(1)}$ and a unit vector $\boldsymbol{e}^{(2)}$ is

$$\begin{aligned} \boldsymbol{A}^{(1)} \cdot \boldsymbol{e}^{(2)} &= A^{(1)}1^{(2)}\boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(2)} = \frac{1}{2}\left(A_X - iA_Y\right)(1 + i) \\ &= \frac{1}{2}\left(A_X - iA_Y + iA_X + A_Y\right) . \end{aligned} \tag{1.1.10}$$

Similarly, as described in Appendix B of Vol. 3, the dot product of a complex circular Pauli matrix $\sigma^{(1)}$ and a unit vector $\boldsymbol{e}^{(2)}$ is

$$\sigma^{(1)} \cdot \boldsymbol{e}^{(2)} = \frac{1}{2}\left(\sigma_X - i\sigma_Y + i\sigma_X + \sigma_Y\right), \tag{1.1.11}$$

as in that Appendix. This procedure leads to the result of that appendix,

$$\left(\sigma^{(1)} \cdot \boldsymbol{e}^{(2)}\right)\left(\sigma^{(2)} \cdot \boldsymbol{e}^{(2)}\right)^{+} = \boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(2)} + i\sigma^{(3)} \cdot \boldsymbol{e}^{(1)} \times \boldsymbol{e}^{(2)}, \tag{1.1.12}$$

which is the equivalent of the Dirac result of the Cartesian basis.

The complex circular basis is a natural description of the observable conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ and thus of $\boldsymbol{B}^{(3)}$.

1.1.3 Dot Product of Two Vectors

The dot product of two vectors when neither is a unit vector is defined as

$$\begin{aligned}\boldsymbol{A} \cdot \boldsymbol{B} &= A^{(1)}B^{(2)}\boldsymbol{e}^{(1)} \cdot \boldsymbol{e}^{(2)} + A^{(2)}B^{(1)}\boldsymbol{e}^{(2)} \cdot \boldsymbol{e}^{(1)} + A^{(3)}B^{(3)}\boldsymbol{e}^{(3)} \cdot \boldsymbol{e}^{(3)} \\ &= A^{(1)}B^{(2)} + A^{(2)}B^{(1)} + A^{(3)}B^{(3)},\end{aligned} \tag{1.1.13}$$

and is the same as the Cartesian dot product,

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_X B_X + A_Y B_Y + A_Z A_Z . \tag{1.1.14}$$

1.1.4 The Del Operator

The del operator in the complex circular basis is a vector operator which can be defined as

$$\begin{aligned}\nabla_X &= \frac{\partial}{\partial X} = \frac{1}{\sqrt{2}}\left(\nabla^{(1)} + \nabla^{(2)}\right), & \nabla^{(1)} &= \frac{1}{\sqrt{2}}\left(\nabla_X - i\nabla_Y\right), \\ \nabla_Y &= \frac{\partial}{\partial Y} = \frac{i}{\sqrt{2}}\left(\nabla^{(1)} - \nabla^{(2)}\right), & \nabla^{(2)} &= \frac{1}{\sqrt{2}}\left(\nabla_X + i\nabla_Y\right), \\ \nabla_Z &= \frac{\partial}{\partial Z} = \nabla^{(3)}, & \nabla^{(3)} &= \nabla_Z .\end{aligned} \tag{1.1.15}$$

1.1.5 Divergence

The divergence in the complex circular basis is defined as

$$\nabla \cdot \boldsymbol{A} = \nabla^{(1)}A^{(2)} + \nabla^{(2)}A^{(1)} + \nabla^{(3)}A^{(3)} . \tag{1.1.16}$$

1.1.6 Gradient

The gradient of a scalar Φ in the complex circular basis is,

$$\nabla\Phi = \nabla^{(1)}\Phi\boldsymbol{e}^{(2)} + \nabla^{(2)}\Phi\boldsymbol{e}^{(1)} + \nabla^{(3)}\Phi\boldsymbol{e}^{(3)} . \tag{1.1.17}$$

1.1.7 Curl

The curl operator in the complex circular basis is defined as,

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \nabla_X & \nabla_Y & \nabla_Z \\ A_X & A_Y & A_Z \end{vmatrix} = -i\begin{vmatrix} \boldsymbol{e}^{(1)} & \boldsymbol{e}^{(2)} & \boldsymbol{e}^{(3)} \\ \nabla^{(1)} & \nabla^{(2)} & \nabla^{(3)} \\ A^{(1)} & A^{(2)} & A^{(3)} \end{vmatrix} . \tag{1.1.18}$$

For example the $\boldsymbol{k}$ component is

$$\left(\nabla_X A_Y - \nabla_Y A_X\right)\boldsymbol{k} = -i\left(\nabla^{(1)} A^{(2)} - \nabla^{(2)} A^{(1)}\right)\boldsymbol{e}^{(3)} . \tag{1.1.19}$$

The $\boldsymbol{i}$ and $\boldsymbol{j}$ components are

$$\begin{aligned} \boldsymbol{i}\left(\nabla_Y A_Z - \nabla_Z A_Y\right) - \boldsymbol{j}\left(\nabla_X A_Z - \nabla_Z A_X\right) &= \frac{1}{\sqrt{2}}\left(\boldsymbol{e}^{(1)} + \boldsymbol{e}^{(2)}\right) \\ &\times \left(\frac{i}{\sqrt{2}}\left(\nabla^{(1)} - \nabla^{(2)}\right) A^{(3)} - \frac{i}{\sqrt{2}} \nabla^{(3)}\left(A^{(1)} - A^{(2)}\right) \right) \\ &- \frac{i}{\sqrt{2}}\left(\boldsymbol{e}^{(1)} - \boldsymbol{e}^{(2)}\right)\left(\frac{1}{\sqrt{2}}\left(\nabla^{(1)} + \nabla^{(2)}\right) A^{(3)} - \frac{\nabla^{(3)}}{\sqrt{2}}\left(A^{(1)} + A^{(2)}\right) \right) \\ &= -i\left(\left(\nabla^{(2)} A^{(3)} - \nabla^{(3)} A^{(2)}\right)\boldsymbol{e}^{(1)} + \left(\nabla^{(3)} A^{(1)} - \nabla^{(1)} A^{(3)}\right)\boldsymbol{e}^{(2)} \right) . \end{aligned} \tag{1.1.20}$$

1.1.8 The Vector Cross Product

The vector cross product in the complex circular basis is by definition,

$$\begin{aligned} \boldsymbol{A} \times \boldsymbol{B} &:= \left(A^{(2)}\boldsymbol{e}^{(1)} + A^{(1)}\boldsymbol{e}^{(2)} + A^{(3)}\boldsymbol{e}^{(3)}\right) \\ &\times \left(B^{(2)}\boldsymbol{e}^{(1)} + B^{(1)}\boldsymbol{e}^{(2)} + B^{(3)}\boldsymbol{e}^{(3)}\right) \\ &= A^{(2)}B^{(1)}\boldsymbol{e}^{(1)} \times \boldsymbol{e}^{((2)} + \ \dots \ = i\boldsymbol{e}^{(3)*}A^{(2)}B^{(1)} + \ \dots \\ &= i \begin{vmatrix} \boldsymbol{e}^{(1)*} & \boldsymbol{e}^{(2)*} & \boldsymbol{e}^{(3)*} \\ A^{(2)} & A^{(1)} & A^{(3)} \\ B^{(2)} & B^{(1)} & B^{(3)} \end{vmatrix} . \end{aligned} \tag{1.1.21}$$

This result can be checked by working out the $\boldsymbol{e}^{(3)}$ component of $\boldsymbol{A} \times \boldsymbol{B}$, $i\boldsymbol{e}^{(3)*}\left(A^{(2)}B^{(1)} - A^{(1)}B^{(2)}\right)$, where,

$$\begin{aligned} A^{(2)} &= \frac{1}{\sqrt{2}}\left(A_X + iA_Y\right) = A^{(1)*} , \\ B^{(1)} &= \frac{1}{\sqrt{2}}\left(B_X - iB_Y\right) = B^{(2)*} . \end{aligned} \tag{1.1.21a}$$

So,

$$\begin{aligned} A^{(2)}B^{(1)} - B^{(2)}A^{(1)} &= -i\left(A_X B_Y - B_X A_Y\right) \\ \text{and } \left(A_X B_Y - A_Y B_X\right)\boldsymbol{k} &= i\boldsymbol{e}^{(3)*}\left(A^{(2)}B^{(1)} - B^{(2)}A^{(1)}\right) . \end{aligned} \tag{1.1.22}$$

The conjugate product can be checked by direct evaluation to be

$$A^{(1)} \times A^{(2)} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_X^{(1)} & A_Y^{(1)} & 0 \\ A_X^{(2)} & A_Y^{(2)} & 0 \end{vmatrix} = \left(A_X^{(1)} A_Y^{(2)} - A_Y^{(1)} A_X^{(2)}\right)\boldsymbol{k} = iA^{(0)}\boldsymbol{A}^{(3)*} . \tag{1.1.23}$$

1.1.9 The Cyclic Relations

The cyclic relations in the ((1), (2), (3)) basis are

$$\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = iA^{(0)}\boldsymbol{A}^{(3)*} , \quad \text{et cyclicum} \tag{1.1.24}$$

and so on for any vector.

1.2 The Electromagnetic Field Tensor in *O(3)* Electrodynamics

The basic concepts of this section were first tried out in Vol. 2 but here we offer a considerable clarification and simplification based on intervening experience and discussion. The basic ideas of general field theory are described for example in Ryder [5] in his Chap. 3, and were first applied to electrodynamics in Vol. 2 in a didactic manner. In order to understand these ideas at all, two concepts in particular are needed which do not exist in *U(1)* electrodynamics: that of the internal space and the covariant derivative as defined in this space. These new and perhaps unfamiliar ideas are best illustrated when it comes to gauge transformation in *O(3)*, which is developed in the next section. This section gives basic ideas and at each stage is careful to spot the difference between *U(1)* and *O(3)*. In this way the interested student can gradually absorb the new material and realize its

advantages. In so doing the need for an *O(3)* electrodynamics becomes ever clearer, and several advantages over *U(1)* start to be defined.

1.2.1 The Internal Space in *O(3)* Electrodynamics

The internal space is defined through the expansion of the potential in space-time to an object which has meaning additionally in a space defined by a particular group structure [1—6]. It becomes necessary to think of the familiar A^{μ} of the received view as a scalar object in this internal space as well as a four-vector in space-time. This idea appears to have been first applied to field theory by Yang and Mills in 1955 [8,9], and has since been developed in many very fruitful ways within the standard model. It was first applied by Barrett [8,9] to classical electrodynamics in the late eighties, and slightly later [1—4] it was shown to lead to the existence of the fundamental field $\boldsymbol{B}^{(3)}$, an object that is missing from the received view. It is not so much that the latter sets $\boldsymbol{B}^{(3)}$ to zero, it is a concept that simply does not appear within its horizon. So it is clear that *O(3)* or *SU(2)* electrodynamics was inferred independently by Barrett [8,9] and by Evans [1—4].

Therefore if A^{μ} is thought of as a scalar object in some internal space, conceptually and empirically ((1), (2), (3)), it becomes possible to write a potential that becomes a vector in the internal space, and whose scalar components in this space are also objects in space-time. This is the basic hypothesis of *O(3)* electrodynamics, and we can write in consequence of this hypothesis,

$$\boldsymbol{A}^{\mu} = A^{\mu(1)}\boldsymbol{e}^{(1)} + A^{\mu(2)}\boldsymbol{e}^{(2)} + A^{\mu(3)}\boldsymbol{e}^{(3)} . \tag{1.1.25}$$

The unit vectors $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$, and $\boldsymbol{e}^{(3)}$ form a complex basis for internal space, and the objects $A^{\mu(1)}$, $A^{\mu(2)}$ and $A^{\mu(3)}$ are scalar coefficients in the internal space of the complete vector $\boldsymbol{A}^{\mu}$. This boldface character therefore denotes an object that is simultaneously a vector in the internal space (a symmetry space [8,9] of gauge field theory) and a four-vector in space-time

(Minkowski space). The indices of the scalar coefficients $A^{\mu(1)}$, $A^{\mu(2)}$ and $A^{\mu(3)}$ must therefore match self consistently.

1.2.1.1 Index Matching

If we consider the received view of ordinary plane waves in space-time [1—4],

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} = \frac{A^{(0)}}{\sqrt{2}} \left(i\boldsymbol{i} + \boldsymbol{j} \right) e^{-i\phi} , \tag{1.1.26}$$

it should be clear that the boldface character $\boldsymbol{A}^{(1)}$ represents a vector in the ordinary space part of space-time. The electromagnetic phase is defined as $\phi := \omega t - \kappa Z$ where ω is the angular frequency at instant t and κ the wave-vector at point Z.

These plane waves are transverse solutions of the received *U(1)* field equations and the d'Alembert wave equation for the free field [1—9]. In order to expand the horizon of the gauge structure of classical electrodynamics from *U(1)* to *O(3)* an additional space-time index must appear in the definition of the plane wave and the (1) and (2) indices must become indices of the internal space. This is achieved by recognizing that:

$$\left.\begin{aligned} A^{1(1)} &= A_X^{(1)} = i\frac{A^{(0)}}{\sqrt{2}} e^{-i\phi} = A^{1(2)*} , \\ A^{2(1)} &= A_Y^{(1)} = \frac{A^{(0)}}{\sqrt{2}} e^{-i\phi} = A^{2(2)*} , \\ A^{0(1)} &= A^{3(1)} = A^{0(2)} = A^{3(2)} = 0 , \end{aligned}\right\} \tag{1.1.27}$$

These equations define two of the scalar coefficients of the complete four-vector A^{μ} ,

$$\left.\begin{aligned} A^{\mu(1)} &= \left(0, \boldsymbol{A}^{(1)}\right) \\ A^{\mu(2)} &= \left(0, \boldsymbol{A}^{(2)}\right) . \end{aligned}\right\} \tag{1.1.28}$$

This process follows from the fact that $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ are transverse, and so can have X and Y components only. The scalar coefficients $A^{\mu(1)}$ and $A^{\mu(2)}$ are light-like invariants [16,17],

$$A^{\mu(1)} A_{\mu}^{(1)} = A^{\mu(2)} A_{\mu}^{(2)} = 0 , \tag{1.1.29}$$

of polar four-vectors in space-time. The third index (3) of the non Abelian theory must therefore be along the direction of propagation of the radiation and must also be a light-like invariant,

$$A^{\mu(3)} A_{\mu}^{(3)} = 0 , \tag{1.1.30}$$

in the vacuum.. It must be light-like because the free field is assumed to propagate, in this classical view, at c in the vacuum..

One possible solution of Eq. (1.1.30) is

$$A^{\mu(3)} = \left(cA^{(0)}, \boldsymbol{A}^{(3)}\right) , \tag{1.1.31}$$

where

$$cA^{(0)} = \left|\boldsymbol{A}^{(3)}\right| . \tag{1.1.32}$$

Such a solution is proportional directly to the wave four-vector,

$$\kappa^{\mu(3)} := \left(c\kappa, \kappa \boldsymbol{e}^{(3)}\right) = eA^{\mu(3)} , \tag{1.1.33}$$

and to the photon energy-momentum,

$$p^{\mu(3)} := \hbar\kappa^{\mu(3)} = eA^{\mu(3)} , \quad (1.1.34)$$

where $\hbar$ is the Dirac constant and $-e$ is the unit of charge, the charge on the electron accelerated to c. Therefore Eq. (1.1.31) quantizes directly to Eq. (1.1.34), giving the Planck Law,

$$En = \hbar\omega = \hbar c\kappa . \quad (1.1.35)$$

This is the same in *O(3)* and *U(1)* electrodynamics. However, the complete vector A_μ in the internal ((1), (2), (3)) space of *O(3)* is the light-like polar vector,

$$A^{\mu} = \left(0, A^{(1)}\right)e^{(1)} + \left(0, A^{(2)}\right)e^{(2)} + \left(cA^{(0)}, A^{(3)}\right)e^{(3)} , \quad (1.1.36)$$

and has time-like, longitudinal and transverse components which are each physical. These concepts do not exist in the *U(1)* hypothesis, in which the time-like and longitudinal components are combined to give what is asserted conventionally to be a physical admixture [5].

To summarize, the differences between the *U(1)* and *O(3)* theories are as follows:

1) In *U(1)*, the physical object that we started with was a transverse plane wave with no longitudinal or time-like components. The internal space was a scalar space, and the physical entity was $A^{\mu} = A^{\mu *}$.

2) In *O(3)*, the physical object has become transverse, longitudinal and time-like, and the internal gauge space has become a vector space with *O(3)* rotation group symmetry. This leads directly to the Planck Law through Eq. (1.31), a concept which does not exist in the classical *U(1)* hypothesis. We begin to see advantages in the *O(3)* hypothesis.

1.2.2 Field Tensor from Field Potential

With these definitions, the rules of general gauge field theory can be applied to electrodynamics. The groundwork for this was provided in Vol. 2 of this series, and the fundamental methods are given by Ryder [5]. It is first necessary to define the field tensor in *O(3)* through the field potential. The field tensor is also a vector in the internal *O(3)* gauge space,

$$\boldsymbol{G}^{\mu\nu} = G^{\mu\nu(1)}\boldsymbol{e}^{(1)} + G^{\mu\nu(2)}\boldsymbol{e}^{(2)} + G^{\mu\nu(3)}\boldsymbol{e}^{(3)} , \quad (1.1.37)$$

and the coefficients $G^{\mu\nu(i)}$, $i = 1, 2, 3$, are scalar coefficients of the internal space. They are also antisymmetric tensors in Minkowski space-time.

General gauge field theory for *O(3)* symmetry [1—9] then gives

$$\begin{aligned}
\boldsymbol{G}^{\mu\nu(1)*} &= \partial^{\mu}\boldsymbol{A}^{\nu(1)*} - \partial^{\nu}\boldsymbol{A}^{\mu(1)*} - ig\boldsymbol{A}^{\mu(2)} \times \boldsymbol{A}^{\nu(3)} , \\
\boldsymbol{G}^{\mu\nu(2)*} &= \partial^{\mu}\boldsymbol{A}^{\nu(2)*} - \partial^{\nu}\boldsymbol{A}^{\mu(2)*} - ig\boldsymbol{A}^{\mu(3)} \times \boldsymbol{A}^{\nu(1)} , \\
\boldsymbol{G}^{\mu\nu(3)*} &= \partial^{\mu}\boldsymbol{A}^{\nu(3)*} - \partial^{\nu}\boldsymbol{A}^{\mu(3)*} - ig\boldsymbol{A}^{\mu(1)} \times \boldsymbol{A}^{\nu(2)} ,
\end{aligned} \quad (1.1.38)$$

which is a relation between vectors in the internal space ((1), (2), (3)). The cross product notation is also a vector notation, for example $\boldsymbol{A}^{\mu(2)} \times \boldsymbol{A}^{\nu(3)}$ is a cross product of a vector $\boldsymbol{A}^{\mu(2)}$ with the vector $\boldsymbol{A}^{\nu(3)}$ in the internal space. In forming this cross product, the Greek indices μ and ν are not transmuted, and the complex basis ((1), (2), (3)) is used, so that the terms quadratic in A become natural descriptions of the empirically observable conjugate product. It will be shown that these terms give rise to vacuum polarization and vacuum magnetization in *O(3)* but not in *U(1)* electrodynamics. The definition (1.1.38) is for the free field in regions free of matter and free of charge/current interaction. The scalar coefficient g is a scalar both in the internal gauge space, a symmetry space, and also in Minkowski space-time. In the vacuum it is given by [1—4],

$$g = \frac{\kappa}{A^{(0)}} = \frac{e}{\hbar} , \quad (1.1.39)$$

and is the inverse of the quantum of magnetic flux, $\hbar/e$. Evidently, Eq. (1.1.39) is a fundamental quantum relation for one photon. In field matter interaction g changes in magnitude and is empirically determined through the Verdet constant in the inverse Faraday effect, and the non linear terms in Eq. (1.1.39) (those quadratic in $\boldsymbol{A}$) become negligible under most conditions [18]. The *O(3)* theory then reduces to the same algebraic form as the *U(1)* theory for $G^{\mu\nu(1)} = G^{\mu\nu(2)*}$, *i.e.,* reduces to the homogeneous and inhomogeneous Maxwell equations for the complex conjugate field tensors $G^{\mu\nu(1)}$ and $G^{\mu\nu(2)}$. This is the linear approximation which neglects all non linear optical phenomena such as the inverse Faraday effect. The latter is described through equations for $G^{\mu\nu(3)}$, which is always quadratic in the potential and always non linear. This tensor, $G^{\mu\nu(3)}$, contains only the $\boldsymbol{B}^{(3)}$ field. Self consistently, therefore, the $\boldsymbol{B}^{(3)}$ field is undefined in the linear approximation, which is Maxwell's theory. Note that g is never zero in free space, however, and in this condition the *O(3)* electrodynamics differs fundamentally from its *U(1)* counterpart because in free space the magnitude of the non linear term is the same as those linear in $\boldsymbol{A}$.

