189(5): Analytical Solution of the Constraint Equation

The first constraint equation is:

$$3\frac{\partial}{\partial r}\left(\frac{1}{m}\frac{\partial m}{\partial r}\right) + \frac{1}{2m^2}\left(\frac{\partial m}{\partial r}\right)^2 = 0 . \quad -(1)$$

Make the substitution:

$$m = e^{2\alpha} \quad -(2)$$

then

$$\frac{1}{m}\frac{\partial m}{\partial r} = \frac{\partial \alpha}{\partial r} \quad -(3)$$

so

$$3\frac{\partial^2 \alpha}{\partial r^2} + \frac{1}{2}\left(\frac{\partial \alpha}{\partial r}\right)^2 = 0 . \quad -(4)$$

A particular solution of this equation has been found by computer algebra. To find the general solution let:

$$\frac{1}{2}\left(\frac{\partial \alpha}{\partial r}\right)^2 = \frac{\partial \alpha}{\partial r} - f(r) \quad -(5)$$

so eq. (4) becomes the inhomogeneous differential equation:

$$3\frac{\partial^2 \alpha}{\partial r^2} + \frac{\partial \alpha}{\partial r} = f(r) \quad -(6)$$

whose reduced equation is:

$$3\frac{\partial^2 \alpha}{\partial r^2} + \frac{\partial \alpha}{\partial r} = 0 \quad -(7)$$

Now let

$$\alpha = e^{xr} \quad -(8)$$

so

$$3x^2 + x = 0 \quad -(9)$$

i.e.

$$\alpha = -\left(\frac{r}{3}\right) \quad -(10)$$

2)

The general solution of eq. (4) is the particular integral plus the complementary function (10). The complementary function in the correct units is:

$$m(r,t) = \exp\left(2\exp\left(-\frac{r}{3R(t)}\right)\right) \quad -(11)$$

The complete solution is this function added to the known particular integral of eq. (4). For example a particular integral of eq. (4) is:

$$m_p(r,t) = e^2 \quad -(12)$$

i.e

$$\alpha = 2 \quad -(13)$$

so the normalised solution is:

$$m(r,t) = \frac{1}{2}\left(1 + \frac{1}{e^2}\exp\left(2\exp\left(\frac{-r}{3R(t)}\right)\right)\right) \quad -(14)$$

$$\rightarrow 1 \text{ as } r \rightarrow \infty$$

Similarly:

$$n(r,t) = \frac{1}{2}\left(1 + \frac{1}{e^2}\exp\left(2\exp\left(-\frac{t}{3\tau(r)}\right)\right)\right) \quad -(15)$$