210 (1) : Proof of the Antisymmetry of the Connection from the Cartan Identity.

The Cartan identity in short hand is:

$$D \wedge T := R \wedge q. \quad -(1)$$

In form notation it is:

$$D \wedge T^a := R^a{}_b \wedge q^b \quad -(2)$$

which in tensor notation is:

$$\partial_\mu T^a{}_{\nu\rho} + \omega^a{}_{\mu b} T^b{}_{\nu\rho} + \partial_\rho T^a{}_{\mu\nu} + \omega^a{}_{\rho b} T^b{}_{\mu\nu} + \partial_\nu T^a{}_{\rho\mu} + \omega^a{}_{\nu b} T^b{}_{\rho\mu} \quad -(3)$$

$$:= R^a{}_{\mu\nu\rho} + R^a{}_{\rho\mu\nu} + R^a{}_{\nu\rho\mu}$$

Here:

$$T^a{}_{\mu\rho} = (\Gamma^\lambda{}_{\mu\rho} - \Gamma^\lambda{}_{\rho\mu}) q^a{}_\lambda \quad -(4)$$

where $\Gamma^\lambda{}_{\mu\rho}$ is the Christoffel connection. The Leibniz theorem means that:

$$\partial_\mu \left((\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}) q^a{}_\lambda \right) = q^a{}_\lambda \partial_\mu (\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}) + (\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}) \partial_\mu q^a{}_\lambda . \quad -(5)$$

Therefore:

$$\partial_\mu T^a{}_{\nu\rho} + \omega^a{}_{\mu b} T^b{}_{\nu\rho} = (\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}) q^a{}_\lambda + (\partial_\mu q^a{}_\lambda + \omega^a{}_{\mu b} q^b{}_\lambda)(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu})$$

$$= \left(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu}\right) q^a_\lambda + \left(\partial_\mu q^a_\lambda + \omega^a_{\mu b}\, q^b_\lambda\right)\left(\Gamma^\lambda_{\nu\rho} - \Gamma^\lambda_{\rho\nu}\right) \quad -(6)$$

and so on.

Now relabel dummy indices:

$$\lambda \rightarrow \sigma \quad -(7)$$

in the second term.

So:

$$\partial_\mu T^a_{\nu\rho} + \omega^a_{\mu b} T^b_{\nu\rho}$$
$$= \left(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu}\right) q^a_\lambda + \left(\partial_\mu q^a_\sigma + \omega^a_{\mu b}\, q^b_\sigma\right)\left(\Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\rho\nu}\right) \quad -(8)$$

The tetrad postulate is:

$$\partial_\mu q^a_\sigma + \omega^a_{\mu b}\, q^b_\sigma = \Gamma^\lambda_{\mu\sigma}\, q^a_\lambda. \quad -(9)$$

Therefore:

$$\partial_\mu T^a_{\nu\rho} + \omega^a_{\mu b} T^b_{\nu\rho}$$
$$= \left(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\mu\sigma}\left(\Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\rho\nu}\right)\right) q^a_\lambda \quad -(10)$$

Using:

$$R^a_{\rho\mu\nu} = R^\lambda_{\rho\mu\nu}\, q^a_\lambda \quad -(11)$$

the identity (1) becomes:

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$$R^{\lambda}_{\mu\nu\rho} + R^{\lambda}_{\rho\mu\nu} + R^{\lambda}_{\nu\rho\mu}$$

$$:= \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\mu \Gamma^{\lambda}_{\rho\nu} + \Gamma^{\lambda}_{\mu\sigma}\left(\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\sigma}_{\rho\nu}\right)$$

$$+ \partial_\rho \Gamma^{\lambda}_{\mu\nu} - \partial_\rho \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\lambda}_{\rho\sigma}\left(\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu}\right)$$

$$+ \partial_\nu \Gamma^{\lambda}_{\rho\mu} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\nu\sigma}\left(\Gamma^{\sigma}_{\rho\mu} - \Gamma^{\sigma}_{\mu\rho}\right) \quad -(12)$$

A solution is:

$$R^{\lambda}_{\mu\nu\rho} = \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\mu \Gamma^{\lambda}_{\rho\nu} + \Gamma^{\lambda}_{\mu\sigma}\left(\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\sigma}_{\rho\nu}\right) \quad -(13)$$

$$= \partial_\mu T^{\lambda}_{\nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}_{\nu\rho}$$

i.e. $$\boxed{R^{\lambda}_{\mu\nu\rho} = \partial_\mu T^{\lambda}_{\nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}_{\nu\rho}} \quad -(14)$$

This means that if the connection is symmetric:

$$\Gamma^{\sigma}_{\nu\rho} \stackrel{?}{=} \Gamma^{\sigma}_{\rho\nu} \quad -(15)$$

then:

$$R^{\lambda}_{\mu\nu\rho} \stackrel{?}{=} 0 \quad -(16)$$

$$T^{\lambda}_{\nu\rho} \stackrel{?}{=} 0 \quad -(17)$$

Therefore $$\boxed{\Gamma^{\sigma}_{\nu\rho} = -\Gamma^{\sigma}_{\rho\nu}} \quad -(18)$$

4) The commutator method produces:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(19)$$

where:

$$R^\rho{}_{\sigma\mu\nu} := \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \quad -(20)$$

and this definition is obtained by arranging terms on the right hand side of eq. (12). For example:

$$R^\lambda_{\mu\nu\rho} = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} \quad -(21)$$

as in UFT 137.

Both eqns. (14) and (21) are _definitions._ Curvature can be defined in more than one way, and in both ways, the Cartan identity is true. However, eq. (14) shows conclusively that if the connection is symmetric the curvature and torsion are zero. Eq. (19) gives the same conclusion precisely. _It_ is _incorrect_ to assert that the torsion is zero and curvature non-zero. For this reason Einsteinian general relativity is incorrect.