The main difference between *O(3)* and *U(1)* in this section are therefore as follows:

1) the field tensor in *U(1)* is well known to be the antisymmetric four-curl:

$$G^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}, \tag{1.1.40}$$

and there is a scalar internal gauge space, *i.e.,* $G^{\mu\nu}$ is a scalar in this space and an antisymmetric tensor in Minkowski space-time. The field tensor is linear in the field potential, and only transverse components are present in *U(1).*

2) The field tensor in *O(3)* is a vector in the internal gauge space ((1), (2), (3)) and is non linear in the field potential. It contains the longitudinal and fundamental magnetic flux density component $\boldsymbol{B}^{(3)}$.

1.2.2.1 Details of Equation (1.1.38)

Equation (1.1.38) is a concise description which contains a considerable amount of information about the *O(3)* theory of electromagnetism in free space. This information is obtainable without assuming any form of field equation, and so, details are given in this section of the correct algebraic methods of reduction. Considering for example the equation,

$$\boldsymbol{G}^{\mu\nu(1)*} = \partial^{\mu}\boldsymbol{A}^{\nu(1)*} - \partial^{\nu}\boldsymbol{A}^{\mu(1)*} - ig\boldsymbol{A}^{\mu(2)} \times \boldsymbol{A}^{\nu(3)} . \tag{1.1.41}$$

$$G^{12(1)*} = \partial^{1}A^{2(1)*} - \partial^{2}A^{1(1)*} - ig\epsilon_{(1)(2)(3)}A^{1(2)}A^{2(3)} , \tag{1.1.42}$$

This equation consists of components such as, where $\epsilon_{(1)(2)(3)}$ is the Levi Civita symbol, defined by

$$\epsilon_{(1)(2)(3)} := 1 = -\epsilon_{(1)(2)(3)} = \epsilon_{(2)(3)(1)} = ... \tag{1.1.43}$$

If we now take the vector potential as defined in Section (1.2.1.1), with

$$\partial^{\mu} := \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla \right) , \tag{1.1.44}$$

then,

$$\begin{aligned} G^{12(1)*} &= \partial^{1}A^{2(1)*} - \partial^{2}A^{1(1)*} \\ &- ig\left(A^{1(2)}A^{2(3)} - A^{1(3)}A^{2(2)}\right) = 0 . \end{aligned} \tag{1.1.45}$$

This is a self-consistent result because there is no Z component of $G^{\mu\nu(1)*}$, which is defined as transverse. Both the linear and non linear components are zero.

We next consider the element,

$$
\begin{aligned}
G^{13(1)*} &= \partial^1 A^{3(1)*} - \partial^3 A^{1(1)*} - ig\epsilon_{(1)(2)(3)} A^{1(2)} A^{3(3)} \\
&= \partial^1 A^{3(2)} - \partial^3 A^{1(2)} - ig\left(A^{1(2)}A^{3(3)} - A^{1(3)}A^{3(2)}\right) \\
&= -\left(\partial^3 + igA^{3(3)}\right)A^{1(2)} = -\left(\partial^3 + i\kappa\right)A^{1(2)},
\end{aligned}
\tag{1.1.46}
$$

where we have used,

$$
g = \frac{\kappa}{A^{(0)}}, \qquad A^{3(3)} = A_Z^{(3)} = A^{(0)} . \tag{1.1.47}
$$

It can be seen that there are two contributions to the field element $G^{13(2)}$, a magnetic field component:

1) the linear contribution, $-\partial^3 A^{1(2)}$;
2) the non-linear contribution, $-igA^{3(3)}A^{1(2)}$.

In vector notation, Eq. (1.1.46) is a component of,

$$
\begin{aligned}
2\boldsymbol{B}^{(1)} &:= \nabla \times \boldsymbol{A}^{(1)} - ig\boldsymbol{A}^{(3)} \times \boldsymbol{A}^{(1)} \\
&= \left(\nabla - ig\boldsymbol{A}^{(3)}\right) \times \boldsymbol{A}^{(1)} \\
&= \nabla \times \boldsymbol{A}^{(1)} - \frac{i}{B^{(0)}}\boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} .
\end{aligned}
\tag{1.1.48}
$$

Furthermore,

$$
\partial^3 A^{1(2)} = i\kappa A^{1(2)}, \tag{1.1.49}
$$

and so it follows that

$$
\boldsymbol{B}^{(1)} = \nabla \times \boldsymbol{A}^{(1)} = -\frac{i}{B^{(0)}}\boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} . \tag{1.1.50}
$$

Similarly,

$$
\boldsymbol{B}^{(2)} = \nabla \times \boldsymbol{A}^{(2)} = -\frac{i}{B^{(0)}}\boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} . \tag{1.1.51}
$$

Therefore the definition of the field tensor in *O(3)* electrodynamics gives the first two components of the B Cyclic Theorem [1—4],

$$
\left.
\begin{aligned}
\boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} &= iB^{(0)}\boldsymbol{B}^{(2)*} \\
\boldsymbol{B}^{(2)} \times \boldsymbol{B}^{(3)} &= iB^{(0)}\boldsymbol{B}^{(1)*}
\end{aligned}
\right\},
\tag{1.1.52}
$$

together with the definition of $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ in terms of the curl of vector potentials $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$,

$$
\left.
\begin{aligned}
\boldsymbol{B}^{(1)} &= \nabla \times \boldsymbol{A}^{(1)}, \\
\boldsymbol{B}^{(2)} &= \nabla \times \boldsymbol{A}^{(2)}.
\end{aligned}
\right\}
\tag{1.1.53}
$$

It is convenient to write this important result as

$$
\boldsymbol{H}(\text{vac.}) = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}(\text{vac.}) , \tag{1.1.54}
$$

where $\boldsymbol{H}(\text{vac})$ is the vacuum magnetic field strength and μ_0 the permeability in vacuo. The object $\boldsymbol{M}(\text{vac})$ does not exist in *U(1)* electrodynamics and is the *vacuum magnetization*, for example,

$$
\boldsymbol{M}^{(1)}(\text{vac}) = -\frac{1}{i\mu_0 B^{(0)}}\boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} . \tag{1.1.55}
$$

The objects $\boldsymbol{M}^{(1)}(\text{vac.})$ and $\boldsymbol{M}^{(2)}(\text{vac.})$ depend on the phase-less vacuum magnetic field $\boldsymbol{B}^{(3)}$ and so does not exist as a concept in *U(1)* electrodynamics. The $\boldsymbol{B}^{(3)}$ field itself is defined through

$$G^{\mu\nu(3)*} = \partial^\mu A^{\nu(3)*} - \partial^\nu A^{\mu(3)*} - igA^{\mu(1)} \times A^{\nu(2)} , \tag{1.1.56}$$

with (3) aligned for convenience in the Z axis. So by definition, the only non zero components are

$$G^{12(3)*} = -G^{12(3)*} = B_Z^{(3)} . \tag{1.1.57}$$

It follows that

$$B_Z^{(3)} = -ig\left(A^{1(1)}A^{2(2)} - A^{1(2)}A^{2(1)}\right) , \tag{1.1.58}$$

or

$$\boldsymbol{B}^{(3)} = \boldsymbol{B}^{(3)*} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = -\frac{i}{B^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} , \tag{1.1.59}$$

giving the third component of the B Cyclic Theorem, $\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*}$, and the vacuum magnetization,

$$\boldsymbol{M}^{(3)*} = -\frac{1}{i\mu_0 B^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} . \tag{1.1.60}$$

These results are all absent from *U(1)* electrodynamics, but we know from Section (1.2.1.1) that they are consistent with the plane waves $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$. We shall return to this point later, in the context of the *O(3)* field equations and their linearization. Note that $\boldsymbol{B}^{(3)}$ is always defined through $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)*}$ and is *not* the curl of $\boldsymbol{A}^{(3)}$. The conjugate product is an observable of magneto-optics and so $\boldsymbol{B}^{(3)}$ is non-zero *empirically* in *O(3)* electrodynamics. In *U(1)* electrodynamics it is rigorously zero, and $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in *U(1)* electrodynamics is considered to be an operator with no Z component. This is in clear conflict with vector algebra, in that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is aligned in the (3) axis. For this reason we prefer to develop

O(3) electrodynamics systematically, and reduce it to the Maxwell equations using linearization approximations where applicable.

To summarize what we have found so far, in *O(3)* electrodynamics (hereinafter frequently referred to just as "*O(3)*") the magnetic part of the complete free field is defined as a sum of the curl of a vector potential and a vacuum magnetization. The latter is inherent in the structure of the B Cyclic Theorem [1—4]. In *U(1)* electrodynamics there is no $\boldsymbol{B}^{(3)}$ field by definition (or more accurately, by hypothesis) and in consequence there is no vacuum magnetization in classical *U(1)* electrodynamics. In *O(3)* the $\boldsymbol{B}^{(3)}$ field is always proportional by hypothesis to the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, which in field matter interaction is an optical observable. The $\boldsymbol{B}^{(3)}$ field is not the curl of a vector potential, and this is a clear departure from the *U(1)* hypothesis of classical electrodynamics. The phase-less $\boldsymbol{B}^{(3)}$ is instead directly proportional in free space to the phase-less $\boldsymbol{A}^{(3)}$ through the scalar relation $B^{(0)} = \kappa A^{(0)}$ [1—4]. These results are obtained self consistently from the definition of the field from the potentials in the *O(3)* gauge theory. We have calculated the field coefficients:

$$\begin{aligned}
G^{01(2)} &= \left(\partial^0 + igA^{0(3)}\right)A^{1(2)} = -G^{10(2)} ,\\
G^{02(2)} &= \left(\partial^0 + igA^{0(3)}\right)A^{2(2)} = -G^{20(2)} ,\\
G^{03(2)} &= 0 ,\\
G^{13(2)} &= -\left(\partial^3 + igA^{3(3)}\right)A^{1(2)} = -G^{31(2)} ,\\
G^{23(2)} &= -\left(\partial^3 + igA^{3(3)}\right)A^{2(2)} = -G^{32(2)} ,\\
G^{12(2)} &= 0 .
\end{aligned} \tag{1.1.61}$$

Similarly,

$$G^{01(1)} = G^{01(2)*} = \left(\partial^0 - igA^{0(3)}\right)A^{1(1)} , \tag{1.1.62}$$

and so on, and,

$$G^{12(3)*} = -G^{21(3)*} = -ig\left(A^{1(1)}A^{2(2)} - A^{1(2)}A^{2(1)}\right) . \tag{1.1.63}$$

The three field tensors are

$$G^{\mu\nu(1)} = G^{\mu\nu(2)*} = \begin{bmatrix} 0 & -E^{1(1)} & -E^{2(1)} & 0 \\ E^{1(1)} & 0 & 0 & cB^{2(1)} \\ E^{2(1)} & 0 & 0 & -cB^{1(1)} \\ 0 & -cB^{2(1)} & cB^{1(1)} & 0 \end{bmatrix}, \tag{1.1.64}$$

the transverse tensor; and the longitudinal

$$G^{\mu\nu(3)*} = G^{\mu\nu(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -cB^{3(3)} & 0 \\ 0 & cB^{3(3)} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.1.65}$$

In classical $O(3)$ electrodynamics there also exists a vacuum polarization, because the complete electric field strength in the vacuum is given by

$$\begin{aligned} 2E^{(2)} &:= -\frac{\partial A^{(2)}}{\partial t} - igcA^{(0)}A^{(2)} \\ &= -\left(\frac{\partial}{\partial t} + igcA^{(0)}\right)A^{(2)} \\ &= 2E^{(1)*} \end{aligned} \tag{1.1.66}$$

Using $g = \kappa / A^{(0)}$,

$$E^{(2)} = -\frac{\partial A^{(2)}}{\partial t} = -ic\kappa A^{(2)} = -i\omega A^{(2)}, \tag{1.1.67}$$

and it is convenient to express this result as

$$\frac{1}{\epsilon_0} D^{(2)}(\text{vac.}) = E^{(2)} + \frac{1}{\epsilon_0} P^{(2)}(\text{vac.}), \tag{1.1.68}$$

where $D^{(2)}(\text{vac.})$ is the electric displacement in vacuo and where the vacuum polarization is $P^{(2)}(\text{vac.}) = -i\epsilon_0 \omega A^{(2)}$, where ϵ_0 is the vacuum permittivity.

The vacuum polarization is well known to have an analogue in quantum electrodynamics: the photon self energy [5]. This has no classical analogue in $U(1)$ electrodynamics, but is clearly defined in $O(3)$ electrodynamics. The classical $O(3)$ vacuum polarization is transverse and vanishes when $\omega = 0$, so has no meaning in electrostatics. This is consistent with the fact that it is the analogue of photon self energy in quantum electrodynamics. Finally, it is pure transverse, because the hypothetical $E^{(3)}$ field is zero in $O(3)$ electrodynamics,

$$\begin{aligned} G^{03(3)*} &= \partial^0 A^{3(3)*} - \partial^3 A^{0(3)*} \\ &- ig\left(A^{0(1)}A^{3(2)} - A^{3(2)}A^{0(1)}\right) = 0, \end{aligned} \tag{1.1.69}$$

and so

$$G^{03(1)} = G^{03(2)} = G^{03(3)} = 0, \tag{1.1.70}$$

in the vacuum. In the presence of field matter interaction this result is no longer true because of the Coulomb field, indicating polarization of matter. Polarization of the vacuum takes place through transverse components only. Again this result is missing from $U(1)$ theory.

1.2.3 Field Matter Interaction

In the presence of field matter interaction the *O(3)* field tensor equivalent to that in Eq. (1.1.38) of Section (1.2.2) becomes

$$\frac{1}{\epsilon_0}H^{\mu\nu(i)*} = F^{\mu\nu(i)*} - \frac{1}{\epsilon_0}M^{\mu\nu(i)*}, \qquad (1.1.71)$$

where $i = 1, 2, 3$. Here,

$$\left.\begin{array}{l} F^{\mu\nu(i)} := \partial^\mu A^{\nu(i)} - \partial^\nu A^{\mu(i)}, \\ M^{\mu\nu(1)} := i\epsilon_0 g' A^{\mu(2)} \times A^{\nu(3)} \end{array}\right], \qquad (1.1.72)$$

in cyclic permutation, with $g' \ll g$ empirically [1—4].

1.2.3.1 Example, the Inverse Faraday Effect

In the inverse Faraday effect we have,

$$F^{\mu\nu(3)*} = \mathbf{0}, \qquad (1.1.75a)$$

$$M^{\mu\nu(3)*} = i\epsilon_0 g' A^{\mu(1)} \times A^{\nu(2)}. \qquad (1.1.75b)$$

Equation (1.1.75a) means that the free space $B^{(3)}$ is zero if we attempt to define it as a conventional *U(1)* four-curl. Equation (1.1.75b) in vector notation is

$$M^{(3)*} = i\epsilon^0 g' A^{(1)} \times A^{(2)}, \qquad (1.1.76)$$

which is the empirically observed phase free magnetization of the inverse Faraday effect [1—4]. This is a small effect and so $g' \ll g$ empirically. The factor g' for field matter interaction is much smaller than g in free space. In other words the covariant derivative changes its nature when there is field matter interaction, and loosely speaking, this is "bending of space-time" in the presence of charge, akin to bending of space-time in the presence of mass in general relativity. (Recall that the idea of covariant derivative is borrowed from general relativity.) In general, g' is relativistic, and an example of its development is given in Vol.1 [1]. We see that the inverse Faraday effect plays a central role in *O(3)* electrodynamics, which is able to describe the phenomenon from the basic definition of the field tensor. It follows that

$$M^{(3)*} = -\epsilon_0 \frac{g'}{g} B^{(3)}, \qquad (1.1.77)$$

for the inverse Faraday effect, which is therefore a direct observation of $B^{(3)}$. Recall that in *U(1)* electrodynamics,

$$A^{(1)} \times A^{(2)}(\,U(1)\,) = \mathbf{0}, \qquad (1.1.78)$$

and so *U(1)* gauge field theory as applied to electrodynamics does not describe the inverse Faraday effect. The phenomenological invocation of non zero $A \times A^* = A^{(1)} \times A^{(2)}$ [1—4] to describe the inverse Faraday effect in *U(1)* theory therefore leads to a paradox, in that the observable does not exist in *U(1)* gauge field theory by definition. The lowest symmetry in which $A^{(1)} \times A^{(2)*}$ exists is *O(3)* [6,8,9], as argued here. The development of *O(3)* = *SU(2)* electrodynamics leads to several major advantages as described by Barrett [8,9] and elsewhere [1—4].

1.2.3.2 Some Conceptual Similarities to and Differences from Yang Mills Theory in High Energy Physics

There are obvious points of similarity between the *O(3)* theory of electrodynamics and conventional Yang-Mills theory in particle physics. Both theories are based on an *SU(2)* = *O(3)* Lagrangian and the structure of the field tensor and field equations is fundamentally the same. However, there are some differences also. One of these is that in *O(3)* electrodynamics the presence of the non-linearity preceded by g or g' in the field tensor definition does not mean that the particle concomitant with the gauge field is a charged particle. In *O(3)* electrodynamics, the field does not act as its own source because the nonlinearities in the definition of the field tensor are interpretable as vacuum polarization and magnetization. The g constant in *O(3)* electrodynamics is proportional to the charge e, (the charge on the proton), but it is well known that the electron accelerated to the speed of light takes on the attributes of a classical electromagnetic *field* as argued by Jackson [19]. This does not mean that the field is charged. It is also well known that the vector potential is C negative, and is proportional to e in the vacuum, but again, A^{μ} is not charged.

As argued in Chap. 2, the internal (gauge) space, and space-time in classical electrodynamics are not independent spaces, they form an extended Lie algebra as defined by Aldrovandi [20] and discussed in detail in Chap. 2. In particle theory the internal space is usually ascribed to an isospin which is independent of space-time. Generally, however, the internal gauge space is a symmetry space and the basis ((1), (2), (3)) has *O(3)* symmetry. Finally, the constant g is defined by Eq. (1.1.47) in free space, but in field-matter interaction is much smaller in magnitude, as determined empirically and from phenomenological, or semi-classical, non-linear optical theory [1—4]. In elementary particle theory the parameter g is usually interpreted as a constant. However, the structure of the gauge field theory is the same for elementary particle theory and electrodynamics. If the latter is quantized, g becomes a constant $e/\hbar$ in free space [1—4], and in field-matter interaction becomes a coefficient proportional to $e/\hbar$. Evidently, the elementary charge

e is the same scalar quantity in both *U(1)* and *O(3)*, *i.e.*, the charge on the proton, the negative of the charge on the electron ($-e$).

