

320(5): Calculation of Ω_θ .

The problem to solve is: - (1)

$$(\dot{r} \underline{e}_r + \omega r \underline{e}_\theta) \times \underline{\Omega} = -\omega^2 r \underline{e}_r$$

where Ω is general:

$$\underline{\Omega} = \Omega_r \underline{e}_r + \Omega_\theta \underline{e}_\theta + \Omega_z \underline{k} \quad - (2)$$

Therefore:

$$\begin{vmatrix} \underline{e}_r & \underline{e}_\theta & \underline{k} \\ \dot{r} & \omega r & 0 \\ \Omega_r & \Omega_\theta & \Omega_z \end{vmatrix} = -\omega^2 r \underline{e}_r \quad - (3)$$

$$\text{i.e. } (\omega r \Omega_z) \underline{e}_r - \dot{r} \Omega_z \underline{e}_\theta + \underline{k} (\dot{r} \Omega_\theta - \omega r \Omega_r) = -\omega^2 r \underline{e}_r$$

$$\text{From eq. (4): } \Omega_z = -\omega \quad - (5)$$

$$\dot{r} \Omega_z = 0 \quad - (6)$$

$$\dot{r} \Omega_\theta = \omega r \Omega_r \quad - (7)$$

$$\text{Therefore } \dot{r} = 0 \quad - (8)$$

$$\text{and } \Omega_r = 0 \quad - (9)$$

It follows that:

$$\underline{\Omega} = \Omega_\theta \underline{e}_\theta + \Omega_z \underline{k} \quad - (10)$$

and

$$\Omega^2 = \Omega_\theta^2 + \Omega_z^2, \quad - (11)$$

so

$$\Omega_\theta^2 = \Omega^2 - \omega^2 \quad - (12)$$

the velocity is

$$\underline{v} = \omega r \underline{e}_\theta \quad - (13)$$

and

$$v = \omega r. \quad - (14)$$

From eqs (1) and (8):

$$\begin{aligned} \underline{v} \times \underline{\Omega} \cdot \underline{v} \times \underline{\Omega} &= \omega^4 r^2 \quad - (15) \\ &= v^2 \Omega^2 - (\underline{\Omega} \cdot \underline{v})^2 \end{aligned}$$

where:

$$\underline{v} = \omega r \underline{e}_\theta \quad - (16)$$

and

$$\underline{\Omega} = \Omega_\theta \underline{e}_\theta + \Omega_z \underline{k} \quad - (17)$$

Therefore:

$$\underline{\Omega} \cdot \underline{v} = \omega r \Omega_\theta \quad - (18)$$

It follows that:

$$v^2 \Omega^2 - \omega^2 r^2 \Omega_\theta^2 = \omega^4 r^2 \quad - (19)$$

i.e.

$$\Omega^2 - \Omega_\theta^2 = \omega^2 \quad - (20)$$

or from (12):

3) Therefore the system is defined by:

$$\Omega_z = -\omega \quad (21)$$

and

$$\Omega^2 - \Omega_\theta^2 = \omega^2 \quad (22)$$

$$\underline{\Omega} = \Omega_\theta \underline{e}_\theta + \Omega_z \underline{k} \quad (23)$$

In order to define Ω_θ uniquely we use the

equation:
$$\nabla \times \underline{\Omega} = \underline{\kappa} \times \underline{\Omega} = \frac{4\pi G}{c^2} \underline{J}_m \quad (24)$$

From the previous paper, $\underline{\kappa}$ was determined in the Newtonian limit, but is general:

$$\underline{\kappa} = \kappa_r \underline{e}_r + \kappa_\theta \underline{e}_\theta + \kappa_z \underline{k} \quad (25)$$

and:

$$\begin{aligned} \nabla \times \underline{\Omega} = & \left(\frac{1}{r} \frac{\partial \Omega_z}{\partial \theta} - \frac{\partial \Omega_\theta}{\partial z} \right) \underline{e}_r + \left(\frac{\partial \Omega_r}{\partial z} - \frac{\partial \Omega_z}{\partial r} \right) \underline{e}_\theta \\ & + \frac{1}{r} \left(\frac{\partial (r \Omega_\theta)}{\partial r} - \frac{\partial \Omega_r}{\partial \theta} \right) \underline{k} \quad (26) \end{aligned}$$

so
$$\frac{1}{r} \frac{\partial \Omega_z}{\partial \theta} - \frac{\partial \Omega_\theta}{\partial z} = A_r \quad (27)$$

$$\frac{\partial \Omega_r}{\partial z} - \frac{\partial \Omega_z}{\partial r} = A_\theta \quad (28)$$

$$\frac{1}{r} \left(\frac{\partial (r \Omega_\theta)}{\partial r} - \frac{\partial \Omega_r}{\partial \theta} \right) = A_z \quad (29)$$

4) where

$$A_r = \frac{4\pi G}{c^2} \bar{J}_{mr} \quad - (30)$$

$$A_\theta = \frac{4\pi G}{c^2} \bar{J}_{m\theta} \quad - (31)$$

$$A_k = \frac{4\pi G}{c^2} \bar{J}_{mz} \quad - (32)$$

where

$$\underline{\bar{J}}_m = \bar{J}_{mr} \underline{e}_r + \bar{J}_{m\theta} \underline{e}_\theta + \bar{J}_{mz} \underline{k} \quad - (33)$$

Here:

$$\underline{\kappa} \times \underline{\Omega} = \begin{vmatrix} \underline{e}_r & \underline{e}_\theta & \underline{k} \\ \kappa_r & \kappa_\theta & \kappa_z \\ \Omega_r & \Omega_\theta & \Omega_z \end{vmatrix}$$

$$= (\kappa_\theta \Omega_z - \kappa_z \Omega_\theta) \underline{e}_r \quad - (34)$$

$$- (\kappa_r \Omega_z - \kappa_z \Omega_r) \underline{e}_\theta$$

$$+ (\kappa_r \Omega_\theta - \kappa_\theta \Omega_r) \underline{k} \quad - (35)$$

So:

$$\frac{1}{r} \frac{\partial \Omega_z}{\partial \theta} - \frac{\partial \Omega_\theta}{\partial z} = \kappa_\theta \Omega_z - \kappa_z \Omega_\theta$$

$$\frac{\partial \Omega_r}{\partial z} - \frac{\partial \Omega_z}{\partial r} = \kappa_z \Omega_r - \kappa_r \Omega_z \quad - (36)$$

$$\frac{1}{r} \left(\frac{\partial (r \Omega_\theta)}{\partial r} - \frac{\partial \Omega_r}{\partial \theta} \right) = \kappa_r \Omega_\theta - \kappa_\theta \Omega_r \quad - (37)$$

5) with: $\Omega_z = -\omega, \Omega_r = 0 - (38)$

If ω is a constant it follows that:

$$-\frac{\partial \Omega_\theta}{\partial z} = -\omega \kappa_\theta - \kappa_z \Omega_\theta - (39)$$

$$\kappa_z \Omega_r = 0 = -\kappa_r \Omega_z - (40)$$

so

$$\kappa_r = 0 - (41)$$

and

$$\frac{1}{r} \frac{d(r\Omega_\theta)}{dr} = 0 - (42)$$

$$i. e. \quad \frac{1}{r} \Omega_\theta + \frac{d\Omega_\theta}{dr} = 0 - (43)$$

$$\text{and} \quad \frac{d\Omega_\theta}{dz} = \omega \kappa_\theta + \kappa_z \Omega_\theta - (44)$$

Eqs. (43) and (44) can be solved numerically for given κ_θ and κ_z .
