1) Notes 108(2): Derivation of the Equation of Motion and Constants of Motion.

In SI units and for a planar orbit, the equation of motion of the binary pulsar is derived from the line element:

$$-c^2 d\tau^2 = -\left(1-\frac{r_s}{r}\right)c^2 dt^2 + \left(1-\frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\phi^2 \quad -(1)$$

Thus:

$$-\epsilon = -c^2\left(\frac{d\tau}{d\lambda}\right)^2 = -\left(1-\frac{r_s}{r}\right)c^2\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{r_s}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 \quad -(2)$$

where ϵ is a constant of motion. From eq. (2):

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{2}\left(1-\frac{r_s}{r}\right)^2 c^2\left(\frac{dt}{d\lambda}\right)^2 - \frac{r^2}{2}\left(1-\frac{r_s}{r}\right)\left(\frac{d\phi}{d\lambda}\right)^2 - \frac{\epsilon}{2}\left(1-\frac{r_s}{r}\right) \quad -(3)$$

From an analysis of Killing vectors (see Carroll chapt. 7) the following two constants of motion are identified:

$$E_r = \left(1-\frac{r_s}{r}\right)\frac{dt}{d\lambda}, \quad L_r = r^2\frac{d\phi}{d\lambda}. \quad -(4)$$

The constants ϵ, E and L remain constants of motion under all conditions, and also for an r_s that is (r, ϕ) dependent.

The equation of motion of the binary pulsar is:

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_r(r) = \frac{1}{2}E_r^2 \quad -(5)$$

and has the S.I. units of energy if

$$\lambda = \tau \quad -(6)$$

and if both sides of eq. (5) are ~~multiplied by mc^2~~.

also in units of joules. To express eq. (5) in units of joules multiply through by m. Thus:

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + m V_r(r) = \frac{1}{2}m E_r^2 c^2 = En \quad -(7)$$

The potential energy of the system is:

$$V = m V_r(r) = \left(\frac{1}{2}\epsilon - \epsilon \frac{r_s}{r} + \frac{L^2}{2r^2} - \frac{r_s L^2}{r^3}\right) m \quad -(8)$$

and the equation of motion in S. I. units is:

$$\boxed{\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + V = En} \quad -(9)$$

with:

$$\epsilon = c^2 \quad -(10)$$

So the potential energy is:

$$V = \frac{1}{2}mc^2 - mc^2\frac{r_s}{r} + \frac{mL^2}{2r^2} - mL^2\frac{r_s}{r^3} \quad -(11)$$

with

$$L = r^2\frac{d\phi}{d\tau} \quad -(12)$$

Various orbits are now constructed from eq. (9) as described by Carroll in chapter 7.

In the standard approach:

$$r_s = \frac{2GM}{c^2} \quad -(13)$$

and this produces the results described by Carroll.

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However, in a binary pulsar, the mean distance between the two stars is decreasing slightly per revolution. This cannot be described by eq. (13). Since r is decreasing and L is a constant, $d\phi / d\tau$ must be increasing for eq. (12). To account for this decrease in orbit, we assume that:

$$r_s = - \frac{T}{R} (r) \quad — (14)$$

or more generally:

$$r_s = - \frac{T}{R} (r, \phi) \quad — (15)$$

To proceed, it is possible to integrate eq. (9) numerically, or to adapt orbital theory for use with eq. (14). In both cases L and E are regarded as constants under all conditions, so the effect of an r dependent r_s can be seen clearly by considering well known orbits such as circular orbits, and by considering the Newtonian limit.

The numerical procedure was described in note 108(1) and the analytical procedure will be described in note 108(3).