1.3 Gauge Transformation in *O(3)* Electrodynamics

There is a profound difference between *U(1)* and *O(3)* electrodynamics in respect of gauge transformation, and so it is important to give considerable calculational detail as in this section. In *U(1)* the potential is subsidiary to the field, as argued by Heaviside and contemporaries in the late nineteenth century. It was Heaviside's avowed intention to murder the potential, but in *O(3)* it springs to life again, as we shall find. In *U(1)*, the gauge transformation process is in the last analysis a mathematical convenience, because the gradient of an arbitrary variable is added to the original $\boldsymbol{A}$. This means that gauge transformation of the second kind essentially adds a random quantity to the electromagnetic phase. In non-Abelian gauge field theory applied to classical electrodynamics, the gauge transformation becomes essentially a geometrical process, and there is a well defined topological phase effect [8,9], related to the Aharonov-Bohm effect [8,9]. This is an observed phase effect, and is not random. There are several other features of *O(3)* which do not occur in *U(1)*, and in respect of gauge transformation, the two theories are very different in nature. The main difference is that the potential in *O(3)* and higher symmetry electrodynamic theories is always a physical object, never a mathematical subsidiary variable. In *O(3)*, gauge transformation is controlled by the rules of general gauge field theory as described for example by Ryder [5]. Such ideas form the basis for contemporary gauge field theories such as instanton theory in high energy physics. They are being applied increasingly to low energy physics and to electrodynamics [8,9]. The careful work by Barrett [8,9] in favor of the physical nature of the *classical* electromagnetic potential, and in favor of *SU(2)* = *O(3)* electrodynamics, appears to be irrefutable to the state of the art, based as it is on several different effects of nature. Since $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is missing by definition [1—4] from *U(1)* gauge field theory applied to classical electrodynamics ("*U(1)*" for short) the various non-linear magneto-optical effects [18] may be added to the list given by Barrett. If so, it follows that $O(3) = SU(2)$ symmetry is to be preferred over $U(1)$ for a more consistent view of optics, in a theoretical framework which envelops both linear and non-linear phenomena. This is a powerful geometrical argument in favor of $O(3)$ because in $U(1)$, the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ must be an operator with no longitudinal component. This makes no sense in three dimensional space, since by analogy with the longitudinally directed Poynting vector, a cross product of transverse field components; $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ must also be longitudinally directed for elementary consistency. Similarly, the angular momentum of a classical electromagnetic beam is longitudinally directed, as argued by Jackson [19]. So the $U(1)$ appellation in classical electrodynamics can refer at best only to the Lagrangian. In other contexts it is self contradictory as evidenced in the vacuum by the Poynting vector, or angular momentum vectors, both of which are perpendicular to the plane of the $O(2) = U(1)$ symmetry group, and both of which are empirical observables in respectively the Lebedev and Beth effects [4]. Similarly for $\boldsymbol{B}^{(3)}$, and $O(3)$ is to be preferred to deal with non linear phenomena within gauge field theory. Such phenomena present an Achilles heel of the standard model as discussed here and elsewhere [1—4]. General gauge field theory has been notably successful in elementary particle theory [5], and may be as successful in classical electrodynamics, but with conceptual differences as discussed already. An important difference appears at present to be that the two spaces in $O(3)$ are not independent. The $O(3)$ hypothesis has the major advantage of being able to incorporate within one structure non-linear and linear phenomena of optics, and also to logically accommodate such quantities as the Poynting vector as just discussed. There is no doubt that this vector is longitudinally directed and outside the plane of definition of $O(2) = U(1)$. It is not consistent to apply $O(2)$ to an energy combination (the Lagrangian) and not to the momentum of the same field, the Poynting vector. In $O(3)$ the Lagrangian and momentum have the symmetry of three dimensional space, the internal gauge space.

In order to progress from $U(1)$ to $O(3)$ the concepts of gauge transformation in general field theory are illustrated in detail in this section to show that the gauge transform process is essentially geometrical. In $U(1)$, the gauge transform is essentially a matter of adding to the magnetic vector potential the gradient of a function which can have any value whatsoever without affecting the original magnetic field. So this is an arbitrary, or random, process in the sense that a random mathematical quantity has no physical meaning unless subjected to thermodynamic averaging. Yet the incorporation of such a quantity is precisely the basis of $U(1)$ gauge transformation of the second kind [6,8,9]. In $U(1)$, the electromagnetic phase is random.

1.3.1 The Fundamental Gauge Transform Equations

In the condensed matrix notation used by Ryder [5], the basic equations of gauge transformation in general field theory are as follows

$$G_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right], \tag{1.1.79a}$$

$$G'_{\mu\nu} = S G_{\mu\nu} S^{-1}, \tag{1.1.79b}$$

$$A'_\mu = \left(S A_\mu - \frac{i}{g}\partial_\mu S \right) S^{-1}. \tag{1.1.79c}$$

In this notation, S is a rotation matrix, A_μ is a matrix generated from the vector potential, and $G_{\mu\nu}$, the field matrix, is defined as the commutator of covariant derivatives, D_μ. Gauge transformation as in Eq. (1.1.79c) is a rotation using curvilinear coordinates, one which changes covariantly. These equations represent physical rotation. If $O(3)$, the rotation group, is used as the background or internal gauge field symmetry of the field theory, the rotation takes place in three dimensions. These ideas have been applied to electromagnetism in previous volumes [1—4], to which the interested reader is referred for more detail. In this section full details of the gauge transform process are given for a rotation about the Z axis.

1.3.2 Background Mathematical Detail

Some background details of the operation of rotation in three dimensional space are given in this section in order to prepare the way for the detailed development of Eqs. (1.1.79a) to (1.1.79c). We consider the Euler angles α, β, and γ and the quaternion coefficients q_0, q_1, q_2 and q_3. Define,

$$\left.\begin{aligned} a &= q_0 + iq_3 = \cos\frac{\beta}{2}\exp\left(\frac{i}{2}(\alpha+\gamma)\right) \\ b &= q_1 - iq_2 = \sin\frac{\beta}{2}\exp\left(\frac{-i}{2}(\alpha-\gamma)\right) \end{aligned}\right\} \tag{1.1.80}$$

with

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 . \tag{1.1.81}$$

Then the spinor rotation in SU(2) is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{1.1.82}$$

with determinant ± 1 and $ad - bc = 1$. This can be re-expressed directly in terms of quaternions by

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} q_0 + iq_3 & q_1 - iq_2 \\ -q_1 - iq_2 & q_0 - iq_3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} . \tag{1.1.83}$$

In *O(3)*, whose covering group is *SU(2)*, the rotation matrix, is

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} e_{1X} & e_{1Y} & e_{1Z} \\ e_{2X} & e_{2Y} & e_{2Z} \\ e_{3X} & e_{3Y} & e_{3Z} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \tag{1.1.84}$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are unit vectors whose components are defined by

$$\left.\begin{aligned} e_{1X} &= q_0^2 + q_1^2 - q_3^2 - q_3^2 = \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta , \\ e_{1Y} &= 2(q_1q_2 + q_0q_3) = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma , \\ e_{1Z} &= 2(q_1q_3 - q_0q_2) = -\sin\beta\cos\gamma , \\ e_{2X} &= 2(q_1q_2 - q_0q_3) = -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma , \\ e_{2Y} &= q_0^2 - q_1^2 + q_2^2 - q_3^2 = -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma , \\ e_{2Z} &= 2(q_2q_3 + q_0q_1) = \sin\beta\sin\gamma , \\ e_{3X} &= 2(q_1q_3 + q_0q_2) = \cos\alpha\sin\beta , \\ e_{3Y} &= 2(q_2q_3 - q_0q_1) = -\sin\alpha\sin\beta , \\ e_{3Z} &= q_0^2 - q_1^2 - q_2^2 + q_3^2 = \cos\beta . \end{aligned}\right\} \tag{1.1.85}$$

Therefore rotation in three dimensions can be represented equivalently in terms of vectors, spinors, quaternions, and Euler angles. Rotation about the *Z* axis is represented by

$$\cos\beta = \cos\gamma = 1 , \qquad \sin\beta = \sin\gamma = 0 , \tag{1.1.86}$$

and so

$$\left.\begin{array}{lll} e_{1X} = \cos\alpha\,, & e_{1Y} = \sin\alpha\,, & e_{1Z} = 0\,, \\ e_{2X} = -\sin\alpha\,, & e_{2Y} = \cos\alpha\,, & e_{2Z} = 0\,, \\ e_{3X} = 0\,, & e_{3Y} = 0\,, & e_{3Z} = 1\,. \end{array}\right\} \tag{1.1.87}$$

A possible description of rotation about the Z axis is $\beta = 0\,, \gamma = 0,$ *i.e.*,

$$q_0 = \cos\frac{\alpha}{2}, \quad q_3 = \sin\frac{\alpha}{2}, \quad q_1 = 0\,, \quad q_2 = 0\,. \tag{1.1.88}$$

The rotation matrices are therefore

$$\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}, \tag{1.1.89}$$

or, in terms of quaternion components or coefficients

$$\begin{bmatrix} q_0^2 - q_3^2 & 2q_0q_3 & 0 \\ -2q_0q_3 & q_0^2 - q_3^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} q_0 + iq_3 & 0 \\ 0 & q_0 - iq_3 \end{bmatrix}. \tag{1.1.90}$$

The self consistency of this process can be checked through the fact that it gives the well known half angle formulae,

$$\begin{aligned} \cos\alpha &= q_0^2 - q_3^2 = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\,, \\ \sin\alpha &= 2q_0q_3 = 2\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}\,. \end{aligned} \tag{1.1.91}$$

Note that the *O(3)* rotation matrix is set up in terms of α and the *SU(2)* in terms of $\alpha/2$. The *O(3)* rotation matrix is real, the *SU(2)* rotation matrix is complex. The same quaternion coefficients appear in *O(3)* and *SU(2)*.

1.3.3 Infinitesimal Rotation Generator in *SU(2)*

Our first example of the development of Eqs. (1.1.79a) to (1.1.79c) uses infinitesimal rotation generators in *SU(2)*. Let

$$R_\alpha(Z) := \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}, \tag{1.1.92}$$

be an *SU(2)* rotation matrix. Its infinitesimal rotation generator is then defined to be

$$\tau_Z := \frac{1}{i}\frac{\partial R_Z}{\partial\alpha}(\alpha)\bigg|_{\alpha=0} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} := \frac{\sigma_Z}{2}\,, \tag{1.1.93}$$

where σ_Z is the third Pauli matrix.

Now apply the Taylor series to the matrix exponential to obtain

$$\begin{aligned} e^{i\sigma_Z\alpha/2} &= 1 + i\sigma_Z\frac{\alpha}{2} - \frac{\sigma_Z^2\,\alpha^2}{2!\;4} - i\frac{\sigma_Z^3\,\alpha^3}{3!\;8} + \ldots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\alpha}{2} - \frac{1}{2!}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\alpha^2}{4} + \ldots. \\ &= \begin{bmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{bmatrix}, \end{aligned} \tag{1.1.94}$$

and therefore

$$R_\alpha(Z) = e^{i\sigma_Z \alpha/2} = q_0 + iq_3 q_Z . \tag{1.1.95}$$

In the small angle limit,

$$q_0 \longrightarrow 1 , \qquad q_3 \longrightarrow \frac{\alpha}{2} , \tag{1.1.96}$$

and

$$e^{i\sigma_Z \alpha/2} \underset{\alpha \to 0}{\longrightarrow} 1 + i\frac{\alpha}{2}\sigma_Z , \tag{1.1.97}$$

which are self consistently the first two terms of a Taylor series.

1.3.3.1 Field Rotation in *SU(2)*

The rotation of a field ψ is now definable by [5],

$$\psi' = e^{i\sigma_Z \alpha/2}\psi = \left(q_0 + iq_3\sigma_2\right)\psi , \tag{1.1.98}$$

and with these components in hand the gauge transformation process in *SU(2)* is based on the idea that the Euler angle α is a function of x^μ, the space-time four-vector. This is a gauge transformation of the second kind, which is underpinned by special relativity [1—5]. The quaternion coefficients become functions of x^μ, and derivatives are replaced by covariant derivatives in *SU(2)* [5]. Under gauge transformation of the second kind, the potential four-vector becomes

$$A'_\mu = SA_\mu S^{-1} - \frac{i}{g}\partial_\mu S S^{-1} , \tag{1.1.99}$$

in which appears an inhomogeneous, purely topological, term, the second term on the right hand side. In our example, this equation is developed as

$$A_\mu := A_\mu^a \frac{\sigma^a}{2} = A_\mu^Z \frac{\sigma^Z}{2} , \qquad S := e^{i\sigma_Z \alpha/2} . \tag{1.1.100}$$

In analogy with Eq. (1.1.1) of Sec. 1.2, the object A_μ is a matrix in an internal gauge space indicated by the superscript a. In this notation, summation is implied over all repeated indices. Greek indices are covariant-contravariant Minkowski space indices. Latin ones denote the internal gauge space. The placement of the Latin indices as subscript or superscript is not significant, because they are not contravariant-covariant indices. For the rotation about the Z axis that we are considering here, $a = Z$. The symbol S is a rotation matrix in *SU(2)* in exponential form. Therefore the symbol A_μ is interpreted as the matrix,

$$A_\mu := \begin{bmatrix} \dfrac{A_\mu^Z}{2} & 0 \\ 0 & -\dfrac{A_\mu^Z}{2} \end{bmatrix} , \tag{1.1.101}$$

for this example of Z axis rotation in *SU(2)* of the field ψ . The *SU(2)* gauge transformation of A_μ is given by Eq. (1.1.99), with its characteristic inhomogeneous or topological term. In a *U(1)* symmetry theory this term is the well known gradient of an arbitrary function first introduced in the late nineteenth century. In *SU(2)* however, it is not arbitrary, and is determined by S, *i.e.*, by a particular Euler angle α , or quaternion component.

1.3.3.2 The Inhomogeneous or Topological Term in *SU(2)*

We use Eq. (1.1.99) with

$$S = e^{i\alpha\sigma_{Z/2}}, \qquad S^{-1} = e^{-i\alpha\sigma_Z/2}, \tag{1.1.102}$$

and

$$\partial_\mu S = \left(i\frac{\sigma_Z}{2}\partial_\mu\alpha \right) S. \tag{1.1.103}$$

Therefore gauge transformation results in

$$A'_\mu = A_\mu - \frac{i^2}{g}\frac{\sigma_Z}{2}\partial_\mu\alpha, \tag{1.1.104}$$

or in matrix form,

$$\begin{bmatrix} \frac{A_\mu^{Z'}}{2} & 0 \\ 0 & \frac{A_\mu^{Z'}}{2} \end{bmatrix} = \begin{bmatrix} \frac{A_\mu^{Z}}{2} & 0 \\ 0 & -\frac{A_\mu^{Z}}{2} \end{bmatrix} + \frac{1}{2g}\begin{bmatrix} \partial_\mu\alpha & 0 \\ 0 & -\partial_\mu\alpha \end{bmatrix}, \tag{1.1.105}$$

where,

$$\alpha = \cos^{-1}\left(q_0^2 - q_3^2\right) = \sin^{-1}\left(2q_0q_3\right). \tag{1.1.106}$$

Therefore ,

$$A_\mu^{Z'} = A_\mu^{Z} + \frac{1}{g}\partial_\mu\alpha. \tag{1.1.107}$$

This is clearly a geometrical result, rotation of the field ψ about the *Z* axis has this effect on the *Z* component of A_μ in the *SU(2)* internal gauge space. We are dealing with curvilinear coordinates because in a flat space-time, $\partial_\mu\alpha = 0$ because α is not a function of x^μ. Terms such as $(1/g)\partial_\mu\alpha$ are the physical bases of effects such as that of Aharonov and Bohm. The latter are usually given in terms of *U(1)* electrodynamics, in which α is effectively an arbitrary function. In *SU(2)*, α is clearly the Euler angle, and a finite rotation must always take place through a finite Euler angle.

In the small angle limit,

$$\frac{\alpha}{2} \sim \sin\frac{\alpha}{2} = q_3, \tag{1.1.108}$$

and so,

$$A_\mu^{Z'} \underset{\alpha \to 0}{\longrightarrow} A_\mu^{Z} + \frac{1}{g}\partial_\mu q_3. \tag{1.1.109}$$

Note that gauge transformation in an *SU(2)* symmetry field theory is a geometrical process. If $\partial\alpha/\partial x^\mu = 0$, A'_Z goes to A_Z, there is no topological term and no Aharonov-Bohm effect. The object A_μ is a physical four-potential in the *classical* field theory. It is not a mathematical subsidiary variable as in a *U(1)* gauge field theory of classical electrodynamics. There is therefore a profound difference between *O(3)* and *U(1)* electrodynamics.

1.3.3.3 Self Consistency of Equation (1.1.107)

There are various ways of self checking Eq. (1.1.107), for example, for small angle rotation in the *O(3)* group, homomorphic [5,8,9] to *SU(2)*, we should obtain the same result. It is convenient to develop the concise description given by Ryder on his p. 119 [5], and to consider the small angle rotation of a field ϕ with components described by ϕ_1, ϕ_2, and ϕ_3, in general, a matter field. In the *O(3)* internal space,

$$\begin{bmatrix} \phi_1' \\ \phi_2' \\ \phi_3' \end{bmatrix} = \begin{bmatrix} 1 & \Lambda_3 & 0 \\ -\Lambda_3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \tag{1.1.110}$$

for a rotation about the small angle Λ_3. This process is

$$\left. \begin{aligned} \phi_1' &= \phi_1 + \Lambda_3 \phi_2 , \\ \phi_2' &= \phi_2 - \Lambda_3 \phi_1 , \\ \phi_3' &= \phi_3 , \end{aligned} \right\} \tag{1.1.111}$$

and is a component of the small angle rotation given by $-\Lambda \times \phi$. When $\Lambda = \Lambda_3 \boldsymbol{k}$ we obtain, self consistently,

$$-\Lambda \times \phi = - \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \Lambda_3 \\ \phi_1 & \phi_2 & \phi_3 \end{vmatrix} = \Lambda_3 \phi_2 \boldsymbol{i} - \Lambda_3 \phi_1 \boldsymbol{j} . \tag{1.1.112}$$

In *O(3)* vector notation,

$$\phi' = e^{iJ \cdot \Lambda} \phi \Leftrightarrow \phi' = \phi - \Lambda \times \phi , \tag{1.1.113}$$

in the small angle limit.

Now apply the formula for gauge transformation,

$$A_\mu' = \left(S A_\mu - \frac{i}{g} \partial_\mu S \right) S^{-1} , \tag{1.1.114}$$

with

$$S A_\mu = \exp \left(i \boldsymbol{J} \cdot \Lambda \right) \boldsymbol{A}_\mu \sim \boldsymbol{A}_\mu - \Lambda \times \boldsymbol{A}_\mu , \tag{1.1.115a}$$

where $\boldsymbol{A}_\mu$ is a vector in the internal *O(3)* group space, with

$$\partial_\mu S = \left(i \partial_\mu \Lambda \right) S , \tag{1.1.115b}$$

to obtain

$$\boldsymbol{A}_\mu' = \left(\boldsymbol{A}_\mu - \Lambda \times \boldsymbol{A}_\mu - \frac{i^2}{g} \partial_\mu \Lambda S \right) S^{-1} , \tag{1.1.116}$$

where

$$\left. \begin{aligned} \boldsymbol{S} &= \boldsymbol{e}^{i\boldsymbol{J} \cdot \Lambda} = 1 + i \boldsymbol{J} \cdot \Lambda + ... \\ \boldsymbol{S}^{-1} &= e^{-i \boldsymbol{J} \cdot \Lambda} = 1 - i \boldsymbol{J} \cdot \Lambda + ... \end{aligned} \right\} \tag{1.1.117}$$

so

$$\boldsymbol{A}_\mu' \sim \boldsymbol{A}_\mu - \Lambda \times \boldsymbol{A}_\mu + \frac{1}{g} \partial_\mu \Lambda + \ ... \ , \tag{1.1.118}$$

which is the Yang-Mills approximation given by Ryder. The small angle gauge transformation in the *O(3)* gauge group's internal space is a geometrical process, not a random process as in the *U(1)* gauge group. Later an example of this different role played by the potential is considered, the gauge transformation of the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. In *O(3)* this object is physical, in *U(1)* it is unphysical. However, it is an observable of magneto-optics, and so empirical data prefer the *O(3)* hypothesis. In *O(3)*, rotation about the *Z* axis, a gauge transformation, leaves $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ unchanged; in *U(1)*, it becomes random, because $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ becomes random..

Returning to the development in this section, then for a *Z* axis rotation,

$$\Lambda_1 = \Lambda_2 = 0, \tag{1.1.119}$$

and

$$-\Lambda \times \boldsymbol{A}_\mu = \Lambda_3 A_{\mu 2}\boldsymbol{i} - \Lambda_3 A_{\mu 1}\boldsymbol{j}, \tag{1.1.120}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are Cartesian unit vectors in the internal space. So,

$$\left.\begin{aligned} A'_{\mu 1} &= A_{\mu 1} + \Lambda_3 A_{\mu 2}, \\ A'_{\mu 2} &= A_{\mu 2} - \Lambda_3 A_{\mu 1}, \\ A'_{\mu 3} &= A_{\mu 3} + \frac{1}{g}\partial_\mu \Lambda_3 . \end{aligned}\right\} \tag{1.1.121}$$

The third of these equations is Eq (1.1.107) in the small angle limit, *QED*.

1.3.4 Gauge Transformation in an *O(3)* Gauge Field Theory

Considering a *Z* axis rotation in an internal *O(3)* space of a gauge field theory governed by Eqs (1.1.79a) to (1.1.79c) we obtain,

$$S = e^{iJ_Z\alpha}, \qquad S^{-1} = e^{-iJ_Z\alpha}, \tag{1.1.122}$$

where J_Z is the infinitesimal rotation generator defined in Ref 5. Thus,

$$S = 1 + iJ_Z\alpha - J_Z^2\frac{\alpha^2}{2!} - iJ_Z^3\frac{\alpha^3}{3!} + \ldots = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.1.123}$$

which is self-consistently the rotation matrix for a rotation about the *Z* axis in an *O(3)* symmetry gauge field theory.

The inverse of *S* is formed by $\alpha \longrightarrow -\alpha$,

$$S^{-1} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = e^{-iJ_z\alpha}, \tag{1.1.124}$$

and it is easily checked that SS^{-1} is the unit 3 x 3 matrix as required.

