189(11): Dependence of $\theta$ on $t$ for the Ellipse

This is given by:

$$\frac{d\theta}{dt} = \frac{L}{md^2}\left(1+\epsilon\cos\theta\right)^2 \quad -(1)$$

i.e.

$$\frac{d\theta}{\left(1+\epsilon\cos\theta\right)^2} = \frac{L}{md^2}\,dt \quad -(2)$$

$$\int\frac{d\theta}{\left(1+\epsilon\cos\theta\right)^2} = \frac{L}{md^2}\,t + C \quad -(3)$$

The integral is best integrated numerically and parameterized. Its analytical solution is:

$$\int\frac{dx}{(a+b\cos x)^2} = \frac{b\sin x}{(b^2-a^2)(a+b\cos x)} - \left(\frac{a}{b^2-a^2}\right)\int\frac{dx}{a+b\cos x};$$

$$\int\frac{dx}{a+b\cos x} = \frac{2}{(a^2-b^2)^{1/2}}\tan^{-1}\left(\frac{a\tan(x/2)+b}{(a^2-b^2)^{1/2}}\right) \quad -(4)$$

where:

$$a = 1,\ b = \epsilon,\ x = \theta \quad -(5)$$

In the solar system:

$$\epsilon \rightarrow 0 \quad -(6)$$

So

$$\int\frac{dx}{(a+b\cos x)^2} \rightarrow -\epsilon\sin\theta + \int\frac{dx}{1+\epsilon\cos x}, \quad -(7)$$

$$\int\frac{dx}{1+\epsilon\cos x} \rightarrow 2\tan^{-1}\tan\theta/2 \rightarrow \theta \quad -(8)$$

So assuming:

$$C = 0 \quad -(9)$$

$$\frac{L}{md^2} t = \theta - \epsilon \sin\theta \quad -(10)$$

For small angular displacements:

$$\theta = \frac{L}{md^2(1-\epsilon)} t \quad -(11)$$

and

$$\boxed{\frac{dr}{dt} = \frac{md^2}{L}(1-\epsilon)\frac{dr}{d\theta}} \quad -(12)$$