189(9): Summary of Calculations

The basic equations are:

$$\Gamma^{\mu}_{\nu\mu} = \frac{1}{2g_{\mu\mu}} \partial_\nu g_{\mu\mu}, \quad \nu \neq \mu \quad -(1)$$

$$D_\mu T^{\kappa\mu\nu} = R^{\kappa\ \ \mu\nu}_{\ \mu} \quad -(2)$$

The relevant connections are:

$$\Gamma^{0}_{10} = -\Gamma^{0}_{01} \quad -(3)$$

$$\Gamma^{1}_{01} = -\Gamma^{1}_{10}. \quad -(4)$$

The torsion is: $T^{\kappa\mu\nu} = 2\Gamma^{\kappa\mu\nu}$ $-(5)$

By definition: $\Gamma^{\kappa\mu\nu} = g^{\mu\alpha}g^{\nu\beta}\Gamma^{\kappa}_{\alpha\beta}$ $-(6)$

From eqs. (3) and (6):

$$\Gamma^{010} = g^{11}g^{00}\Gamma^{0}_{10} \quad -(7)$$

From eqs (2) and (3)

$$D_1 T^{010} = R^{0\ 10}_{\ 1} \quad -(8)$$

i.e. $$D_1\left(g^{11}g^{00}T^{0}_{10}\right) = g^{11}g^{00}R^{0}_{110}. \quad -(9)$$

By metric compatibility:

$$D_\mu g^{\nu\rho} = 0 \quad -(10)$$

so $$D_1(g^{11}g^{00}) = g^{11}D_1 g^{00} + g^{00}D_1 g^{11} = 0 \quad -(11)$$

2)

i.e $$D_1 T^0{}_{10} = R^0{}_{110} \quad -(12)$$

As in previous work:
$$\partial_1 T^0{}_{10} = 2\Gamma^0{}_{10} = R^0{}_{110} \quad -(13)$$

The curvature tensor is:
$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \quad -(14)$$

i.e $$R^0{}_{110} = \partial_1 \Gamma^0_{01} - \partial_0 \Gamma^0_{11} + \Gamma^0_{1\lambda}\Gamma^\lambda_{01} - \Gamma^0_{0\lambda}\Gamma^\lambda_{11} \quad -(15)$$

with $$\lambda = 0. \quad -(16)$$

so $$R^0{}_{110} = \partial_1 \Gamma^0_{01} + \Gamma^0_{10}\Gamma^0_{01}. \quad -(17)$$

Therefore:
$$2\partial_1 \Gamma^0_{10} = -\partial_1 \Gamma^0_{10} - \Gamma^0_{10}\Gamma^0_{10} \quad -(18)$$

i.e. $$3\partial_1 \Gamma^0_{10} + \left(\Gamma^0_{10}\right)^2 = 0 \quad -(19)$$

where $$\Gamma^0_{10} = \frac{1}{2m}\frac{\partial m}{\partial r}, \quad -(20)$$

$$m = m(r,t). \quad -(21)$$

Therefore:
$$3\frac{\partial}{\partial r}\left(\frac{1}{m}\frac{\partial m}{\partial r}\right) + \frac{1}{2}\frac{1}{m^2}\left(\frac{\partial m}{\partial r}\right)^2 = 0. \quad -(22)$$

Let $$m(r,t) = \exp\left(2\alpha(r,t)\right) \quad -(23)$$

to obtain $$3\frac{\partial^2 \alpha}{\partial r^2} + \frac{1}{2}\left(\frac{\partial \alpha}{\partial r}\right)^2 = 0. \quad -(24)$$

3) whose reduced equation is found by the change of variable:

$$\frac{1}{2}\left(\frac{\partial \alpha}{\partial r}\right)^2 = \frac{\partial \alpha}{\partial r} - f(r). \quad -(25)$$

From eq (25) in (24):

$$3\frac{\partial^2 \alpha}{\partial r^2} + \frac{\partial \alpha}{\partial r} = f(r). \quad -(26)$$

The reduced equation of eq. (26) is:

$$3\frac{\partial^2 \alpha}{\partial r^2} + \frac{\partial \alpha}{\partial r} = 0. \quad -(27)$$

Let
$$\alpha = e^{xr} \quad -(28)$$

then
$$x = -\frac{1}{3}. \quad -(29)$$

The complementary function is the solution of eq (27):

$$\alpha = \exp\left(-\frac{r}{3R(t)}\right) \quad -(30)$$

where R is not a function of r. The complementary function is therefore:

$$m(r,t) = \exp\left(2\exp\left(-\frac{r}{3R(t)}\right)\right) \quad -(31)$$

which is normalized as:

$$m(r,t) = \frac{1}{e^2}\exp\left(2\exp\left(-\frac{r}{3R(t)}\right)\right) \quad -(32)$$

t)

so:

$$m(r,t) \to 1 \text{ as } r \to \infty \quad -(33)$$

giving the correct flat spacetime metric.