The existence of the term $\partial_\mu S$ depends on α being a function of x^μ, since α is the only independent variable in *S*. So,

$$\partial_\mu S = \partial_\mu \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{1.1.125}$$

Now use the calculus,

$$\frac{dy}{dx} = \frac{df}{dx}\frac{dy}{df}, \tag{1.1.126}$$

so if $y = \cos\left(f(x)\right)$ for example, then,

$$\frac{dy}{dx} = -f'(x)\sin\left(f(x)\right). \tag{1.1.127}$$

We obtain

$$\left.\begin{aligned} \partial_\mu\left(\cos\alpha(x^\mu)\right) &= -\partial_\mu\alpha\sin\alpha, \\ \partial_\mu\left(\sin\alpha(x^\mu)\right) &= \partial_\mu\alpha\cos\alpha, \end{aligned}\right] \tag{1.1.128}$$

and

$$\partial_\mu S = \partial_\mu\alpha\begin{bmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.1.129}$$

The existence of $\partial_\mu S$ depends directly on that of $\partial_\mu\alpha$ and on the postulate that α is a function of x^μ; a postulate that springs directly from special relativity via type two gauge transform theory [5], or gauge transformation of the second kind.

1.3.4.1 Definition of A_μ

In $O(3)$ symmetry gauge field theory the object A_μ is expressed as a matrix,

$$A_\mu = J^a A_\mu^a, \tag{1.1.130}$$

where J^a are the three infinitesimal rotation generator matrices of $O(3)$ [1—9] and where the double indexed A_μ^a are scalar coefficients of the internal space, a vector space. For Z axis rotation,

$$A_\mu = J^Z A_\mu^Z. \tag{1.1.131}$$

In this notation, the placing of Z as an upper or lower index has no algebraic significance, as discussed already, whereas μ is covariant-contravariant. Thus, for Z axis rotation,

$$A_\mu = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} A_\mu^Z, \tag{1.1.132}$$

The inhomogeneous term in Eq (1.1.114) is also directly dependent on the existence of $\partial_\mu\alpha$,

$$\partial_\mu S S^{-1} = \partial_\mu \alpha \begin{bmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \partial_\mu \alpha \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \qquad (1.1.133)$$

So,

$$-\frac{i}{g}\partial_\mu S S^{-1} = \frac{J_Z}{g}\partial_\mu \alpha . \qquad (1.1.134)$$

Note that this is the topological term responsible for the Aharonov-Bohm effect and so forth [1—9]. The scalar g is a dimensionality coefficient introduced as such in the definition of the covariant derivative [5]. The operator J_Z is the infinitesimal rotation generator of *O(3)* about *Z*. The existence of this term in the gauge transform of A_μ is the direct result of special relativity, of gauge transformation of the second kind. In a *U(1)* gauge field theory the equivalent of α is arbitrary, and has no geometrical meaning as we have argued already. In the *O(3) = SU(2)* version it is an Euler angle which is a function of x^μ for a given rotation, $\alpha(x^\mu)$ is clearly finite and well defined, being a physical Euler angle in curvilinear coordinates necessitated by special relativity.

The above calculation can be checked for self consistency using the operator formalism. If,

$$S = \exp\left(iJ_Z \alpha \right) , \quad \text{then} \quad \partial_\mu S = iJ_Z \partial_\mu \alpha S , \qquad (1.1.135)$$

and, QED,

$$\partial_\mu S S^{-1} = iJ_Z \partial_\mu \alpha . \qquad (1.1.136)$$

1.3.4.2 The Term $SA_\mu S^{-1}$

This is also a matrix given by,

$$SA_\mu S^{-1} = -iA_\mu^Z$$

$$\times \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (1.1.137)$$

$$= A_\mu^Z J_Z = A_\mu .$$

The overall result of the gauge transformation is therefore,

$$A_\mu^Z J_Z \rightarrow \left(A_\mu^Z + \frac{1}{g}\partial_\mu \alpha \right) J_Z , \qquad (1.1.138)$$

i.e.,

$$A_\mu^Z \rightarrow A_\mu^Z + \frac{1}{g}\partial_\mu \alpha . \qquad (1.1.139)$$

Self consistently, this is Eq. (1.1.107) of Sec. 1.3.3.1. The *O(3)* and *SU(2)* symmetry theories give the same result for the scalar A_μ^Z of the internal gauge space. If the space is such that α has no dependence on x^μ, the A_μ^Z is unchanged by rotation about *Z*. Self-consistently, this is Euclidean space, in which rotation about Z does not change the direction or magnitude of a vector component aligned in Z.

1.3.5 Transformation of the Field Tensor

The rule for transformation of the field tensor in general gauge field theory is,

$$G'_{\mu\nu} = S G_{\mu\nu} S^{-1} . \tag{1.1.140}$$

The inhomogeneous term does not appear and the transformation takes place covariantly rather than invariantly as in *U(1)* [5]. The $\boldsymbol{B}^{(3)}$ field transforms as follows, for a Z axis rotation and in matrix algebra,

$$\begin{bmatrix} 0 & -B_Z & 0 \\ -B_Z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -B_Z & 0 \\ B_Z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.1.141}$$

$$= \begin{bmatrix} 0 & -B_Z & 0 \\ B_Z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

i.e.,

$$B_Z \rightarrow B_Z . \tag{1.1.142}$$

The $\boldsymbol{B}^{(3)}$ field is therefore self-consistently invariant under rotation about the Z axis, and the *O(3)* gauge transform is a rotation which produces,

$$\left.\begin{aligned} A_Z &\rightarrow A_Z + \frac{1}{g}\partial_Z \alpha , \\ B_Z &\rightarrow B_Z . \end{aligned}\right\} \tag{1.1.143}$$

In *U(1)* these concepts do not arise because both A_Z and B_Z are zero, and the idea of gauge transformation being a rotation through a physical Euler angle does not exist.

1.3.6 *O(3)* Gauge Transformation of the Optical Conjugate Product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$

The optical conjugate product is a well accepted physical observable of the semi classical, phenomenological, theory of non-linear optics [1—4]. As argued in several ways [1—4] already this observable is identically zero by definition in *U(1)*. In *O(3)* it is identically non-zero by definition and proportional to $\boldsymbol{B}^{(3)}$ by definition. To check the consistency of the result (1.1.142) of the preceding section this section is devoted to the details of gauge transformation of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in *O(3)*. Since $\boldsymbol{B}^{(3)}$ is invariant under *O(3)* gauge transformation defined as a rotation about Z, so should be $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. In order for this to be so, we shall see that the gauge transformation in *O(3)* must generate an electromagnetic phase shift defined in terms of the physical angle of rotation. This result is akin to the topological phase [8,9] and the Aharonov-Bohm effect [8,9] as discussed lucidly by Barrett. It means that there exists an *optical* Aharonov-Bohm effect which is measurable in principle by this phase shift. In *U(1)*, as argued already, the electromagnetic phase is random because of the random nature of gauge transformation of the second kind in *U(1)* [1—4]. To see this result in *O(3)*, the gauge transformation rules must be applied carefully to $A^{(1)}$ and to $A^{(2)}$ as follows,

$$\left.\begin{aligned} A^{(1)} &\rightarrow SA^{(1)}S^{-1} - \frac{i}{g}\partial_\mu S S^{-1}, \\ A^{(2)} &\rightarrow SA^{(2)}S^{-1} + \frac{i}{g}\left(\partial_\mu S S^{-1}\right)^{*}. \end{aligned}\right] \quad (1.1.144)$$

In vector notation, the $A^{(1)}$ and $A^{(2)}$ components are complex conjugates such as,

$$A^{(1)} = \frac{A^{(0)}}{\sqrt{2}}(\boldsymbol{i} - i\boldsymbol{j})e^{i\phi}, \qquad A^{(2)} = \frac{A^{(0)}}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})e^{-i\phi}, \quad (1.1.145)$$

$$:= A_X^{(1)}\boldsymbol{i} + A_Y^{(1)}\boldsymbol{j}, \qquad := A_X^{(2)}\boldsymbol{i} + A_Y^{(2)}\boldsymbol{j}.$$

Therefore in matrix form,

$$\left.\begin{aligned} A^{(1)} &= A_X^{(1)}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} + A_Y^{(1)}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \\ A^{(2)} &= A_X^{(2)}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} + A_Y^{(2)}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}. \end{aligned}\right\} \quad (1.1.146)$$

Rotation of these terms about the *Z* axis produces results such as the following,

$$SA_X^{(1)}S^{-1}$$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} A_X^{(1)} \quad (1.1.147)$$

$$= \begin{bmatrix} 0 & 0 & -i\sin\alpha \\ 0 & 0 & -i\cos\alpha \\ i\sin\alpha & i\cos\alpha & 0 \end{bmatrix} A_X^{(1)}.$$

Therefore the vector $A^{(1)}$ is changed by an *O(3)* gauge transformation defined as a rotation about the *Z* axis. This is self consistent because $A^{(1)}$ has *X* and *Y* components only.

Similarly,

$$SA_Y^{(1)}S^{-1} = A_Y^{(1)}\begin{bmatrix} 0 & 0 & i\cos\alpha \\ 0 & 0 & -i\sin\alpha \\ -i\cos\alpha & i\sin\alpha & 0 \end{bmatrix}. \quad (1.1.148)$$

It can now be checked that the commutator $[SA_X^{(1)}S^{-1}, SA_Y^{(1)}S^{-1}]$, is a *Z* axis rotation as required,

$$\left[SA_X^{(1)}S^{-1},\, SA_Y^{(1)}S^{-1}\right] = \left(SA_X^{(1)}S^{-1}\,SA_Y^{(1)}S^{-1} - SA_Y^{(1)}S^{-1}\,SA_X^{(1)}S^{-1}\right)$$

$$= A_X^{(1)}A_Y^{(1)}\begin{bmatrix} 0 & 0 & -i\sin\alpha \\ 0 & 0 & -i\cos\alpha \\ i\sin\alpha & i\cos\alpha & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & i\cos\alpha \\ 0 & 0 & -i\sin\alpha \\ -i\cos\alpha & i\sin\alpha & 0 \end{bmatrix}$$

$$-A_X^{(1)}A_Y^{(1)}\begin{bmatrix} 0 & 0 & i\cos\alpha \\ 0 & 0 & -i\sin\alpha \\ -i\cos\alpha & i\sin\alpha & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & -i\sin\alpha \\ 0 & 0 & -i\cos\alpha \\ i\sin\alpha & i\cos\alpha & 0 \end{bmatrix} \qquad (1.1.149)$$

$$= A_X^{(1)}A_Y^{(1)}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = iA_X^{(1)}A_Y^{(1)}J_Z .$$

The overall result is that a rotation about the Z axis changes the X and Y components of the potentials $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$, but leaves $\boldsymbol{B}^{(3)}$ unchanged. However, the polar longitudinal component A_Z, (which has no existence in *U(1)*), is changed by the same gauge transform process to $A_Z + \partial_Z\alpha$. Therefore, $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is self consistently proportional to $\boldsymbol{B}^{(3)}$ in *O(3)*. In *U(1)*, as we have seen, $\boldsymbol{B}^{(3)}$ is zero and $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is randomized by the *U(1)* gauge transformation of the second kind because random quantities are added to $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ (gradients of arbitrary scalars). It seems clear that *O(3)* is the more consistent theory on these arguments alone, because $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is an optical *observable*. In *U(1)*, the potential is never an observable according to the Heaviside interpretation, it is strictly a mathematical subsidiary. The latter conclusion has been shown to be false by Barrett [8,9] using half a dozen phenomena of nature. The Heaviside

view was criticized by Ritz as early as 1908 [13] on the grounds that the classical potential denoted delayed action at a distance as advocated by Schwarzschild in 1902 [12], and so must be physical.

As a further check on self consistency of Eq. (1.1.142) we can calculate the commutator,

$$\begin{aligned}&\left[SA^{(1)}S^{-1},\, SA^{(2)}S^{-1}\right] \\ &= SA^{(1)}S^{-1}\,SA^{(2)}S^{-1} - SA^{(2)}S^{-1}\,SA^{(1)}S^{-1},\end{aligned} \qquad (1.1.150)$$

where

$$A^{(1)} = A_X^{(1)}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} + A_Y^{(1)}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & -0 \\ -i & 0 & 0 \end{bmatrix}, \qquad (1.1.151)$$

and

$$SA^{(1)}S^{-1} = SA_X^{(1)}S^{-1} + SA_Y^{(1)}S^{-1}$$

$$= \frac{A^{(0)}}{\sqrt{2}}e^{i\phi}\begin{bmatrix} 0 & 0 & -\sin\alpha + \cos\alpha \\ 0 & 0 & -i\cos\alpha - \sin\alpha \\ i\sin\alpha - \cos\alpha & i\cos\alpha + \sin\alpha & 0 \end{bmatrix}. \qquad (1.1.152)$$

Similarly,

$$SA^{(2)}S^{-1} = SA_X^{(1)}S^{-1} + SA_Y^{(2)}S^{-1}$$

$$= \frac{A^{(0)}}{\sqrt{2}} e^{-i\phi} \begin{bmatrix} 0 & 0 & -\sin\alpha - \cos\alpha \\ 0 & 0 & -i\cos\alpha + \sin\alpha \\ i\sin\alpha + \cos\alpha & i\cos\alpha - \sin\alpha & 0 \end{bmatrix} . \tag{1.1.153}$$

Straightforward algebra then shows that,

$$\left[SA^{(1)}S^{-1},\ SA^{(2)}S^{-1} \right] = -A^{(0)2} J_Z , \tag{1.1.154}$$

or in vector notation, we obtain the self consistent result,

$$\boldsymbol{B}^{(3)} = -i \frac{\kappa}{A^{(0)}} \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} , \tag{1.1.155}$$

which again shows that the cross product of two polar vectors, $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$, is the axial vector $\boldsymbol{B}^{(3)}$. (Recall that the cross product of two polar or of two axial vectors both give rise to an axial vector, not to a polar vector [1—4].) Therefore the longitudinal polar vector potential A_Z can be a component of the overall potential four-vector, but cannot be generated by the cross product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. The latter always generates an axial vector proportional to $\boldsymbol{B}^{(3)}$.

Now evaluate the commutator $\left[A^{(1)},\ A^{(2)} \right]$, with,

$$A^{(1)} = \frac{A^{(0)}}{\sqrt{2}} e^{i\phi} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{bmatrix} , \qquad A^{(2)} = \frac{A^{(0)}}{\sqrt{2}} e^{-i\phi} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ 1 & i & 0 \end{bmatrix} ; \tag{1.1.156}$$

to obtain

$$\left[A^{(1)},\ A^{(2)} \right] = -A^{(0)2} J_Z , \tag{1.1.157}$$

and so,

$$\left[A^{(1)},\ A^{(2)} \right] = \left[SA^{(1)}S^{-1},\ SA^{(2)}S^{-1} \right] . \tag{1.1.158}$$

This result means that an *O(3)* gauge transformation defined as a Z axis rotation changes $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$ but leaves $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ unchanged. This is an obvious and simple geometrical result which is physically meaningful as a geometric rotation in three dimensions, and which is self consistent with the invariance of $\boldsymbol{B}^{(3)}$ under such a gauge transformation. These concepts do not exist in *U(1)*.

1.3.7 The Topological or Inhomogeneous Term: The Optical Aharonov-Bohm Effect and Topological Phase Effect in *O(3)*

The complete gauge transformation is,

$$\left. \begin{aligned} A^{(1)} &\rightarrow SA^{(1)}S^{-1} - \frac{i}{g}\left(\partial_\mu SS^{-1}\right)^{(1)} , \\ A^{(2)} &\rightarrow SA^{(2)}S^{-1} + \frac{i}{g}\left(\partial_\mu SS^{-1}\right)^{(2)} , \end{aligned} \right\} \tag{1.1.159}$$

and for a Z axis rotation,

$$\left[A^{(1)},\ A^{(2)} \right] = \left[SA^{(1)}S^{-1},\ SA^{(2)}S^{-1} \right] , \tag{1.1.160}$$

$$-\frac{i}{g}\left(\partial_\mu SS^{-1}\right)^{(1)} = \frac{1}{g}\partial_\mu\alpha\begin{bmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = \frac{J_Z}{g}\partial_\mu\alpha\,, \tag{1.1.161}$$

$$\frac{i}{g}\left(\partial_\mu SS^{-1}\right)^{(2)} = \frac{1}{g}\partial_\mu\alpha\begin{bmatrix}0 & i & 0\\ -i & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = -\frac{J_Z}{g}\partial_\mu\alpha\,, \tag{1.1.162}$$

From Eq. (1.1.160) we know that the sum generated by the commutator of inhomogeneous terms and cross terms on the right hand side must be zero. The commutator of inhomogeneous terms is indeed zero,

$$-\frac{1}{g^2}(\partial_\mu\alpha)(\partial_\mu\alpha)^*\left(\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right) = 0. \tag{1.1.163}$$

Therefore the sum of cross terms must be zero,

$$\begin{aligned}&\left(SA^{(1)}S^{-1}\right)\left(\partial_\mu SS^{-1}\right)^{(2)} - \left(\partial_\mu SS^{-1}\right)^{(1)}\left(SA^{(2)}S^{-1}\right)\\ &+\left(SA^{(2)}S^{-1}\right)\left(\partial_\mu SS^{-1}\right)^{(1)} - \left(\partial_\mu SS^{-1}\right)^{(2)}\left(SA^{(1)}S^{-1}\right) = 0\end{aligned} \tag{1.1.164}$$

After some elementary algebra the result reduces to

$$e^{i\phi}\begin{bmatrix}0 & 0 & ie^{-i\alpha}\\ 0 & 0 & e^{-i\alpha}\\ -ie^{-i\alpha} & -e^{-i\alpha} & 0\end{bmatrix} + e^{-i\phi}\begin{bmatrix}0 & 0 & ie^{i\alpha}\\ 0 & 0 & -e^{i\alpha}\\ -ie^{i\alpha} & e^{i\alpha} & 0\end{bmatrix} = 0\,, \tag{1.1.165}$$

i.e.,

$$e^{i(\phi-\alpha)} = -e^{-i(\phi-\alpha)}, \tag{1.1.165a}$$

or,

$$\cos(\phi-\alpha) = 0\,, \tag{1.1.165b}$$

$$\phi \rightarrow \alpha \pm (2n+1)\frac{\pi}{2}\,, \tag{1.1.165c}$$

Therefore the $O(3)$ gauge transformation produces a topologically induced change in the electromagnetic phase. A rotation through the angle produces a change $\alpha\pm(2n+1)\pi/2$ in the phase. This is also a polarization change because for instance,

$$(\boldsymbol{i}+i\boldsymbol{j})e^{i\phi} \rightarrow (\boldsymbol{i}+i\boldsymbol{j})e^{i(\alpha\pm(2n+1)\pi/2)}\,, \tag{1.1.166}$$

and using the angle formulae,

$$\left.\begin{aligned}\cos(A\pm B) &= \cos A\cos B \mp \sin A\sin B\,,\\ \sin(A\pm B) &= \sin A\cos B \pm \cos A\sin B\,,\end{aligned}\right\} \tag{1.1.167}$$

it follows that

$$\left.\begin{array}{l} Re\left((\boldsymbol{i}+i\boldsymbol{j})e^{i\phi}\right) = \cos\phi\,\boldsymbol{i} - \sin\phi\,\boldsymbol{j} \\ \quad\longrightarrow \pm\left(\sin\phi_0\boldsymbol{i} + \cos\phi_0\boldsymbol{j}\right), \end{array}\right\} \tag{1.1.168}$$

where $\phi_0 = \alpha$.

This result clearly shares the features of the topological phase effect, for example, winding an optical fiber on a drum, and sending a linearly polarized laser beam through it produces a rotation of the linear polarization plane [8,9]. This is in *O(3)* an optical Aharonov Bohm effect as argued. In *U(1)* the same effect is random, and unphysical. This seems to be further clear empirical reason for preferring *O(3)* to *U(1)* and the observation of the topological phase in this manner is also an observation of the optical Aharonov-Bohm effect. For example, a rotation of $3\pi/2$ increases ϕ in Eq. (1.1.165) by the same amount, $3\pi/2$, and changes the *polarization* of the light beam. For example, for $n = 0$,

$$\left.\begin{array}{ll} \sin\left(\alpha + \dfrac{\pi}{2}\right) = \cos\alpha & (\neq \sin\alpha \text{ in general}), \\ \cos\left(\alpha + \dfrac{\pi}{2}\right) = -\sin\alpha & (\neq \cos\alpha \text{ in general}). \end{array}\right\} \tag{1.1.169}$$

Since gauge transformation in *O(3)* is a physical (or geometrical) rotation, the rotation of the direction of the light beam as it propagates through an optical fiber wound about the *Z* axis as a helix [8,9] is a geometrical process that is a gauge transformation, one which can be observed empirically to change the polarization of that light beam, *QED*. The geometrical details are different, because the helical rotation of a beam propagating within a fiber is not the same as a straightforward rotation of that beam about its own propagation axis *Z* when the latter is held constant, but the overall result is the same, a change of polarization of the light beam. Such a phenomenon has no existence in *U(1)*.

