189(3) · Development of the Constraint Equations

For power notes the only non-zero antisymmetric connection of a spherical spacetime are:

$$\Gamma^0_{\ 10} = -\Gamma^0_{\ 01} = \frac{1}{2}\frac{1}{m}\frac{\partial m}{\partial r}, \quad -(1)$$

$$\Gamma^1_{\ 01} = -\Gamma^1_{\ 10} = \frac{1}{2}\frac{1}{n}\frac{1}{c}\frac{\partial n}{\partial t} \quad -(2)$$

where in general m and n are functions of t and r. If m and n are functions only of r, then there is only one connection, $\Gamma^0_{\ 10}$.

The latter is constrained by the identity:

$$D_\mu T^{\kappa\mu\nu} := R^{\kappa\ \ \mu\nu}_{\ \ \mu} \quad -(3)$$

where

$$T^{\kappa\mu\nu} = \Gamma^{\kappa\mu\nu} - \Gamma^{\kappa\nu\mu}. \quad -(4)$$

So:

$$\kappa = 0, \mu = 1, \nu = 0, \quad -(5)$$

and

$$\boxed{D_1 T^{010} = R^{0\ \ 10}_{\ 1}} \quad -(6)$$

By metric compatibility eq. (6) is:

$$D_1 T^0_{\ 10} = R^0_{\ 110} \quad -(7)$$

By definition:

$$D_\mu T^\kappa_{\ \nu\sigma} = \partial_\mu T^\kappa_{\ \nu\sigma} + \Gamma^\kappa_{\mu\lambda} T^\lambda_{\ \nu\sigma} - \Gamma^\lambda_{\mu\nu} T^\kappa_{\ \lambda\sigma} - \Gamma^\lambda_{\mu\sigma} T^\kappa_{\ \nu\lambda}$$

$$-(8)$$

So:

$$\therefore D_1 T^{0}{}_{10} = \partial_1 T^{0}{}_{10} + \Gamma^{0}{}_{1\lambda} T^{\lambda}{}_{10} - \Gamma^{\lambda}{}_{11} T^{0}{}_{\lambda 0} - \Gamma^{\lambda}{}_{10} T^{0}{}_{1\lambda}$$

$$= \partial_1 T^{0}{}_{10} + \Gamma^{0}{}_{10} T^{0}{}_{10} - \Gamma^{0}{}_{10} T^{0}{}_{10}$$

$$= \partial_1 T^{0}{}_{10} \quad -(9)$$

So

$$\partial_1 T^{0}{}_{10} = R^{0}{}_{110} \quad -(10)$$

The curvature tensor is:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma} \quad -(11)$$

Therefore:

$$R^{0}{}_{110} = \partial_1 \Gamma^{0}{}_{01} + \Gamma^{0}{}_{10}\Gamma^{0}{}_{01} \quad -(12)$$

$$= -\partial_1 \Gamma^{0}{}_{10} - \Gamma^{0}{}_{10}\Gamma^{0}{}_{10}$$

From eqs. (10) and (12):

$$\partial_1 T^{0}{}_{10} + \partial_1 \Gamma^{0}{}_{10} + \Gamma^{0}{}_{10}\Gamma^{0}{}_{10} = 0 \quad -(13)$$

i.e.

$$\boxed{3\partial_1 \Gamma^{0}{}_{10} + \Gamma^{0}{}_{10}\Gamma^{0}{}_{10} = 0} \quad -(14)$$

From eqs. (1) and (14):

$$\boxed{3\frac{\partial}{\partial r}\left(\frac{1}{m(r)}\frac{\partial m(r)}{\partial r}\right) + \frac{1}{4m^2(r)}\left(\frac{\partial m(r)}{\partial r}\right)^2 = 0} \quad -(15)$$

This is a non-trivial second order differential equation for $m(r)$.

Schwarzschild Metric

In this case:

$$m = 1 - \frac{r_0}{r} \quad -(16)$$

and

$$\Gamma^0_{10} = \frac{1}{2}\left(1 - \frac{r_0}{r}\right)^{-1} \frac{r_0}{r^2}$$

$$= \frac{r_0}{2r(r-r_0)} \quad -(17)$$

From eq. (17) in eq. (14):

$$-\frac{3r_0}{r(r-r_0)^2} + \frac{r_0^2}{4r^2(r-r_0)^2} = 0 \quad -(18)$$

The only reasonable solution in physics is:

$$\boxed{\frac{r_0}{r} \longrightarrow 0} \quad -(19)$$

which is consistent with the fact that the SM is a vacuum metric:

$$r \longrightarrow \infty. \quad -(20)$$

It is known that:

$$m = \frac{1}{n} = 1 - \frac{r_0}{r} \quad -(21)$$

is an accurate description of the relativistic Kepler orbits of the solar system.

4)

Discussion

The use of antisymmetric connections in cosmology has greatly simplified the problem of describing the solar system. There is only one connection:

$$\Gamma^0_{10} = -\Gamma^0_{01} = \frac{1}{2}\frac{1}{m}\frac{\partial m}{\partial r} \quad -(22)$$

and as

$$r \rightarrow \infty, \quad -(23)$$

$$\Gamma^0_{10} = \frac{r_0}{r(r-r_0)} \sim \frac{r_0}{r^2} \quad -(24)$$

The rigorous equation for $m(r)$ is eq. (15), with initial conditions and boundary conditions. Experimental data show that the boundary condition is:

$$\Gamma^0_{10} \rightarrow \frac{r_0}{r(r-r_0)} \quad \text{as } r \rightarrow \infty. \quad -(25)$$

However, computer algebra can be used to explore small perturbations to the orbits of planets, for example:

$$m(r) = 1 - r_0\left(\frac{1}{r} + \frac{1}{r^2} + \dots + \frac{1}{r^n}\right) \quad -(26)$$

i.e. can eq. (26) be a solution of eq. (15) and if not, can other perturbations of $m(r)$ be solutions?
