1) 159(1): A Self Consistent Definition of the Metric

Consider the cylindrical polar coordinates:

$$\underset{\sim}{r} = X\underline{i} + Y\underline{j} + Z\underline{k} \quad -(1)$$
$$= r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j} + Z\underline{k}$$

(E. G. Milewski, ed., "The Vector Analysis Problem Solver" (Research and Education Association, New York, 1987)).

Then:

$$dX = \frac{\partial X}{\partial r}dr + \frac{\partial X}{\partial \theta}d\theta = \cos\theta\,dr - r\sin\theta\,d\theta \quad -(2)$$

$$dY = \frac{\partial Y}{\partial r}dr + \frac{\partial Y}{\partial \theta}d\theta = \sin\theta\,dr + r\cos\theta\,d\theta \quad -(3)$$

and

$$dX^2 + dY^2 = (\cos\theta\,dr - r\sin\theta\,d\theta)(\cos\theta\,dr - r\sin\theta\,d\theta)$$
$$+ (\sin\theta\,dr + r\cos\theta\,d\theta)(\sin\theta\,dr + r\cos\theta\,d\theta)$$
$$= dr^2 + r^2 d\theta^2 \quad -(4)$$

So:

$$ds^2 = dX^2 + dY^2 + dZ^2 \quad -(5)$$
$$= dr^2 + r^2 d\theta^2 + dZ^2$$

is the infinitesimal line element.

By definition:

$$d\underline{r} = \frac{\partial \underline{r}}{\partial r}dr + \frac{\partial \underline{r}}{\partial \theta}d\theta + \frac{\partial \underline{r}}{\partial Z}dZ \quad -(6)$$

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So:

$$\frac{\partial \underline{r}}{\partial r} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \qquad -(7)$$

$$\frac{\partial \underline{r}}{\partial \theta} = -r\sin\theta\,\underline{i} + r\cos\theta\,\underline{j} \qquad -(8)$$

$$\frac{\partial \underline{r}}{\partial z} = \underline{k} \qquad -(9)$$

The scale factors are:

$$h_1 = h_r = \left|\frac{\partial \underline{r}}{\partial r}\right| = 1 \qquad -(10)$$

$$h_2 = h_\theta = \left|\frac{\partial \underline{r}}{\partial \theta}\right| = r \qquad -(11)$$

$$h_3 = h_z = \left|\frac{\partial \underline{r}}{\partial z}\right| = 1 \qquad -(12)$$

The metric elements are:

$$g_{11} = h_1^2,\ g_{22} = h_2^2,\ g_{33} = h_3^2 \qquad -(13)$$

so

$$g_{11} = 1,\ g_{22} = r^2,\ g_{33} = 1 \qquad -(14)$$

and so

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad -(15)$$

The coordinates are:

$$1 = r,\ 2 = \theta,\ 3 = z \qquad -(16)$$

The unit vectors are

$$\underline{e}_1 = \underline{e}_r = \frac{1}{h_1}\frac{\partial \underline{r}}{\partial r} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j}$$

$$-(17)$$

$$\underline{e}_2 = \underline{e}_\theta = \frac{1}{h_2}\frac{\partial \underline{r}}{\partial \theta} = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j} \quad -(18)$$

$$\underline{e}_3 = \underline{e}_Z = \frac{1}{h_3}\frac{\partial \underline{r}}{\partial Z} = \underline{k} \quad -(19)$$

Therefore in the Cartesian representation of three dimensional space:

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(20)$$

but in the cylindrical polar representation of three dimensional space:

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(21)$$

It is not convenient to have two different metrics representing the same space, and the presence of r^2 causes confusion. For example some Christoffel symbols of the Schwarzschild metric appear not to vanish in a flat spacetime, in which all connection must vanish.

Unfortunately a metric of type (21) was used in the twentieth century literature on general relativity, and in consequence Christoffel symbols and curvature elements are full of idiosyncracies and self contradictions. The root cause of this is the presence of r in the scale factor h_θ of eq. (11). The metric element g_{22} was defined as:

$$g_{22} = h_\theta^2 = r^2 \quad -(22)$$

These peculiarities are completely eliminated if the metric is defined as:

$$\boxed{g_{ij} = \underline{e}_i \cdot \underline{e}_j} \quad -(23)$$

for a diagonal metric. So in both the Cartesian and cylindrical polar systems:

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(24)$$

and the Minkowski metric is:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(25)$$

in both Cartesian and cylindrical polar coordinates.

The metric of the spherical spacetime is:

$$g_{\mu\nu} = \begin{bmatrix} m(r,t) & 0 & 0 & 0 \\ 0 & -n(r,t) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(26)$$

Using these fundamental definitions there are only two antisymmetric connections of the spherical spacetime

$$\boxed{\begin{aligned}\Gamma^{0}_{10} = -\Gamma^{0}_{01} &= \frac{1}{2m(r,t)}\frac{\partial m(r,t)}{\partial r} \quad -(27)\\ \Gamma^{1}_{01} = -\Gamma^{1}_{10} &= \frac{1}{2n(r,t)}\frac{1}{c}\frac{\partial n(r,t)}{\partial t} \quad -(28)\end{aligned}}$$

constrained by the Evans identity:

$$D_\mu T^{\kappa\mu\nu} := R^{\kappa}{}_{\mu}{}^{\mu\nu}. \quad -(29)$$

In the old system the Minkowski metric in cylindrical polar coordinates was:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(30)$$

according to definitions (10) to (14). Sometimes it was written as:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix} \quad -(31)$$

However, eq. (31) implies:

$$h_3 \overset{?}{=} r\sin\theta \quad -(32)$$

but the correct result is eq. (12):

$$h_3 = 1. \quad -(33)$$

The use of eq. (31) means that connection and curvature in the twentieth century were idiosyncratic