References

[1] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $\boldsymbol{B}^{(3)}$ (Kluwer Academic, Dordrecht, 1995).

[2] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).

[3] M. W. Evans , J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).

[4] M. W. Evans, J.-P. Vigier, and S.Roy, *The Enigmatic Photon, Vol. 4: New Directions* (Kluwer Academic, Dordrecht, 1998).

[5] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).

[6] M. W. Evans, *Physica B*, **182**, 227 (1992).

[7] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, 1962).

[8] T.W. Barrett and D. M. Grimes, eds., *Advanced Electromagnetism, Foundations, Theory and Applications*, Chap. 1 (World Scientific, Singapore, 1995).

[9] T. W. Barrett in A. Lakhtakia, ed. *Essays on the Formal Aspects of Electromagnetic Theory*, (World Scientific, Singapore, 1993).

[10] R. Aldrovandi, Ref. 8, pp. 3 ff.

[11] T. E. Bearden, personal communications.

[12] R. S. Fritzius, Critical Researches on General Electrodynamics *Apeiron*, 1998.

[13] W. Ritz, *Ann. Chim. Phys.*, **13**, 145 (1908).

[14] M. W. Evans in I. Prigogine and S. A. Rice, eds., *Advances in Chemical Physics*, Vol. 81, pp 361—702 (Wiley, New York, 19 92).

[15] B. L. Silver, *Irreducible Tensor Theory* (Academic, New York, 1976).

[16] A. O. Barut, Electrodynamics and Classical Theory of Fields and Particles (MacMillan, New York, 1964).

[17] D. Corson and P. Lorrain, *Introduction to Electromagnetic Fields and Waves* (W. H. Freeman & Co., San Francisco, 1962).

[18] M. W. Evans and S. Kielich eds., *Modern Nonlinear Optics*, Vols. 85(1), 85(2), 85(3) of Advances in Chemical Physics, I. Prigogine and S. A. Rice, eds. (Wiley Interscience, New York, 1993/1994/1997).
[19] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).
[20] Definitions in Ref. 10.

Chapter 2

The Geometry of Gauge Fields

Contemporary bundle tangent theory is able to establish the basic structure of any gauge theory from pure geometry. It can be shown [1] that the internal space is a symmetry space. Vector fields, forms and tensors on the basic manifold are related to their correspondents on the bundle. Vector fields are lifted by a section to certain fields on the bundle and this is pure geometry as is well known in contemporary mathematical physics. A frame $\left[e_\mu\right]$ on the basic space will be taken by a section σ into a set of basic fields $X_\mu = \sigma\left(e_\mu\right)$. Around any point of the bundle there exists a *separated* basis, called a direct product basis, formed by the basic fields x_μ and the fundamental fields X_a. In this (direct product) basis the commutation relations are [1],

$$\left[X_\mu, X_\nu\right] = C^\lambda_{\mu\nu} X_\lambda , \tag{1.2.1}$$

$$\left[X_\mu, X_a\right] = 0 , \tag{1.2.2}$$

$$\left[X_a, X_b\right] = f^c_{ab} X_c , \tag{1.2.3}$$

As described in Ref. 1, Eq. (1.2.2) establishes the independence of the algebra of the fundamental and basic fields.

2.1 Application to Electromagnetic Theory with Internal Space ((1), (2), (3))

Usually, electromagnetism is described as a gauge theory with *U(1)* internal symmetry [2—4]. However, in Chap. 1 we have developed a gauge theory of electromagnetism with an internal space ((1), (2), (3)) which is a physical space with *O(3)* symmetry. Does this gauge theory comply with Eqs. (1.2.1) to (1.2.3)? This question can be tested with a particular choice of generators for the X_μ fields. If, for example, we choose the X_μ to be rotation generators of the Lorentz group we obtain [5—8] for Eq. (1.2.1),

$$\left.\begin{aligned} \left[J_1, J_2\right] &= iJ_3 , \\ \left[J_2, J_3\right] &= iJ_1 , \\ \left[J_3, J_1\right] &= iJ_2 \end{aligned}\right\} \tag{1.2.4}$$

so $f_{ab}^c = i$, $a = X$, $b = Y$, $c = Z$. Similarly for Eq. (1.2.3),

$$\left[J^{(1)}, J^{(2)}\right] = -J^{(3)*} , \tag{1.2.5}$$

et cyclicum, so $f_{ab}^c = -1$, $a = (1)$, $b = (2)$, $c = (3)$. However, when we come to test Eq. (1.2.2), we obtain results such as

$$\left[J_1, J^{(2)}\right] = \frac{1}{\sqrt{2}} J^{(3)*} , \tag{1.2.6}$$

and so forth. This simply means that the internal gauge space as used in Chap. 1 is not independent of space-time, as in Eqs. (1.2.1), (1.2.2), and (1.2.3). How are we to justify the gauge theory of Chap. 1 therefore in the general structure described for example by Aldrovandi [1]? It turns out that the answer is to be found in the theory of extended Lie algebra.

Note firstly that the X^μ vector field symbols used by Aldrovandi [1] can represent space-time translation generators, for example, or rotation

generators. In a simple or direct product Lie algebra he proves that all gauge theories must have the following structure (his Eqs. (61) to (63)) [1],

$$\left[X_\mu^{\prime}, X_\nu^{\prime}\right] = C_{\mu\nu}^{\lambda} X_\lambda^{\prime} - F_{\mu\nu}^{a} X_a , \tag{1.2.7}$$

$$\left[X_\mu^{\prime}, X_a\right] = 0 , \tag{1.2.8}$$

$$\left[X_a, X_b\right] = f_{ab}^{c} X_c , \tag{1.2.9}$$

where $X_\mu^{\prime} = X_\mu - A_\mu^a X_a$. If X_μ is a translation generator proportional to ∂_μ [4], then $X_\mu^{\prime}$ is a covariant derivative. The X_a generators can be matrix generators for example.

In simple gauge theories such as these, Eq. (1.2.8) shows that the internal (gauge) space and the external space are independent. The connection A_μ^a modifies the translation generator $X_\mu := \partial_\mu$ and alters space-time homogeneity [1]. This is a key point in all gauge theories applied to electrodynamics. Essentially, the non-linearities in electrodynamics become a property of space-time in the spirit of general relativity. The *O(3)* symmetry electrodynamics is only one example of many possible self consistent theories of non-linear optics, all of which reduce to the Maxwellian formalism in the linear limit. In a simple gauge theory such as the Yang-Mills theory the underlying group Jacobi identities completely determine the structure of the theory, which is fixed by Eqs. (1.2.7) to (1.2.9). The Bianchi identity for example is formed from the Jacobi identity through the use of covariant derivatives [1], and all of these results are purely geometrical. They are of the type used in particle physics, where it is usually assumed that the internal space is independent of Minkowski space-time.

2.1.1 Extended Lie Algebra

A rigorous geometrical basis for the theory used in Chap. 1 can be given using a simple example of the Lie algebra extension theory given by Aldrovandi [2.8] in his section 6.1. This is developed in this section using the Lie algebra of rotation generators in the basis (X, Y, Z) and the basis ((1), (2), (3)). The L algebra [1] is defined by

$$[J_X, J_Y]_L = iJ_Z, \tag{1.2.10}$$

and the V algebra by:

$$[J^{(1)}, J^{(2)}]_V = -J^{(3)*}. \tag{1.2.11}$$

Given a Lie algebra L and a representation ρ of L on another algebra V we produce a joint algebra E encompassing L and V, following the methods given by Aldrovandi [1]. The algebra E is an extension of L by V through ρ. The extension of V to E is an inclusion such that

$$[J^{(1)}, J^{(2)}]_V = [J^{(1)}, J^{(2)}]_E = f^{(3)*}_{(1)(2)} J^{(3)*}, \tag{1.2.12}$$

so, $f^{(3)*}_{(1)(2)} = -1$. The extension of L to E is a mapping such that,

$$\left.\begin{aligned} &\sigma : L \rightarrow E, \\ &\sigma : J_\mu \rightarrow J^{(a)}; \qquad (a) = (1), (2), (3), \end{aligned}\right\} \tag{1.2.13}$$

and,

$$[J^{(1)}, J^{(2)}]_E = iC^{(3)*}_{(1)(2)} J^{(3)*}, \tag{1.2.14}$$

where

$$[J_\mu, J_\nu] = C^\rho_{\mu\nu} J_\rho. \tag{1.2.15}$$

Therefore $C^{(3)*}_{(1)(2)} = i$ and,

$$iC^{(3)*}_{(1)(2)} = f^{(3)*}_{(1)(2)}. \tag{1.2.16}$$

The mapping (1.2.13) means that the Cartesian space of J_μ in L is extended to a complex spherical space in E.

The Lie algebras L and V have been combined into a Lie algebra E with an underlying vector space $L \oplus V$, the direct *sum* of those of L and V. In general, L and V can be combined [1] to give many different extended algebras E. In this case E is an algebra that incorporates the Cartesian basis (L) and the spherical basis V where L describes a Cartesian basis and V a spherical basis only. For rotation generators, the extended Lie algebra E is given by

$$[J^{(1)}, J^{(2)}]_E = [J^{(1)}, J^{(2)}]_V = f^{(3)*}_{(1)(2)} J^{(3)*}, \tag{1.2.17}$$

$$[J^{(1)}, J^{(2)}]_E = \rho(J_\mu) J^{(2)} = iC^{(3)*}_{(1)(2)} J^{(3)*}, \tag{1.2.18}$$

$$[J_\mu, J_\nu]_E = C^\rho_{\mu\nu} J_\rho - \beta^{(3)*}_{\mu\nu} J^{(3)*}, \tag{1.2.19}$$

where the constants $\beta^{(3)*}_{\mu\nu}$ measure the departure from homomorphism [1].

2.1.2 Extended Lie Algebra with Connections

Equation (1.2.18) above measures that the coupling between the internal space ((1), (2), (3)) and the extended space of the E Lie algebra. This can be made clear by writing the commutator on the left hand side of Eq. (1.2.18) as [5—8],

$$\left[J^{(1)}, J^{(2)}\right]_E = \frac{1}{i}\left[\frac{1}{\sqrt{2}}\left(J_X - iJ_Y\right), J^{(2)}\right]. \tag{1.2.20}$$

This means that the theory developed in Chap. 1 is an extended Lie algebra with connections, which is described in general gauge theory by Aldrovandi on his page 39 [1], his Eqs. (105) to (111). In this extended Lie algebra, the connection is denoted [1] B_μ^a and

$$X'_\mu = X_\mu - B_\mu^a X_a. \tag{1.2.21}$$

The commutator relations become

$$\left.\begin{aligned} \left[X'_\mu, X'_\nu\right] &= C^\rho_{\mu\nu} X'_\rho - \beta'^c_{\mu\nu} X_c, \\ \left[X'_\mu, X_b\right] &= C'^c_{\mu b} X_c, \\ \left[X_a, X_b\right] &= f^c_{ab} X_c, \end{aligned}\right\} \tag{1.2.22}$$

where [1],

$$\left.\begin{aligned} \beta'^c_{\mu\nu} &= \beta^c_{\mu\nu} + K^c_{\mu\nu}, \\ K^c_{\mu\nu} &= C^c_{\mu a} B^a_\nu - C^c_{\nu a} B^a_\mu - B^c_\rho C^\rho_{\mu\nu} - f^c_{ab} B^a_\mu B^b_\nu \\ C'^c_{\mu b} &= C^c_{\mu b} - B^a_\mu f^c_{ab}. \end{aligned}\right\} \tag{1.2.23}$$

If $C'^c_{\mu b} = \beta'^c_{\mu\nu} = 0$ there is no extension [1].

We are now in a position to check this extended gauge theoretical structure against the *O(3)* gauge theory given by Ryder [4] in his Chap. 3. This theory is in turn the basis of our development in Chap. 1. We shall first show that even *U(1)* electrodynamics, seen as a gauge theory, is a special

case of Eq. (1.2.22), but with $C'^c_{\mu b}$ *not* equal to zero. This means that the Lie algebra underlying ordinary *U(1)* electrodynamics is an *extended* Lie algebra, and so the spaces *L* and *V* are *not* independent, even in Maxwellian electrodynamics seen as a gauge field theory. So application of gauge theory, with affine algebra, to electrodynamics is different in principle from its application to elementary particle physics, if- when the latter takes the two spaces to be independent. The only idea in common is that space-time is made inhomogeneous.

The special case of *U(1)* electromagnetism can be recovered from Eqs. (1.2.22) as follows. Firstly define the covariant derivatives by taking X'_μ and X'_ν to be extended translation generators,

$$X'_\nu = \partial_\mu - igA_\mu, \qquad X'_\nu = \partial_\nu - igA_\nu, \tag{1.2.24}$$

where $g = e$, the charge on the proton, and A_μ and A_ν are the *U(1)* four potentials [4]. Then,

$$\left[X'_\mu, X'_\nu\right] = -ie\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = -ieF_{\mu\nu}, \tag{1.2.25}$$

where $F_{\mu\nu}$ is the ordinary *U(1)* field tensor. From Aldrovandi's Eq. (99),

$$\left[X_\mu, X_\nu\right]_E = C^\rho_{\mu\nu} X_\rho = 0, \tag{1.2.26}$$

because $X_\mu = \partial_\mu$; $X_\nu = \partial_\nu$; $X_\rho = \partial_\rho$ are translation generators within a proportionality factor [4]. Therefore,

$$C^\rho_{\mu\nu} = 0, \tag{1.2.27}$$

and

$$\beta'^c_{\mu\nu} = igF_{\mu\nu}. \tag{1.2.28}$$

Since we are dealing with an affine algebra, we obtain in Aldrovandi's Eq. (100),

$$\beta^{c}_{\mu\nu} = 0 . \tag{1.2.29}$$

Furthermore, in *U(1)*,

$$X_a = X_b = X_c = 1 , \tag{1.2.30}$$

being interpreted as rotation generators of *U(1)* [4]. Therefore,

$$f^{c}_{ab} = 0 , \tag{1.2.31}$$

and

$$\beta^{\prime c}_{\mu\nu} = K^{c}_{\mu\nu} = C^{c}_{\mu a} B^{a}_{\nu} - C^{c}_{\nu a} B^{a}_{\mu} . \tag{1.2.32}$$

We can identify:

$$\left.\begin{array}{ll} C^{c}_{\mu a} := ig\partial_{\mu} , & C^{c}_{\nu a} := ig\partial_{\nu} , \\ B^{a}_{\nu} := A_{\nu} , & B^{a}_{\mu} := A_{\mu} . \end{array}\right\} \tag{1.2.33}$$

As described by Aldrovandi the *C's* are interpreted as matrix operators which in *U(1)* are 1×1 matrices, *e.g.*,

$$C^{b}_{\mu c} = \left(X_{\mu}\right)^{b}_{c} = \partial_{\mu} \quad \text{etc.} . \tag{1.2.34}$$

When we come to examine the commutator (1.2.22b) we find,

$$\left[X^{\prime}_{\mu}, X_b\right] = \left[\partial_{\mu} - igA_{\mu} , 1\right] = C^{\prime c}_{\mu b} X_c = 0 . \tag{1.2.35}$$

We have shown that $f^{c}_{ab} = 0$, so,

$$C^{\prime c}_{\mu b} = C^{c}_{\mu b} = \partial_{\mu} , \tag{1.2.36}$$

and

$$C^{\prime c}_{\mu b} X_c = \partial_{\mu} 1 = 0 . \tag{1.2.37}$$

So Eq. (1.2.35) becomes

$$0 = C^{c}_{\mu b} 0 , \qquad C^{c}_{\mu b} \neq 0 , \tag{1.2.38}$$

and since the spaces L and V decouple if and only if $C^{c}_{\mu b} = 0$, *the U(1) theory of electrodynamics is a gauge theory in an extended Lie algebra.* So the fundamental and basic fields in *U(1)* electrodynamics occur in an E space; $E = L \oplus V$, the direct sum of L and V. The internal *U(1)* gauge space is *not* independent of the space-time of the gauge theory. This is of course a physical result, because the potential four-vector A_{μ} of the *U(1)* theory is well defined in both L and V.

These points appear not to have been realized hitherto, or not made clear. The application as exemplified by Ryder, of gauge field theory to classical electrodynamics implies an extended Lie algebra whose two spaces are *not* independent. If elementary particle theory is to be defined as a gauge field theory then it is usually assumed that the two spaces are independent. This hypothesis is justified by its success in particle physics, but evidently, the photon does not fit into a gauge field theory whose spaces are independent, even in the *U(1)* linear approximation. This questions the standard model again at a fundamental level, and questions the assertion that the photon is a particle, or at least the same type of particle as for example a quark or electron. This is a rigorous result of pure geometry applied as we have just demonstrated to *U(1)* electrodynamics, the kind of electrodynamics that is usually quantized to give the photon.

On the classical level, the same geometrical methods of advanced fiber bundle theory show [1] that there is no conceptual problem whatsoever in replacing *U(1)* by *O(3)* in classical electrodynamics, we are simply changing the symmetry of an internal space which by definition is a symmetry space if we are to apply group theoretic restrictions to the covariant derivative. (More generally we can lift these restrictions and make the covariant derivative a Taylor series for example.) Extended gauge theory [1] gives as rigorous a basis for *O(3)* as it does for *U(1)*, or any other group theoretic restriction on the covariant derivative. In the last analysis, such a description is a guess about the vacuum, or in-homogeneity of space-time, and shifts the description of what mediates interaction between two charges from the field to the potential and to space-time itself, in the spirit of general relativity. In this scenario, the classical electromagnetic field is the result of a round trip with covariant derivatives [4]. If the round trip has a physical effect, the field is not zero. The *U(1)* hypothesis makes the covariant derivative linear in the potential four-vector. A round trip produces the familiar four-curl, but from a theory akin to *general* relativity [4] in which space-time itself is given a structure. This simple linear hypothesis results in Maxwell's equations. The next simplest guess, or hypothesis, is *O(3)* electrodynamics, in which the covariant derivative contains rotation generators of *O(3)*, and in which the field is non-linear in the potential as developed in Chap 1. The *O(3)* guess results in a theory which is already much richer than *U(1)*, but is still a simple guess. Proceeding in this way there emerges a set of classical electrodynamic theories, each member of which is as rigorous as *U(1)*. The differences between each member of this set of theories show up most vividly in the vacuum. For example, as we have seen in Chap. 1, *O(3)* gives vacuum polarization and magnetization (the B cyclic theorem), which are all missing from *U(1)*. Similarly, an *SU(3)* group theoretic guess will bring out a far richer structure than *O(3)* and so on. In this way we can begin to describe the various non linear optical phenomena [5—8] which have no existence at all in *U(1)*. The conventional [9] phenomenological approach simply inserts non-linear terms into *U(1)* through the classical constitutive relations: a hybrid, self-contradictory, approach. (Recall, for example, that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ has no existence in *U(1)*,

but is thrown ad hoc into the theory in order to be able to describe the inverse Faraday effect.)

The *O(3)* theory emerges from the general geometrical equations if the internal space V has this symmetry, so that X_a, X_b and X_c become rotation generators of *O(3)*,

$$\left.\begin{aligned} X_a := J_a, \quad X_b := J_b, \quad X_c := J_c, \\ \left[J_a, J_b\right] = f_{ab}^c J_c, \quad f_{ab}^c \neq 0. \end{aligned}\right\} \tag{1.2.39}$$

The *O(3)* covariant derivatives are extended space-time translation generators of the general theory given by Aldrovandi [1],

$$\left.\begin{aligned} X_\mu^{\prime} = \partial_\mu - igA_\mu^a J_a := \partial_\mu - igA_\mu := D_\mu, \\ X_\nu^{\prime} = \partial_\nu - igA_\nu^a J_a := \partial_\nu - igA_\nu := D_\nu, \end{aligned}\right\} \tag{1.2.40}$$

and g is proportional to the elementary charge e through different coefficients in free space and in the presence of matter. This has no effect on the *O(3)* symmetry of the internal gauge space. So [5—8],

$$\left[X_\mu^{\prime}, X_\nu^{\prime}\right]_E = -ig\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]\right), \tag{1.2.41}$$

and the commutator $\left[A_\mu, A_\nu\right]$ is non-zero, making the theory non-linear.

Note that this is still a theory of electromagnetism, the elementary charge e still appears in it, and the potentials are electromagnetic potentials. The theory is, in the last analysis, a purely geometrical description of classical electromagnetism.