The solution of eq. (26) is the same as the solution of eq. (24), and is the complementary function plus a particular integral of eq. (24). Chose the particular integral to be:

$$\alpha_p \to -\infty \quad -(34)$$

so

$$m_p(r,t) \to 0. \quad -(35)$$

The complete solution is therefore eq. (33).

The equation of orbits is found from eq. (1):

$$\partial_0 g_{00} = 0 \quad -(36)$$

i.e

$$\frac{\partial}{\partial t} m(r,t) = 0. \quad -(37)$$

Let

$$m(r,t) = \frac{1}{e^2} \exp(f(t)) \quad -(38)$$

so

$$\frac{\partial m}{\partial t} = \frac{f'(t)}{e^2} \exp(f(t)) = 0 \quad -(39)$$

i.e

$$f'(t) = \frac{\partial f}{\partial t} = 0 \quad -(40)$$

or

$$\frac{\partial}{\partial t}\left(\frac{r}{R(t)}\right) = 0. \quad -(41)$$

$$= \frac{1}{R^2(t)}\left(R(t)\frac{dr}{dt} - r\frac{dR}{dt}\right)$$

5)

so
$$\boxed{\frac{\partial r}{\partial t} = \frac{r}{R(t)} \cdot \frac{\partial R(t)}{\partial t}} \quad -(42)$$

This is an equation for all orbits.

It is convenient to illustrate the meaning of $R(t)$ with a logarithmic spiral orbit (Marion and Thornton chapter 7):

$$r = k \exp(\alpha\theta) \quad -(43)$$

for which:

$$r(t) = \left(\frac{2\alpha L}{\mu} t + k^2 C\right)^{1/2} \quad -(44)$$

i.e. r is expressed as a function of t. So:

$$\frac{dr}{dt} = -\frac{\alpha L}{\mu}\left(\frac{2\alpha L t}{\mu} + k^2 C\right)^{-3/2} \quad -(45)$$

and

$$\frac{1}{r}\frac{dr}{dt} = -\frac{\alpha L}{\mu}\left(\frac{2\alpha L t}{\mu} + k^2 C\right)^{-2}$$

$$= \frac{1}{R(t)}\frac{dR(t)}{dt} \quad -(46)$$

It is seen that $R(t)$ is a function of t.

The Newtonian orbit is the ellipse:

$$r = \frac{\alpha}{1 + \epsilon\cos\theta} \quad -(47)$$

so

$$\frac{dr}{d\theta} = \frac{\alpha\epsilon\sin\theta}{(1+\epsilon\cos\theta)^2} \quad -(48)$$

Therefore

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad -(49)$$

also:

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2} \quad -(50)$$

$$= \frac{L}{\mu d}\left(1+\epsilon\cos\theta\right)^2$$

So

$$\frac{dr}{dt} = \left(\frac{L\epsilon}{\mu d}\right)\sin\theta(t) \quad -(51)$$

and

$$\boxed{\frac{1}{R(t)}\frac{dR(t)}{dt} = \frac{1}{r}\frac{dr}{dt} = \left(\frac{1+\epsilon\cos\theta(t)}{d}\right)\left(\frac{L\epsilon}{\mu d}\right)\sin\theta(t)}$$

$$-(52)$$

In general $d\theta/dt$ and $dr/d\theta$ are found from the line element:

$$ds^2 = n(r,t)\,c^2dt^2 - n(r,t)\,dr^2 - r^2d\theta^2$$

$$-(53)$$

in the plane

$$dZ^2 = 0. \quad -(54)$$

For the solar system to an excellent approximation:

$$n(r,t) = \frac{1}{n(r,t)} = 1 - \frac{r_0}{r}$$

$$-(55)$$