113(5): Derivation of the Gravitational Red Shift from the Theorem of orbits.

Consider the line element:

$$ds^2 = \left(1+\frac{\mu}{r}\right)c^2 dt^2 - \left(1+\frac{\mu}{r}\right)^{-1} dr^2 - r^2 d\phi^2 - dZ^2 \quad -(1)$$

in cylindrical polar coordinates. Rotate it as in paper 110:

$$\phi' = \phi + \omega t \quad -(2)$$

so:

$$ds'^2 = \left(1+\frac{\mu}{r}\right)c^2 dt^2 - \left(1+\frac{\mu}{r}\right)^{-1} dr^2 - r^2 d\phi'^2 - dZ^2 - (2)$$

where:

$$d\phi' = d\phi + \omega dt \quad -(3)$$

$$d\phi'^2 = d\phi^2 + 2\omega d\phi dt + \omega^2 dt^2 \quad -(4)$$

Thus:

$$ds'^2 = \left(\left(1+\frac{\mu}{r}\right)c^2 - r^2\omega^2\right)dt^2 - \left(1+\frac{\mu}{r}\right)^{-1} dr^2$$
$$- 2\omega r^2 d\phi dt - r^2 d\phi^2 - dZ^2 - (5)$$

$$= \left(1+\frac{\mu}{r} - \frac{v^2}{c^2}\right)\left(c^2 dt^2 - 2r^2\Omega d\phi dt\right)$$
$$- \left(1+\frac{\mu}{r}\right)^{-1} dr^2 - r^2 d\phi^2 - dZ^2 - (6)$$

where

$$v = r\omega. \quad -(7)$$

As in paper 110 identify:

2)

$$\Omega = \omega\left(1+\frac{\mu}{r}-\frac{v^2}{c^2}\right)^{-1}, \quad -(8)$$

$$d\tau = \left(1+\frac{\mu}{r}-\frac{v^2}{c^2}\right)^{1/2} dt \quad -(9)$$

The change of phase, or precession, upon rotating by 2π radians is:

$$\alpha = \Omega\, d\tau - \omega\, dt \quad -(10)$$

$$\boxed{\alpha = 2\pi\left(\left(1+\frac{\mu}{r}-\frac{v^2}{c^2}\right)^{-1/2} - 1\right)} \quad -(11)$$

As

$$v \to 0 \quad -(12)$$

$$\alpha \to 2\pi\left(\left(1+\frac{\mu}{r}\right)^{-1/2} - 1\right). \quad -(12)$$

For almost all orbits:

$$\mu = -\frac{2mG}{c^2} \quad -(13)$$

so:

$$\alpha = 2\pi\left(\left(1-\frac{2mG}{rc^2}\right)^{-1/2} - 1\right) \quad -(14)$$

The factor in brackets is <u>the gravitational</u> <u>red shift</u>.