As in *U(1)* theory,

$$C_{\mu\nu}^{\rho} = 0, \tag{1.2.42}$$

because unextended translation generators commute, *e.g.* $\left[\partial_\mu , \partial_\nu\right] = 0$ as used for example by Ryder [4] in an *SU(2)* symmetry gauge field theory of elementary particles. So,

$$\left[X'_\mu , X'_\nu\right] = -\beta'^{c}_{\mu\nu} X_c \,, \tag{1.2.43}$$

indicating the existence of an internal vector space as in the notation of Eqs. (1.1.38) of Chap. 1; and so, as in Eq. (3.169) of Ryder [4],

$$\beta'^{c}_{\mu\nu} = ig\left(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu - ig\,\epsilon_{cab} A^a_\mu A^b_\nu\right) . \tag{1.2.44}$$

If we assume no departure from homomorphism [1], *i.e.*, that,

$$\beta^c_{\mu\nu} = 0 \,, \tag{1.2.45}$$

we obtain,

$$\beta'^{c}_{\mu\nu} = C^c_{\mu a}\beta^a_\nu - C^c_{\nu a}\beta^a_\mu - f^c_{ab} B^a_\mu B^b_\nu \,, \tag{1.2.46}$$

and so,

$$\left.\begin{aligned} C^c_{\mu a} B^a_\nu &:= ig\partial_\mu A^c_\nu \,, \qquad C^c_{\nu a} B^a_\mu := ig\partial_\nu A^c_\mu \\ igA^a_\mu &:= B^a_\mu \,, \qquad igA^a_\nu := B^a_\nu \,, \\ f^c_{ab} &= i\epsilon_{cab} \,. \end{aligned}\right\} \tag{1.2.47}$$

The spaces L and V are connected into an extended Lie algebra because

$$\begin{aligned} \left[X'_\mu , X_b\right]_E &= \left[\partial_\mu - igA_\mu , J_b\right] = \left[\partial_\mu , J_b\right] - ig\left[A_\mu , J_b\right] \\ &= \partial_\mu J_b - igA^a_\mu\left[J_a , J_b\right] \\ &= \partial_\mu J_b - igA^a_\mu f^c_{ab} J_c \,. \end{aligned} \tag{1.2.48}$$

This result, in Aldrovandi's notation, is identified as,

$$\begin{aligned} C^c_{\mu b} X_c &:= \partial_\mu J_b = 0 \,, \\ B^a_\mu f^c_{ab} X_c &:= igA^a_\mu f^c_{ab} J_c \end{aligned} . \tag{1.2.49}$$

The *O(3)* theory of classical electromagnetism is therefore an example of a gauge field theory in an affine space with $E = L \oplus V$. So all the development by Aldrovandi in his pages 39 ff. can be taken over unchanged as a description of *O(3)* electrodynamics. This means that all the insights on pp. 39 ff. of Aldrovandi [1] can be implemented, including those in unified field theory. The result is a powerful support for *O(3)* electrodynamics based on pure geometry. No physics has yet entered the scene [1]. In other words we have guessed that space-time can be made inhomogeneous by the imposition of an internal *O(3)* symmetry in a gauge field theory. Metaphorically, Maxwell guessed that this symmetry is *U(1)*. (Historically, gauge field theories were not, of course, available to him.)

2.2 The Geometrical Meaning of *O(3)* Electrodynamics

The results of Sec. 2.1 mean that *O(3)* electrodynamics is *completely* defined in contemporary geometrical theories, provided that $E = L \oplus V$, *i.e.*, that E is the direct sum of L and V. For example, *O(3)* electrodynamics can be fully developed using exterior derivatives in an anholonomic basis, extending the Maurer-Cartan equations as described in Aldrovandi's Sec. 6.2. The *O(3)* electrodynamics can also be developed as a field algebra on manifolds, leading to a similarity with gravitational theories as developed in Vol. 4 of this series [8]. The key empirical difference between *U(1)* and *O(3)* electrodynamics is that the commutator $\left[A_\mu, A_\nu\right]$ is non zero in *O(3)*, as observed in the inverse Faraday effect of nonlinear magneto-optics [9].

2.3 The Field Equations of *O(3)* Electrodynamics

Physics enters the scene when we come to consider field equations [1]. In this section, their complete self-consistency in classical *O(3)* electrodynamics is demonstrated for the free field and in the presence of field matter interaction.

2.3.1 The *O(3)* Field Equations in Free Space

It is argued in this section that the following *O(3)* free space field equations are rigorously self-consistent,

$$D_\nu \tilde{\boldsymbol{G}}^{\mu\nu} := \boldsymbol{0}, \tag{1.2.50}$$

$$D_\nu \boldsymbol{G}^{\mu\nu} = \frac{\boldsymbol{J}\left(\text{vac}\right)}{\epsilon_0}, \tag{1.2.51}$$

where $\tilde{\boldsymbol{G}}^{\mu\nu}$ is the dual of the *O(3)* field tensor defined in Chap. 1,

$$\tilde{\boldsymbol{G}}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\boldsymbol{G}_{\rho\sigma}, \tag{1.2.52}$$

and where $\boldsymbol{J}^\mu\left(\text{vac}\right)$ is a vacuum Noether current, or helicity current, to be defined. Eq. (1.2.50) is the Feynman-Jacobi identity for an *O(3)* symmetry

gauge field theory [4], one in which the covariant derivative is defined in terms of *O(3)* rotation generators. The *O(3)* field tensor $\boldsymbol{G}^{\mu\nu}$ is defined and discussed in detail in Chap. 1. The equations (1.2.50) and (1.2.51) are therefore,

$$\partial_\nu \tilde{\boldsymbol{G}}^{\mu\nu} + g\boldsymbol{A}_\nu \times \tilde{\boldsymbol{G}}^{\mu\nu} = \boldsymbol{0}, \tag{1.2.53}$$

$$\partial_\nu \boldsymbol{G}^{\mu\nu} + g\boldsymbol{A}_\nu \times \boldsymbol{G}^{\mu\nu} = \frac{\boldsymbol{J}^\mu\left(\text{vac}\right)}{\epsilon_0}, \tag{1.2.54}$$

for the free classical field. Equations (1.2.53) and (1.2.54) use the same notation as in Ryder's discussion of Yang-Mills theory [8], but as discussed, form an extended Lie algebra. The coefficient g for the free field is,

$$g = \frac{\kappa}{A^{(0)}} = \frac{e}{\hbar}, \tag{1.2.55}$$

and is proportional to the elementary charge e after quantization [5—8],

$$e = \hbar\left(\frac{\kappa}{A^{(0)}}\right). \tag{1.2.56}$$

This concept of photon momentum $\hbar\kappa$ occurs after quantization of the *U(1)* theory, but usually, it is not clear that this quantum of momentum, the photon momentum, is equal to $eA^{(0)}$ for the free field. The conceptual problem posed by Eq. (1.2.55) is the presence of e in the free field, and as argued already, its presence does not mean that the field is charged. It means that the field is C negative. Therefore non-Abelian gauge field theory such as *O(3)* classical electrodynamics allows charge quantization, the elementary charge e being that on the proton, minus the charge on the electron. Equation (1.2.56) is similar to Planck quantization, $En = \hbar\omega$, of the energy. The self consistency of this result is illustrated through the fact that an electron accelerated to c becomes the electromagnetic field in free space as described by Jackson [2], and charge conservation means that e is present in the free field. On an elementary physical level, the electromagnetic field must be C negative for one charge to influence another through the field. In action at a distance theories the same must apply, for example in Schwarzschild's delayed action at a distance theory of 1902 [10] the potential is C negative. On the classical level, the factor g in free space is the wavevector magnitude divided by $A^{(0)}$, and so g is C negative as required. In electrostatics, g goes to zero, and the *O(3)* theory takes on a linear form, giving the Coulomb, Gauss and Ampère Laws of electrostatics and magnetostatics. This is easy to see because g goes to zero gives a linear theory which has the same structure as the familiar Maxwellian theory, except for the presence of indices (1) and (2), indicating complex conjugation. In the static limit however, we can use a real potential four-vector, so that the indices (1) and (2) are equal. (Complex conjugation does not affect a real valued variable.)

More subtly, we must consider whether the elementary magnetic flux density on one photon, which is the elementary magnetic fluxon [8], divided by a quantization volume, is localized or not after quantization. It is well known that the photon, the quantum of energy, is not localized, and that the photon can be created and destroyed with creation and annihilation operators without affecting the principle of conservation of energy. These are features of the quantized *U(1)* electromagnetic field. After quantization of *O(3)* however, we find Eq. (1.2.56), and it seems that the fluxon $\hbar/e$ may share these properties of being non localized with the quantized unit of energy, the conventional photon, $\hbar\omega$. It then seems appropriate to ask whether $\hbar/e$ divided by the quantization volume can also be created and destroyed statistically within the quantized *O(3)* field without violating the principle of conservation of charge. This is not a feature of the quantized *U(1)* field.

2.3.2 Self Consistency of Eqs. (1.2.50) and (1.2.51)

Equations (1.2.50) and (1.2.51) are self consistent and consistent with the definition of $\boldsymbol{A}_\mu$ and $\boldsymbol{G}^{\mu\nu}$ used in Chap. 1. The detailed proof of this self consistency is given in this section. The solution of the two field equations (1.2.50) and (1.2.51) must be consistent with the fact that the B cyclic theorem is produced from the fundamental definition of the field tensors appearing in the field equations. This can be so if and only if,

$$\boldsymbol{A}_\nu \times \tilde{\boldsymbol{G}}^{\mu\nu} = \boldsymbol{0}\,, \tag{1.2.57}$$

$$\boldsymbol{A}_\nu \times \boldsymbol{G}^{\mu\nu} = \frac{\boldsymbol{J}(\text{vac})}{\epsilon_0}\,, \tag{1.2.58}$$

These conditions give the *O(3)* field equations in the form,

$$\partial_\nu \tilde{\boldsymbol{G}}^{\mu\nu} = \boldsymbol{0} \tag{1.2.59}$$

$$\partial_\nu \boldsymbol{G}^{\mu\nu} = \boldsymbol{0}\,, \tag{1.2.60}$$

where

$$\tilde{\boldsymbol{G}}^{\mu\nu} = \tilde{G}^{\mu\nu(1)}\boldsymbol{e}^{(1)} + \tilde{G}^{\mu\nu(2)}\boldsymbol{e}^{(2)} + \tilde{G}^{\mu\nu(3)}\boldsymbol{e}^{(3)}\,, \tag{1.2.61}$$

$$\boldsymbol{G}^{\mu\nu} = G^{\mu\nu(1)}\boldsymbol{e}^{(1)} + G^{\mu\nu(2)}\boldsymbol{e}^{(2)} + G^{\mu\nu(3)}\boldsymbol{e}^{(3)}\,, \tag{1.2.62}$$

so we obtain the equations for indices (1) and (2),

$$\partial_\nu \tilde{G}^{\mu\nu(1)} = \partial_\nu \tilde{G}^{\mu\nu(2)} = 0\,, \tag{1.2.63}$$

$$\partial_\nu G^{\mu\nu(1)} = \partial_\nu G^{\mu\nu(2)} = 0\,, \tag{1.2.64}$$

and the equations for the $\boldsymbol{B}^{(3)}$ field,

$$\partial_\nu \tilde{G}^{\mu\nu(3)} = \partial_\nu G^{\mu\nu(3)} = 0\,. \tag{1.2.65}$$

Equations (1.2.63) and (1.2.64) are formally identical with the Maxwell equations in free space for the complex field tensor components $G^{\mu\nu(1)} = G^{\mu\nu(2)*}$ and its dual. Equation (1.2.65) is not present in *U(1)* electrodynamics and in vector notation gives the $\boldsymbol{B}^{(3)}$ field equation in free space,

$$\nabla \times \boldsymbol{B}^{(3)} = \boldsymbol{0}\,, \qquad \frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \boldsymbol{0}\,. \tag{1.2.66}$$

It will be shown that Eq. (1.2.57) produces the B cyclic equations self consistently. Equation (1.2.58) produces the helicity current, which depends on $\boldsymbol{B}^{(3)} \neq \boldsymbol{0}$; $\boldsymbol{A}^{(3)} \neq \boldsymbol{0}$. These concepts are not available in *U(1)* electromagnetism. Equations (1.2.57) and (1.2.58) also produce the vacuum Maxwell equations given, self-consistently, that the plane waves $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ are solutions of the Maxwell equations and that $\boldsymbol{B}^{(3)}$ is a solution of Eqs. (1.2.65) and (1.2.66), being phase free. A third self-consistency check is that $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ are linked by the B cyclic theorem which is given by Eq. (1.2.57). A fourth check for self consistency is given by the fact that the *O(3)* electrodynamical equations in free space give $\boldsymbol{E}^{(3)} = \boldsymbol{0}$. The $\boldsymbol{B}^{(3)}$ field is not accompanied by an $\boldsymbol{E}^{(3)}$ field [5—8] as shown by Eq. (1.2.57) to (1.2.66), and as shown empirically by Raja *et al.* [8] and Compton *et al.* [8]. There is no Faraday induction due to $\boldsymbol{B}^{(3)}$ and it is observed experimentally through $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in field-matter interaction, for example the magnetization of the inverse Faraday effect is due to $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ as shown in Chap. 1. Note that $\boldsymbol{B}^{(3)}$ is therefore a fundamental field of *O(3)* electrodynamics, but does not occur in *U(1)* electrodynamics. As shown earlier in this chapter, both theories are rigorous gauge field theories in an extended Lie algebra $E = T \oplus V$. However, *O(3)* has several advantages over *U(1)* and gives insights and concepts which *U(1)* does not. For example, the commutator $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is zero by definition in *U(1)* electrodynamics, yet $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is an empirical observable of the inverse Faraday effect. This is a sure indicator of the need for an *O(3)* or other non-linear electrodynamics.

There is no known electric analogue of the inverse Faraday effect, suggesting that there is no $\boldsymbol{E}^{(3)}$ field as indicated by *O(3)* electrodynamics. The effect, as for its famous counterpart, the Faraday effect, is magnetic in nature and is mediated by the same Verdet constant [9].

The Stokes Theorem applied to $\boldsymbol{B}^{(3)}$ is clearly not to be found in the *U(1)* electrodynamics, and is the integral form of the $\boldsymbol{B}^{(3)}$ curl equation, $\nabla \times \boldsymbol{B}^{(3)} = \boldsymbol{0}$. This is simply a consequence of the fact that $\boldsymbol{B}^{(3)}$ is irrotational, and that the Stokes Theorem means that the curl of an irrotational vector field vanishes for any contour [8]. Again, such a result will not occur in *U(1)* electrodynamics because $\boldsymbol{B}^{(3)}$ is not defined there.

2.3.2.1 Self Consistency of Equations (1.2.57) and (1.2.50)

In order for the linearization scheme leading to Eqs. (1.2.59) and (1.2.60) to be applicable the solutions of Eq. (1.2.57) must be consistent with Eq. (1.2.50). In order to show this we write the covariant derivative as [5—8],

$$D_\mu = \partial_\mu - igM^a A_\mu^a\,, \tag{1.2.67}$$

so its action on the general m component field ψ_m is,

$$D_\mu \psi_m = \partial_\mu \psi_m - ig\left(M^a\right)_{mn} A_\mu^a \psi_n = \partial_\mu \psi_m - g\epsilon_{amn} A_\mu^a \psi_n\,. \tag{1.2.68}$$

Therefore the Feynman-Jacobi identity is

$$\partial_\mu \tilde{G}_m^{\mu\nu} - g\,\epsilon_{amn} A_\mu^a \tilde{G}_n^{\mu\nu} = 0 . \tag{1.2.69}$$

We know that the B cyclic theorem is constructed from,

$$\partial_\mu \tilde{G}_m^{\mu\nu} = 0 , \qquad m = (1), (2), (3) , \tag{1.2.70}$$

and that this implies Eq. (1.2.57), *i.e.*,

$$\epsilon_{amn} A_\mu^a \tilde{G}_n^{\mu\nu} = 0 . \tag{1.2.71}$$

At this point it is necessary to verify that Eq. (1.2.70) is self consistent with Eq. (1.2.71). For $m = (3)$, for example,

$$\partial_\nu \tilde{G}^{\mu\nu(3)*} - ig\,\epsilon_{(1)(2)(3)} A_\nu^{(1)} \tilde{G}^{\mu\nu(2)} = 0 , \tag{1.2.72}$$

with,

$$\tilde{G}^{03(3)*} = -\tilde{G}^{30(3)*} \neq 0 , \tag{1.2.73}$$

and with all other components zero. Therefore,

$$\partial_0 \tilde{G}^{03(3)*} - ig\left(A_0^{(1)} \tilde{G}^{03(2)} - A_0^{(2)} \tilde{G}^{03(1)} \right) = 0 . \tag{1.2.74}$$

This equation is consistent with,

$$\tilde{G}^{03(2)} = \tilde{G}^{03(1)} = 0 , \qquad A_0^{(1)} = A_0^{(2)} = 0 , \tag{1.2.75}$$

qed. For $m = 1$, Eq. (1.2.71) gives,

$$\epsilon_{(2)(1)(3)} A_\mu^{(2)} \tilde{G}_{(3)}^{\mu\nu} + \epsilon_{(3)(1)(2)} A_\mu^{(3)} \tilde{G}_{(2)}^{\mu\nu} = 0 , \tag{1.2.76}$$

i.e.,

$$A_\mu^{(3)} \tilde{G}_{(2)}^{\mu\nu} = A_\mu^{(3)} \tilde{G}_{(1)}^{\mu\nu} . \tag{1.2.77}$$

For $m = (2)$,

$$A_\mu^{(1)} \tilde{G}_{(3)}^{\mu\nu} = A_\mu^{(3)} \tilde{G}_{(1)}^{\mu\nu} . \tag{1.2.78}$$

In vector notation, Eq. (1.2.78) gives,

$$\boldsymbol{A}^{(1)} \cdot \boldsymbol{B}^{(3)} = \boldsymbol{A}^{(3)} \cdot \boldsymbol{B}^{(1)} , \tag{1.2.79}$$

and

$$- A_0^{(1)} c\boldsymbol{B}^{(3)} + \boldsymbol{A}^{(1)} \times \boldsymbol{E}^{(3)} = - A_0^{(3)} c\boldsymbol{B}^{(1)} + \boldsymbol{A}^{(3)} \times \boldsymbol{E}^{(1)} . \tag{1.2.80}$$

Equation (1.2.79) is consistent with the fact that $\boldsymbol{A}^{(1)}$ is perpendicular to $\boldsymbol{B}^{(3)}$ and $\boldsymbol{A}^{(3)}$ to $\boldsymbol{B}^{(1)}$. Equation (1.2.80) simplifies to

$$\boldsymbol{A}^{(3)} \times \boldsymbol{E}^{(1)} = c A_0^{(3)} \boldsymbol{B}^{(2)*} , \tag{1.2.81}$$

which is consistent with the fact that the cross product of two polar vectors, $\boldsymbol{A}^{(3)}$ and $\boldsymbol{E}^{(1)}$, gives an axial vector $\boldsymbol{B}^{(2)*} = \boldsymbol{B}^{(1)}$ multiplied by $cA_0^{(3)}$, a scalar. To show that Eq. (1.2.81) is a component of the B cyclic theorem, and consistent with the fundamental definitions given in Chap. 1, use

$$\boldsymbol{A}^{(3)} = A_0^{(3)}\boldsymbol{k} = A_0^{(3)}\boldsymbol{e}^{(3)}, \tag{1.2.82}$$

so,

$$\boldsymbol{A}^{(3)} \times \boldsymbol{E}^{(1)} = A_0^{(3)}\left(\boldsymbol{k} \times \boldsymbol{E}^{(1)}\right) = cA_0^{(3)}\boldsymbol{B}^{(1)}, \tag{1.2.83}$$

a result which is consistent with

$$c\boldsymbol{B}^{(1)} = \boldsymbol{k} \times \boldsymbol{E}^{(1)}, \tag{1.2.84}$$

which in turn is a plane wave relation consistent with the fact that the B cyclic theorem is constructed from plane waves. This result is enough to show that Eq. (1.2.57) is consistent with Eq. (1.2.50). To reduce Eq. (1.2.83) to the B cyclic theorem use,

$$\boldsymbol{E}^{(1)} \rightarrow -ic\boldsymbol{B}^{(1)}, \quad \boldsymbol{A}^{(3)} \rightarrow \frac{\boldsymbol{B}^{(3)}}{\kappa}. \tag{1.2.85}$$

The arrows are used in the above equations to denote that $\boldsymbol{E}^{(1)}$ is numerically the same as $-ic\boldsymbol{B}^{(1)}$ and that $\boldsymbol{A}^{(3)}$ is numerically the same as $\boldsymbol{B}^{(3)}/\kappa$. We do not use $\boldsymbol{E}^{(1)} = -ic\boldsymbol{B}^{(1)}$ because a polar vector cannot be equal to an axial vector in *O(3)*. Equations (1.2.85) are then examples of duality transformations, rather than equations. Equation (1.2.85a) occurs in the *U(1)* theory, *i.e.,* for transverse plane waves, Eq. (1.2.85b) occurs only in *O(3)* theory. Use of the transformations (1.2.85) in Eq. (1.2.83) produces

$$\boldsymbol{B}^{(3)} \times \boldsymbol{B}^{(1)} = iB^{(0)}\boldsymbol{B}^{(2)*}, \tag{1.2.86}$$

which is one component of the B cyclic theorem, *qed.* The overall result is that the B cyclic theorem linearizes Eq. (1.2.50) to Eq. (1.2.59) in a rigorously self consistent way.

