## 210(2): Further Proof of the Antisymmetry of the Connection.

Consider the Cartan identity:

$$\boxed{R^\lambda_{\ \mu\nu\rho}} + R^\lambda_{\ \rho\mu\nu} + R^\lambda_{\ \nu\rho\mu}$$

$$:= \boxed{\partial_\mu \Gamma^\lambda_{\nu\rho}} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \boxed{\Gamma^\lambda_{\mu\sigma}\left(\Gamma^\sigma_{\nu\rho}} - \Gamma^\sigma_{\rho\nu}\right)$$

$$+ \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\rho \Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\rho\sigma}\left(\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}\right)$$

$$+ \partial_\nu \Gamma^\lambda_{\rho\mu} - \boxed{\partial_\nu \Gamma^\lambda_{\mu\rho}} + \boxed{\Gamma^\lambda_{\nu\sigma}}\left(\Gamma^\sigma_{\rho\mu} - \boxed{\Gamma^\sigma_{\mu\rho}}\right). \qquad -(1)$$

The second Cartan structure equation is:

$$R^\lambda_{\ \mu\nu\rho} = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} \qquad -(2)$$

It is seen that eq. (2) is derived from choosing the combination of terms in eq. (1). These are the dotted terms. The second Cartan structure equation is one possible solution of the Cartan identity.

Now consider the sum:

$$R^\lambda_{\ \mu\nu\rho} - R^\lambda_{\ \mu\rho\nu} = 2R^\lambda_{\ \mu\nu\rho} \qquad -(3)$$

2) It follows that:

$$
\begin{aligned}
2R^{\lambda}_{\mu\nu\rho} &= \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} \\
&\quad - \partial_\mu \Gamma^{\lambda}_{\rho\nu} + \partial_\nu \Gamma^{\lambda}_{\rho\mu} - \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\rho\mu} \\
&= \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\mu \Gamma^{\lambda}_{\rho\nu} + \Gamma^{\lambda}_{\mu\sigma}\left(\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\sigma}_{\rho\nu}\right) \\
&\quad + \partial_\nu \Gamma^{\lambda}_{\rho\mu} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\nu\sigma}\left(\Gamma^{\sigma}_{\rho\mu} - \Gamma^{\sigma}_{\mu\rho}\right)
\end{aligned}
\qquad -(4)
$$

So the second Cartan structure equation is:

$$
\boxed{
\begin{aligned}
R^{\lambda}_{\mu\nu\rho} &= \frac{1}{2}\Big(\partial_\mu T^{\lambda}_{\nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}_{\nu\rho} \\
&\qquad + \partial_\nu T^{\lambda}_{\rho\mu} + \Gamma^{\lambda}_{\nu\sigma} T^{\sigma}_{\rho\mu}\Big) \\
&= \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho}
\end{aligned}
}
\qquad -(5)
$$

If $\Gamma^{\sigma}_{\nu\rho} = ?\ \Gamma^{\sigma}_{\rho\nu}$ $\quad -(6)$

then $R^{\lambda}_{\mu\nu\rho} = ?\ 0$ $\quad -(7)$

Q.E.D.