2.3.2.2 Tensor to Vector Notation

Considering for example,

$$A_\mu^{(2)}\tilde{G}^{(1)\mu\nu} = A_\mu^{(1)}\tilde{G}^{(2)\mu\nu}, \tag{1.2.87}$$

and taking $\nu = 0$,

$$\begin{aligned} &A_0^{(2)}\tilde{G}^{(1)00} + A_1^{(2)}\tilde{G}^{(1)10} + A_2^{(2)}\tilde{G}^{(1)20} + A_3^{(2)}\tilde{G}^{(1)30} \\ &= A_0^{(1)}\tilde{G}^{(2)00} + A_1^{(1)}\tilde{G}^{(2)10} + + A_2^{(1)}\tilde{G}^{(2)20} + A_3^{(1)}\tilde{G}^{(2)30}, \end{aligned} \tag{1.2.88}$$

i.e.,

$$\boldsymbol{A}^{(2)} \cdot \boldsymbol{B}^{(1)} = \boldsymbol{A}^{(1)} \cdot \boldsymbol{B}^{(2)}, \tag{1.2.89}$$

in vector form..

Similarly, for $\nu = 1$,

$$\begin{aligned} &A_0^{(2)}\tilde{G}^{(1)01} + A_1^{(2)}\tilde{G}^{(1)11} + A_2^{(2)}\tilde{G}^{(1)21} + A_3^{(2)}\tilde{G}^{(1)31} \\ &= A_0^{(1)}\tilde{G}^{(2)01} + A_1^{(1)}\tilde{G}^{(2)11} + + A_2^{(1)}\tilde{G}^{(2)21} + A_3^{(1)}\tilde{G}^{(2)31}, \end{aligned} \tag{1.2.90}$$

i.e.,

$$-A_0^{(2)}B^{(1)1} - A_2^{(2)}E^{(1)3} + A_3^{(2)}E^{(1)2}$$

$$= -A_0^{(1)}B^{(2)1} - A_2^{(1)}E^{(2)3} + A_3^{(1)}E^{(2)2} \quad (1.2.91)$$

$$-A_0^{(2)}\boldsymbol{B}^{(1)} + \boldsymbol{A}^{(2)} \times \boldsymbol{E}^{(1)} = -A_0^{(1)}\boldsymbol{B}^{(2)} + \boldsymbol{A}^{(1)} \times \boldsymbol{E}^{(2)}$$

$$i.e., \ \boldsymbol{A}^{(2)} \times \boldsymbol{E}^{(1)} = \boldsymbol{A}^{(1)} \times \boldsymbol{E}^{(2)} .$$

This result is consistent with the fact that

$$\boldsymbol{B}^{(2)} = \nabla \times \boldsymbol{A}^{(2)} , \quad (1.2.92)$$

qed.

Similarly self consistent results are found for $\nu = 2$ and for $\nu = 3$, demonstrating the rigorous self-consistency of *O(3)* electrodynamics in free space for the special case of plane waves $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ and for longitudinal phaseless $\boldsymbol{B}^{(3)}$.

2.3.3 Self Consistency of Equation (1.2.58)

Examination of the self consistency of Eq. (1.2.58) leads to the definition of a vacuum current that has no existence in *U(1)* theory. Linearization of Eq. (1.2.58) proceeds on the basis that in free space

$$\partial_\nu \boldsymbol{G}^{\mu\nu} = \boldsymbol{0} , \quad (1.2.93)$$

i.e.,

$$\partial_\nu G^{\mu\nu(1)} = \partial_\nu G^{\mu\nu(2)} = \partial_\nu G^{\mu\nu(3)} = 0 , \quad (1.2.94)$$

which is consistent with transverse plane waves; a phaseless longitudinal $\boldsymbol{B}^{(3)}$, and $\boldsymbol{E}^{(3)} = \boldsymbol{0}$ in free space.

This result implies that

$$g\boldsymbol{A}_\nu \times \boldsymbol{G}^{\mu\nu} = \frac{\boldsymbol{J}^\mu(\mathrm{vac})}{\epsilon_0} , \quad (1.2.95)$$

where $\boldsymbol{J}^\mu(\mathrm{vac})$ is a conserved current caused by the $\boldsymbol{B}^{(3)}$ field in free space and which does not exist in *U(1)* theory. We refer to it as the vacuum current, a polar vector in the *O(3)* symmetry internal gauge space whose scalar components in this space are polar four-vectors:

$$\left.\begin{aligned} J^{\mu(1)*} &= -ig\epsilon_0\left(A_\nu^{(2)}G^{\mu\nu(3)} - A_\nu^{(3)}G^{\mu\nu(2)}\right) \\ J^{\mu(2)*} &= -ig\epsilon_0\left(A_\nu^{(3)}G^{\mu\nu(1)} - A_\nu^{(1)}G^{\mu\nu(3)}\right) \\ J^{\mu(3)*} &= -ig\epsilon_0\left(A_\nu^{(1)}G^{\mu\nu(2)} - A_\nu^{(2)}G^{\mu\nu(1)}\right) . \end{aligned}\right\} \quad (1.2.96)$$

Using Eqs. (1.1 .64) and (1.1.65) we can proceed to investigate the above cyclic relations for each μ.

For $\mu = 0$ we obtain, for example,

$$A_\nu^{(1)}G^{0\nu(3)} - A_\nu^{(3)}G^{0\nu(1)} = \frac{i}{g\epsilon_0}J^{0(2)*} = 0 , \quad (1.2.97)$$

because all terms of

$$A_\nu^{(1)}G^{0\nu(3)} = A_\nu^{(3)}G^{0\nu(1)} , \quad (1.2.98)$$

are zero on both sides.

For $\mu = 1$,

$$A_\nu^{(1)} G^{1\nu(3)} - A_\nu^{(3)} G^{1\nu(1)} = \frac{i}{g\epsilon_0} J^{1(2)*}, \tag{1.2.99}$$

which reduces to

$$cA_Y^{(1)} B_Z^{(3)} = E_X^{(1)} A_0^{(3)} - cB_Y^{(1)} A_Z^{(3)} + \frac{i}{g\epsilon_0} J_X^{(1)}, \tag{1.2.100}$$

Using

$$E_X^{(1)} A_0^{(3)} = cB_Y^{(1)} A_Z^{(3)}, \tag{1.2.101}$$

we obtain

$$J_X^{(1)} = -icg\epsilon_0 A_Y^{(1)} B_Z^{(3)} = -i\epsilon_0 c \frac{A_Y^{(1)}}{A^{(0)}} B_Z^{(3)}, \tag{1.2.102}$$

which is a transverse current whose phase average is zero. It exists if and only if $\boldsymbol{B}^{(3)}$ is non-zero, and can be interpreted as a type of helicity current [8].

For μ = 2 we obtain

$$A_\nu^{(1)} G^{2\nu(3)} - A_\nu^{(3)} G^{2\nu(1)} = \frac{i}{g\epsilon_0} J^{2(2)*}, \tag{1.2.103}$$

which reduces to

$$cA_1^{(1)} B^{3(3)} - A_0^{(3)} E^{2(1)} + cA_3^{(3)} B^{1(1)} = \frac{i}{g\epsilon_0} J^{2(1)}, \tag{1.2.104}$$

i.e.,

$$\frac{i}{g\epsilon_0} J^{2(1)} + A_0^{(3)} E^{2(1)} = cA_1^{(1)} B^{3(3)} + cA_3^{(3)} B^{1(1)}, \tag{1.2.105}$$

or

$$\frac{i}{g\epsilon_0} J_Y^{(1)} + A_0^{(3)} E_Y^{(1)} = -2cA_X^{(1)} B_Z^{(3)}, \tag{1.2.106}$$

and using

$$E_Y^{(1)} = -cB_X^{(1)}, \tag{1.2.107}$$

we obtain

$$J_Y^{(1)} = igc\epsilon_0 A_X^{(1)} B_Z^{(3)}. \tag{1.2.108}$$

Therefore Eq. (1.2.51) linearizes to Eq. (1.2.94) provided that there is a vacuum current which phase averages to zero. This can be identified as a helicity current due to $\boldsymbol{A}^{(1)}$ and $\boldsymbol{B}^{(3)}$. Equation (1.2.93) is a vector equation in the *O(3)* internal space and produces three scalar equations in this space, Eqs. (1.2.94). Those for indices (1) and (2) are the inhomogeneous Maxwell equations in free space and the third in vector form is Eq. (1.2.66), a result which shows that $\boldsymbol{B}^{(3)}$ is irrotational if $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ are plane waves or if $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ have imaginary phases opposite in sign. This is consistent with the B cyclic theorem, or vacuum magnetization.

To summarize, a theory of electrodynamics has been proposed based on the structure of general gauge field theory as used in particle and high energy physics. This theory linearizes self-consistently to the homogeneous and inhomogeneous Maxwell equations giving in the process a phase dependent vacuum current proportional directly to $\boldsymbol{B}^{(3)}$. This vacuum current is therefore zero in the *U(1)* theory: a self inconsistency of Maxwellian electrodynamics in the received view because if the electromagnetic field is assumed to propagate in vacuo at c, it must carry a

C negative influence at a finite velocity from one charge to another. This is a current which is always non-zero, yet which is set to zero in "charge free regions". In *O(3)* electrodynamics there is always a vacuum current which depend on the non-zero $\boldsymbol{B}^{(3)}$ component of the field. Therefore there are several ways in which the *O(3)* hypothesis is more self-consistent than the *U(1)* hypothesis. In other words, linearization as in Maxwellian electrodynamics removes a great deal of information and leads to self inconsistencies. The simplest possible type of non linear theory, based on the *O(3)* group symmetry, produces non-linear effects which are missing from the *U(1)* theory, and which are reinstated in that theory by hand. One of these is the vacuum current as demonstrated in this section; other examples include vacuum polarization and magnetization.

References

[1] R. Aldrovandi, in T.W. Barrett and D. M. Grimes, eds., *Advanced Electromagnetism, Foundations, Theory and Applications*, Chap. 1 (World Scientific, Singapore, 1995).

[2] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).

[3] W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, 1962).

[4] L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, 1987).

[5] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 1: The Field* $\boldsymbol{B}^{(3)}$ (Kluwer Academic, Dordrecht, 1995).

[6] M. W. Evans and J.-P. Vigier, *The Enigmatic Photon, Vol. 2: Non-Abelian Electrodynamics* (Kluwer Academic, Dordrecht, 1995).

[7] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Vol. 3: Theory and Practice of the* $\boldsymbol{B}^{(3)}$ *Field* (Kluwer Academic, Dordrecht, 1996).

[8] M. W. Evans, J.-P. Vigier, and S.Roy, *The Enigmatic Photon, Vol. 4: New Directions* (Kluwer Academic, Dordrecht, 1998).

[9] M. W. Evans and S. Kielich eds., *Modern Nonlinear Optics*, Vols. 85(1), 85(2), 85(3) of Advances in Chemical Physics, I. Prigogine and S. A. Rice, eds. (Wiley Interscience, New York, 1993/1994/1997).

[10] K. Schwarzschild, *Göttinger Nachr., Math.-Phys. Klasse*, 1903, p.126

Chapter 3

Field-Matter Interaction

3.1 Introduction

In this chapter we demonstrate the fundamental $\boldsymbol{B}^{(3)}$ to one fermion interaction that leads to the phenomenon of radiatively induced electron spin resonance (*ESR*) and nuclear magnetic resonance (*NMR*). The Dirac equation was first solved to show the existence of radiatively induced fermion resonance (*RFR*) as reported in the third volume of this series [1]. The term responsible for the effect was isolated to be the novel interaction energy, the real valued and physical expectation value,

$$En = i\frac{e^2}{2m}\,\sigma\cdot\boldsymbol{A}\times\boldsymbol{A}^*, \tag{1.3.1}$$

where e/m is the charge to mass ratio of a fermion (electron or proton) and $i\boldsymbol{A}\times\boldsymbol{A}^*$ the real valued conjugate product of complex vector potentials in a circularly polarized electromagnetic field, considered to be classical in the manner first proposed by Dirac [2]. In Eq. (1.3.1), σ is the Z component of the Pauli matrix [3—6]. The interaction energy can be expressed in terms of the $\boldsymbol{B}^{(3)}$ field of the radiation as [1]

$$En = -\frac{e\hbar}{2m}\,\sigma\cdot\boldsymbol{B}^{(3)}, \tag{1.3.2}$$

and this is an *ESR* or *NMR* equation with the static magnetic field replaced by $\boldsymbol{B}^{(3)}$. All known *ESR* and *NMR* effects can therefore be induced by radiation rather than by a static magnetic field. The technique produces unprecedented resolving power because the resonance frequencies are proportional to I/ω^2 where I is the beam power density (or intensity in $W m^{-2}$) and ω beam angular frequency. With moderate microwave pumping, fermion resonance can be induced in theory in the visible, and picked up with an ordinary Fourier transform infra red-visible spectrometer acting as probe. This produces in theory a resolving power about one thousand to ten thousand times that available with magnet based *ESR* or *NMR* of any kind (including multi dimensional *ESR* and *NMR*) because the visible range is that much higher in frequency than the microwave (or gigahertz) range in which the current instruments operate.

This result indicates the existence and usefulness of the $\boldsymbol{B}^{(3)}$ field and is the fundamental spin-spin coupling between the photomagneton [7] (the photon's $\boldsymbol{B}^{(3)}$ field) and the fermion's half integral spin $\boldsymbol{B}^{(3)}$ proposed by Pauli [8] and Dirac [9]. Indications of the existence of $\boldsymbol{B}^{(3)}$ open the road to non-Abelian electrodynamics and non-local and superluminal interpretations [10] unknown in the traditional view [11].

In this chapter the above result is reproduced with several equations of motion, beginning with the Newton equation of a classical charged particle in a classical electromagnetic field; and ending with the quantum relativistic van der Waerden equation [12] for a two component spinor. The complete hierarchy of known equations of motion in physics produces the same *RFR* term, Eq. (1.3.1). It is a real, non-zero and physical ground state term in Rayleigh-Schrödinger perturbation theory [13]. The same type of coupling appears to have been recognized in principle by Pershan *et al.* [14] in 1966, during their establishment of the inverse Faraday effect, but these authors used higher order perturbation theory near optical resonance as did Li *et al.* [15] and others [16—20] in recent papers confirming the original proposal of *RFR* [21]. The key $\sigma\cdot\boldsymbol{A}\times\boldsymbol{A}^*$ coupling in higher order perturbation theory is clearly represented in Ref. [14] Eq. (8.6) and was confirmed by them empirically in paramagnetic, rare earth doped glass samples. These authors did not appear to realize however that the ground state term (1.3.1) is non-zero. This is the fundamental $\boldsymbol{B}^{(3)}$ term discussed in this chapter and occurs independently of any optical resonance, as in

ordinary magnet based *ESR* and *NMR*. The great beauty of the new theory therefore is that one merely replaces $\boldsymbol{B}$ of the magnet by $\boldsymbol{B}^{(3)}$ of the electromagnetic field [1]. One can then proceed to understand the gallery of consequences as in the highly developed theory of *ESR* and *NMR*, but with a potential resolving power up to ten thousand times greater. In analogy, successful development would be the metaphorical equivalent of replacing the optical with the scanning tunneling electron microscope.

3.2 Classical Non-Relativistic Physics

In order to derive Eq. (1.3.1) in Newtonian physics, write the kinetic energy in *SU(2)* topology through the use of the Pauli matrix σ [8] and describe the field to particle interactions with the minimal prescription applied to a complex valued $\boldsymbol{A}$ representing the magnetic vector potential of the electromagnetic field [11]. Finally use ordinary complex algebra to extract the real valued and physically meaningful interaction kinetic energy corresponding to Eq. (1.3.1). The Newtonian kinetic energy of a classical charged particle interacting with the classical electromagnetic field in *SU(2)* topology is therefore the real part of

$$\begin{aligned} H_{\text{int}} &= \frac{1}{2m}\sigma\cdot(\boldsymbol{p}-e\boldsymbol{A})\sigma\cdot(\boldsymbol{p}-e\boldsymbol{A}^*) = \frac{1}{2m}\sigma\cdot\boldsymbol{p}\sigma\cdot\boldsymbol{p} \\ &\quad - \frac{e}{2m}(\sigma\cdot\boldsymbol{A}\,\sigma\cdot\boldsymbol{p} + \sigma\cdot\boldsymbol{p}\,\sigma\cdot\boldsymbol{A}^*) \\ &\quad + \frac{e^2}{2m}\sigma\cdot\boldsymbol{A}\sigma\cdot\boldsymbol{A}^* . \end{aligned} \tag{1.3.3}$$

Using the results,

$$\left.\begin{aligned} \sigma\cdot\boldsymbol{A}\sigma\cdot\boldsymbol{p} &= \boldsymbol{A}\cdot\boldsymbol{p} + i\sigma\cdot\boldsymbol{A}\times\boldsymbol{p}, \\ \sigma\cdot\boldsymbol{p}\sigma\cdot\boldsymbol{A}^* &= \boldsymbol{p}\cdot\boldsymbol{A}^* + i\sigma\cdot\boldsymbol{p}\times\boldsymbol{A}^*, \end{aligned}\right\} \tag{1.3.3a}$$

we can isolate the following terms from the right hand side of Eq. (1.3.3).

1) *Magnetic Dipole Term*

$$H_1 = -\frac{e}{2m} \boldsymbol{p} \cdot (\boldsymbol{A} + \boldsymbol{A}^*) \tag{1.3.3b}$$

$$= \frac{e}{2m} \boldsymbol{m}_0 \cdot Re\, \boldsymbol{B} \,, \tag{1.3.3c}$$

where $\boldsymbol{m}_0$ is the magnetic dipole moment of the electron or proton and *Re* $\boldsymbol{B}$ is the real magnetic component of the electromagnetic field.

2) *Spin-Flip Term*

$$H_2 = -i\frac{e}{2m} \sigma \cdot \boldsymbol{p} \times (\boldsymbol{A}^* - \boldsymbol{A}) \,, \tag{1.3.3d}$$

which for an electron or proton moving initially in the Z axis can be expressed as

$$H_2 = -e\frac{A^{(0)}}{\sqrt{2}} p_Z \sigma \cdot (\boldsymbol{j} \cos\phi + \boldsymbol{i} \sin\phi) \,, \tag{1.3.3e}$$

where

$$\phi = \omega t - \kappa Z = \omega \left(t - \frac{Z}{c} \right) . \tag{1.3.3f}$$

If initially $\phi = 0$ the spin σ points in the Y axis; when $\phi = \pi/2$ it points in the X axis; when $\phi = \pi$ in the $-Y$ axis; when $\phi = 3\pi/2$ in the $-X$ axis and when $\phi = 2\pi$ back in the Y axis. So this confirms that H_2 is the spin-flip term used in all Fourier transform *ESR* and *NMR* instruments.

3) *Polarizability Term*

This is,

$$H_3 = \frac{e^2}{2m} \boldsymbol{A} \cdot \boldsymbol{A}^* = \frac{e^2}{2m} A^{(0)2} \,, \tag{1.3.3g}$$

and is the basis of susceptibility theory [13].

4) *The RFR Term*

The *RFR* term, finally, is,

$$\left. \begin{aligned} H_4 &= i\frac{e^2}{2m} \sigma \cdot \boldsymbol{A} \times \boldsymbol{A}^* \\ &= -\frac{e^2}{2m} A^{(0)2} \sigma \cdot \boldsymbol{k} \,. \end{aligned} \right\} \tag{1.3.3h}$$

All four terms have been observed empirically. Terms 1) to 3) are well known and term 4) was observed by Pershan et al. [14] in the paramagnetic inverse Faraday effect.

Thus Eq. (1.3.3) contains the spin-flip and *RFR* term in addition to the familiar and observable *O(3)* terms as found in a text such as that by Pike and Sarkav [22]. These terms rely for their existence on topology rather than quantum mechanics. It is well known [23] that *SU(2)* is homomorphic with *O(3)*, the usual rotation group of three dimensional space in Newtonian physics. However, the Clifford algebra underlying *SU(2)* gives more information, as advocated by Bearden *et al.* [24]. Our Newtonian result is consistent with the fact that Eq. (1.3.1) was obtained in the non-relativistic limit of the Dirac equation as a real expectation value [1].

3.3 Classical Relativistic Physics

It is a straightforward matter to repeat this simple exercise for classical relativistic physics because one can use the same minimal prescription in the Einstein equation written in *SU(2)* topology. For a free classical particle, the latter is

$$\gamma^\mu p_\mu \gamma^\mu p_\mu = m^2 c^2 , \tag{1.3.4}$$

where γ^μ is the Dirac matrix, p_μ the energy momentum four-vector, and c the speed of light in vacuo. The interaction of the classical electromagnetic field with the classical, relativistic, particle is described therefore by the equation of motion,

$$\gamma^\mu (p_\mu - eA_\mu)\gamma^\mu (p_\mu - eA_\mu^*) = m^2 c^2 , \tag{1.3.5}$$

which in Feynman's slash notation becomes [1]

$$(\not{p} - e\not{A})(\not{p} - e\not{A}^*) = m^2 c^2 . \tag{1.3.6}$$

The *RFR* term is [1] the real valued interaction energy,

$$En := \frac{e^2}{m}\not{A}\not{A}^* , \tag{1.3.7}$$

which includes term (1.3.1) of this chapter as part of a fully relativistic treatment,

$$\not{A}\not{A}^* = A_0 A_0^* - (\sigma \cdot \boldsymbol{A})(\sigma \cdot \boldsymbol{A}^*) = A_0 A_0^* - \boldsymbol{A} \cdot \boldsymbol{A}^* - i\sigma \cdot \boldsymbol{A} \times \boldsymbol{A}^* . \tag{1.3.8}$$

3.4 Non-Relativistic Quantum Physics

We can consider the Schrödinger Pauli equation [8],

$$\hat{H}\psi = En\,\psi , \tag{1.3.9}$$

in which the classical kinetic energy becomes an operator on a wavefunction which is a two component spinor in *SU(2)* topology. The usual operator replacements are used as follows:

$$\begin{aligned}
&p^\mu \to i\hbar\partial^\mu , && p_\mu \to i\hbar\partial_\mu , \\
&En \to i\hbar\frac{\partial}{\partial t} , && \boldsymbol{p} \to -i\hbar\nabla , \\
&p^\mu := \left(\frac{En}{c}, \boldsymbol{p}\right) , && p_\mu := \left(\frac{En}{c}, -\boldsymbol{p}\right) , \\
&\partial^\mu := \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) , && \partial_\mu := \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) .
\end{aligned} \tag{1.3.10}$$

It is interesting to note that for a real valued $\boldsymbol{A}$ (static magnetic field problem of ordinary *ESR* and *NMR* [13]) the Schrödinger-Pauli equation produces the famous real expectation value,

$$En = -\frac{e\hbar}{2m}\sigma \cdot \boldsymbol{B} , \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} , \tag{1.3.11}$$

where $\hbar$ is the Dirac constant. This is the fundamental *ESR* or *NMR* term obtained in the non-relativistic quantum limit and has no classical equivalent because it depends for its existence on the operator rules (1.3.10). The Hamiltonian operator that produces result (1.3.11) is

$$\hat{H} = \frac{1}{2m}\sigma \cdot (\hat{p} - eA)\sigma \cdot (\hat{p} - eA) + V, \qquad \hat{p} = -i\hbar\hat{\nabla}, \tag{1.3.12}$$

where V is a potential energy.

In order to obtain the new *RFR* term (1.3.1) this operator becomes

$$\hat{H} = \frac{1}{2m}\sigma \cdot (\hat{p} - eA)\, \sigma \cdot (\hat{p} - eA^*) + V, \tag{1.3.13}$$

and leads to the classical real valued term,

$$\hat{H}_{\text{RFR}}\, \psi = i\frac{e^2}{2m}\, \sigma \cdot A \times A^*\, \psi, \tag{1.3.14}$$

which obviously has the same expectation value, Eq. (1.3.1). Therefore, unlike ordinary *ESR* and *NMR*, *RFR* depends on a term which *does* have a classical equivalent if we treat the field classically as did Dirac [2].

3.5 Relativistic Quantum Physics

The most straightforward route to relativistic quantum mechanics is through the replacement of p_μ in the Einstein equation (1.3.4) by its operator equivalent to give the van der Waerden equation of motion as detailed by Sakurai [8] for example,

$$\left(i\gamma^\mu\partial_\mu\right)\left(i\gamma^\mu\partial_\mu\right)\psi = \left(\frac{mc}{\hbar}\right)^2 \psi. \tag{1.3.15}$$

Here ψ is a two component spinor and the equation is well known to be equivalent to the much better known Dirac equation involving a four component spinor. The *RFR* term emerges from the van der Waerden equation in the form,

$$\gamma^\mu\left(i\partial_\mu - eA_\mu\right)\gamma^\mu\left(i\partial_\mu - eA_\mu^*\right)\psi = \left(\frac{mc}{\hbar}\right)^2 \psi. \tag{1.3.16}$$

The real and classical $e^2 \not{A}\not{A}^*$ is a simple multiplicative operator on the two component spinor, with the same, real, expectation value. This is also the case for the Dirac equation as given in Ref. 1, and in general for all *SU(2)* topology quantum mechanical equations.

3.6 Rayleigh-Schrödinger Perturbation Theory

In perturbation theory [13] the *RFR* term is a non-zero ground state term,

$$En = i\frac{e^2}{2m}\langle 0|\sigma \cdot A \times A^*|0\rangle + \text{second order terms.} \tag{1.3.17}$$

As shown recently by Li *et al.* [15] and by others [16—20] small second order *RFR* shifts also occur in second order corrections in perturbation theory, but term (1.3.17) is of far greater practical interest, because as shown in Ref. 1, it produces fermion resonances in the visible. Second order perturbation theory was also used by Pershan *et al.* [14] to produce the paramagnetic inverse Faraday effect, which they confirmed experimentally.

3.7 Discussion

In free space, the novel $\boldsymbol{B}^{(3)}$ field of *O(3)* symmetry electrodynamics is defined for one photon by,

$$\boldsymbol{B}^{(3)} := -i\frac{e}{\hbar}\boldsymbol{A} \times \boldsymbol{A}^*, \tag{1.3.18}$$

where e is the elementary charge [1]. Substituting this definition into Eq. (1.3.17) we find that the *RFR* term takes the same form precisely as the spin Zeeman effect produced by a static magnetic field,

$$En_{(\mathrm{RFR})} = -\frac{e\hbar}{2m}\langle 0|\sigma \cdot \boldsymbol{B}^{(3)}|0\rangle. \tag{1.3.19}$$

We need only replace $\boldsymbol{B}$ by $\boldsymbol{B}^{(3)}$ as defined in Eq. (1.3.18). Equation (1.3.19) is the fundamental spin-spin interaction between one photon and one fermion. For a free electron, the resonance frequency is straightforwardly calculated [1] from Eq. (1.3.19) to be

$$\omega_{\mathrm{res}} = \left(\frac{e^2\mu_0 c}{\hbar m}\right)\frac{I}{\omega^2} = 1.007 \times 10^{28}\frac{I}{\omega^2}, \tag{1.3.20}$$

where I is the pump beam power density in watts m^{-2} (10,000 watts m^{-2}) = 1.0 watt cm^{-2}), μ_0 the free space permeability in *SI* units. For the H atom, the Hamiltonian operator is well known to be,

$$\hat{H}_{(H\ \mathrm{atom})} = -\frac{\hbar^2}{2\mu}\nabla^2 + V, \tag{1.3.21}$$

where V denotes the classical Coulomb interaction between electron and proton and μ is the reduced mass,

$$\mu = \frac{m_e m_p}{m_e + m_p} \sim m_e, \tag{1.3.22}$$

where m_e and m_p are respectively the electron and proton masses. The resonance frequency in atomic H from Eq. (1.3.20) is therefore slightly shifted away from the free electron resonance frequency because the reduced mass is slightly different from the electron mass. The Hamiltonian operator

(1.3.22) for a monovalent alkali metal atom such as sodium (*Na*) must take account of the fact that there are several protons, neutrons and electrons arranged in orbitals according to the Pauli exclusion principle [13]. This atomic structure gives rise to the possibility of spin orbit coupling, spin-spin coupling between electrons, Fermi contact splitting, and hyperfine splitting as in *ESR* or *NMR* [13]. However, as a rule of thumb estimate, the outer or valence electron can be considered as superimposed on closed shells of inner electrons and a nucleus made up of protons and neutrons of a given reduced mass. To a first approximation, the Hamiltonian (1.3.22) can be used in which the sodium atom's reduced mass is slightly different from the free electron mass. This means that the main *RFR* resonance frequency in sodium is well estimated by Eq. (1.3.20) and so sodium vapor can be used in the experiment to detect *RFR*.

In order to detect *RFR* experimentally adjust conditions in the first instance so that,

$$\omega_{\mathrm{res}} = \omega, \tag{1.3.23}$$

which is the auto-resonance condition in which the pump beam is absorbed at resonance because the pump frequency matches the resonance frequency precisely. Equation (1.3.20) simplifies to

$$\omega_{\mathrm{res}}^3 = 1.007 \times 10^{28} I. \tag{1.3.24}$$

Therefore we can either tune ω_{res} for a given I or vice versa. Since auto-resonance must appear in the *GHz* if the pump frequency is in this range it is convenient to slightly modify the set up used by Deschamps *et al.* [25] in their detection of the inverse Faraday effect in plasma. They used a pulsed microwave signal at 3.0 *GHz* from a klystron delivering megawatts of power over 12 microseconds with a repetition rate of 10 *Hz*. The TE_{11} Mode was circularly polarized with a polarizer placed inside a circular waveguide of 7.5 *cm* diameter. The plasma sample was created by a very intense microwave pulse and held in a pyrex tube inserted coaxially in the waveguide of 6.5 *cm* diameter and length 20.0 *cm*. The section of the waveguide surrounding the tube was made of nylon internally coated with a 20 *micron* layer of copper. The inverse Faraday effect was then picked up with Faraday induction [25].

To detect *RFR* change the sample to sodium vapor, which is easily prepared and held in the sample tube. Equation (1.3.24) predicts that resonance occurs at 3.0 *GHz* if I is tuned to 0.0665 *watts* cm^{-2}. For a circular waveguide of 7.5 *cm* diameter this requires only 2.94 *watts* of *CW* power from the klystron at 3.0 *GHz*. In deriving Eq. (1.3.20) it has been assumed that [1,26]

$$I = \frac{c}{\mu_0} B^{(0)2} . \tag{1.3.25}$$

This is a simple theoretical estimate and it is strongly advisable that I can be tuned over a considerable range around 2.94 *watts* to allow for unforeseen discrepancies between Eq. (1.3.25) and the actual experimental beam intensity generated by the apparatus. Once the main resonance is detected however, further refinements can follow, making full use of contemporary electronics. To repeat the experiment with atomic *H* or with the free electron gas is likely to be more difficult purely because of sample handling problems. The experiment should be repeated after auto-resonanc is detected to demonstrate the major advantage of *RFR* by pulsing the pump beam for increased power density at the same frequency and by using Eq. (1.3.24) to estimate the resonance frequency. A sample of expected results is given in Table 3.1. As can be seen it is possible in theory to produce *ESR* (and *NMR*) in the visible range, with a four-order of magnitude increase in resolving power over current magnet based techniques.

Table 1. *RFR* Frequencies for a 3.0 GHz Pump for given I

Pump Intensity I (*watts* cm^{-2})	Resonance Frequency
10.0	15.04 cm^{-1} (Far infra red)
100.0	150.4 cm^{-1} (Far infra red)
1,000.0	1,504 cm^{-1} (Infra red)
10,000.0	15,040 cm^{-1} (Visible)
100,000.0	150,040 cm^{-1} (Ultra violet-X ray)

For a 3.0 *GHz* circularly polarized pump pulse of 10 *kwatts* cm^{-2} the *RFR* frequency is at 15,040 cm^{-1} in the visible, and can be detected with a Fourier transform infra red-visible spectrometer such as a fully computerized Bruker *IFS 113v*. The detector of the spectrometer must be fast enough to record an interferogram during the microsecond interval of the microwave pulse. Therefore pulse repetition and computer based refinement is necessary for good quality data. The pump should be kept as homogeneous and noise free as possible, but because of the I/ω^2 dependence of *RFR* , simple Maxwell-Boltzmann theory [1] shows that conditions can be adjusted to produce a much larger population difference between up and down fermion spins than in magnet based *ESR* or *NMR*. Therefore this alleviates the well known problem of magnet homogeneity in magnet based *ESR* and *NMR*, a problem which is due to a small (one part in a million) population difference. In *RFR* the latter can easily exceed 20% [1] at a conservative estimate for moderate pump power of ten *watts* order of magnitude. The complete *ESR* spectrum of sodium vapor can therefore be taken, in theory, in the infra red or visible. This is terra incognita in magnet based technology, which is reaching its design limit. The whole process can then be repeated for *NMR* and *MRI*.

The characteristic and key I/ω^2 coefficient of our theory [1] appears also in the second order perturbation theory of Harris and Tinoco [17], their p. 9291, second column, premultiplied by a factor. These authors miss the first order or ground state term (1) and in consequence their theory falls short of empirical indications by Warren *et al.* [27] by eight orders of magnitude. Straightforward estimates [1] based on Eq. (1.3.1) applied to *NMR* fall in the order of magnitude of the data obtained by Warren *et al.* [27] by visible frequency irradiation of molecular liquids with various circularly polarized lasers, including an argon ion laser at 528.7 *nm*, 488 *nm*, and 476.5 *nm*. Accounting simply for the different g factors of the proton and electron, Eq. (1.3.1) applied to *NMR* [1] produces very tiny shifts of 0.12, 0.10 and 0.098 *Hz* respectively for the three argon ion laser frequencies quoted above and for an intensity of ten *watts* per square *centimeter*, approaching the highest *CW* intensities used by Warren *et al.* [27] in important and pioneering experiments at Princeton following our early theory [21,28] which also missed the key term (1.3.1) introduced finally in Ref. 1. Equation (1.3.1) now shows now why Warren *et al.* [27] were not able to obtain more than indications of *RFR* shifts, both in proton and ^{13}C Fourier transform and two dimensional *NMR*. In ^{13}C *NMR* the mass of the ^{13}C nucleus is an order of magnitude heavier than in ^{1}H *NMR* and the shifts from Eq. (1.3.1), all other factors being equal, are in consequence an order of magnitude smaller, in the 0.01 *Hz* range — too small to be detected, as found experimentally [27]. The remedy is also given by Eq. (1.3.1), which is to replace the lasers with pulsed or *CW* microwave generators for about the same I. Their effort [27] nevertheless remains as a landmark in the field.

Finally, Li *et al.* [15] have shown that even in second order perturbation theory of the type used by Harris and Tinoco [16,17], or Buckingham *et al.* [18,19], large *RFR* shifts of up to 10 *MHz* are possible using pump lasers tuned near to optical resonance. Systematic development of *RFR*, first proposed by the present author in Ref. 21 and in several consequent papers [28], is clearly going to be highly beneficial to chemical physics and medicine unless all the equations of physics are misleading or unless some unforeseen technical difficulty occurs. With contemporary technology it is unlikely that such a difficulty, if it occurred, could not be overcome. Philosophically the whole process can be thought of as stemming from the $\boldsymbol{B}^{(3)}$ (Evans-Vigier) field of *O(3)* electrodynamics [1], which for one photon, is the fundamental photomatic [29].

References

[1] M. W. Evans, J.-P. Vigier, S. Roy, and S. Jeffers, *The Enigmatic Photon, Volume Three: Theory and Practice of the $\boldsymbol{B}^{(3)}$ Field*, Chaps. 1 and 2,. (Kluwer Academic, Dordrecht, 1996).

[2] P. A. M. Dirac, *Quantum Mechanics,* 4th edn. revised, (Oxford University Press, Oxford, 1974).

[3] L. H. Ryder, *Quantum Field Theory,* 2nd edn., (Cambridge University Press, Cambridge,1987).

[4] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw Hill, New York, 1964).

[5] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw Hill, New York, 1980).

[6] R. R. Ernst, G. Bodenhausen, and A. Wokaun, *Principles of Nuclear Magnetic Resonance in One and Two Dimensions* (Oxford University Press, Oxford, 1987).

[7] M. W. Evans, *Physica B* **182**, 227, 237 (1992); **183**, 103 (1993); **190**, 310 (1993); *Physica A* **215**, 605 (1995).

[8] J. J. Sakurai, *Advanced Quantum Mechanics* 11th printing, (Addison Wesley, New York, 1967) Chap. 3.

[9] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevski, *Relativistic Quantum Theory* (Pergamon, Oxford, 1971).

[10] M. W. Evans, J.-P. Vigier, and M. Meszaros, eds., *The Enigmatic Photon, Volume Five: Part 2, Selected papers of Erasmo Recami*, (Kluwer Academic, Dordrecht, *in preparation*).

[11] J. D. Jackson, *Classical Electrodynamics,* 1st edn. (Wiley, New York, 1962).

[12] B. L. van der Waerden, *Group Theory and Quantum Mechanics* (Springer Verlag, Berlin, 1974).

[13] P. W. Atkins, *Molecular Quantum Mechanics*, 2nd edn., (Oxford University Press, Oxford, 1983), pp. 383 ff.

[14] P. S. Pershan, J. van der Ziel, and L. D. Malmstrom, *Phys. Rev.* **143**, 574 (1966).

[15] L. Li, T. He, X. Wang, and F.-C. Liu, *Chem. Phys. Lett.* **268**, 549 (1997).

[16] R. A. Harris and I. Tinoco, *Science* **259**, 835 (1993).

[17] R. A. Harris and I. Tinoco, *J. Chem. Phys.* **101**, 9289 (1994).

[18] A. D. Buckingham and L. C. Parlett, *Chem. Phys. Lett.* **243**, 15 (1995).

[19] A. D. Buckingham and L. C. Parlett, *Science* **264**, 1748 (1994).

[20] M. W. Evans, *Found. Phys. Lett.* **8**, 563 (1995).

[21] M. W. Evans, *J. Mol. Spect.* **143**, 327 (1990); *Chem. Phys.* **157**, 1 (1991); *J. Phys. Chem.* **95**, 2256 (1991); *J. Mol. Liq.* **49**, 77 (1991); *Int. J. Mod. Phys. B* **5**, 1963, invited rev., (1991); *J. Mol. Spect.* **146**, 143 (1991); *Physica B* **168**, 9 (1991); *Adv. Chem. Phys.* **51**, 361—702 , invited rev. (1992); M. W. Evans, S. Woźniak, and G. Wagnière, *Physica B* **173**, 357 (1991); **175**, 416 (1991).

[22] E. R. Pike and S. Sarkav, *The Quantum Theory of Radiation*, Example 3.4, (Oxford University Press, Oxford, 1995).

[23] T. W. Barrett and D. M. Grimes, eds., *Advanced Electromagnetism, Foundations, Theory and Applications* (World Scientific, Singapore, 1995).

[24] T. W. Bearden, www.europa.com/~ rsc/physics, 1997 to present.

[25] J. Deschamps, M. Fitaire, and M. Lagoutte, *Phys. Rev. Lett.* **25**, 1330 (1970).

[26] L. D. Barron, *Molecular Light Scattering and Optical Activity*, Eq. (2.2.16) (Cambridge University Press, Cambridge, 1982).

[27] W. S. Warren, S. Mayr, D. Goswami, and A. P. West., Jr., *Science* **255**, 1683 (1992); **259**, 836 (1993).

[28] M. W. Evans and S. Kielich, eds., *Modern Nonlinear Optics*, Vol. 85(2) of *Advances in Chemical Physics,* I. Prigogine and S. A. Rice, eds., pp. 51 ff (Wiley Interscience, New York, 1992, 1993, and 1997 (paperback printing)).; also M. W. Evans, *Physica B* **182**, 118 (1992); **176**, 254 (1992); **179**, 157 (1992); S. Wozniak, M. W. Evans, and G. Wagniere, *Mol. Phys.* **75**, 81, 99 (1992).

[29] M. W. Evans and A. A. Hasanein, *The Photomagneton in Quantum Field Theory* (World Scientific, Singapore, 1994); M. W. Evans and

J.-P. Vigier, *Classical Electrodynamics and the* $\boldsymbol{B}^{(3)}$ *Field* (World Scientific, *in preparation*).