1)

# PROOF OF THE TETRAD POSTULATE FROM THEIS EQN. (3.62)

The reference is to a course by Prof. Dr. U. Theis of the Institute for theoretical Physics, F. Schiller Univ., Jena, eqn (3.62):

$$D_\mu V_\nu = q^a_\nu D_\mu V_a. \quad -(1)$$

This equation implies that:

$$\partial_\mu V^a + \omega^a_{\mu b} V^b = q^a_\nu \left( \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \right) \quad -(1a)$$

where

$$V^a = q^a_\nu V^\nu \quad -(2)$$

From eqn (2)

$$\partial_\mu V^a = \partial_\mu \left( q^a_\nu V^\nu \right)$$
$$= V^\nu \partial_\mu q^a_\nu + q^a_\nu \partial_\mu V^\nu \quad -(3)$$

and

$$\omega^a_{\mu b} V^b = \omega^a_{\mu b} q^b_\nu V^\nu \quad -(4)$$

Add (3) and (4):

$$\partial_\mu V^a + \omega^a_{\mu b} V^b$$
$$= q^a_\nu \partial_\mu V^\nu + V^\nu \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu V^\nu \quad -(5)$$

Comparing (1a) and (5)

$$q^a_\nu \Gamma^\nu_{\mu\lambda} V^\lambda = V^\nu \left( \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu \right)$$

2) $= V^\lambda \left( \partial_\mu q^a_{\ \lambda} + \omega^a_{\ \mu b} q^b_{\ \lambda} \right)$

$$\Rightarrow \boxed{\partial_\mu q^a_{\ \lambda} + \omega^a_{\ \mu b} q^b_{\ \lambda} - q^a_{\ \nu} \Gamma^\nu_{\mu\lambda} = 0} \quad -(6)$$

Theis' eqn. (1) comes from the Leibniz rule:

$$D_\mu \left( q^a_{\ \nu} V^\nu \right) = \left( D_\mu q^a_{\ \nu} \right) V^\nu + q^a_{\ \nu} D_\mu V^\nu \quad -(7)$$

i.e. $\boxed{D_\mu q^a_{\ \nu} = 0} \quad -(8)$

From (6) and (8) we obtain the tetrad postulate:

$$\boxed{D_\mu q^a_{\ \nu} = \partial_\mu q^a_{\ \lambda} + \omega^a_{\ \mu b} q^b_{\ \lambda} - q^a_{\ \nu} \Gamma^\nu_{\mu\lambda} = 0} \quad -(9)$$

Q.E.D.

Reference

U. Theis, Inst. for Theoretical Physics, Friedrich Schiller Univ., Jena (ulrich.theis@uni-jena.de), a course in supersymmetry, eqn. (3.62).

As explained by Theis, eqn (9) means that the tetrad is fully covariantly constant, parallel transport does not change the length of vectors. Eqn (9) is fully compatible with the

3) <u>metric compatibility condition</u>:

$$D_\rho g^{\mu\nu} = 0 \quad -(10)$$

or

$$D_\rho g_{\mu\nu} = 0. \quad -(11)$$

<u>DISCUSSION</u>

It can be seen from eqn (9) that $q^a_\nu$ is a mixed index tensor, and as such <u>does not obey the rules for covariant derivative of a vector</u>:

$$D_\mu \bar{V}^a = \partial_\mu \bar{V}^a + \omega_\mu{}^a{}_b \bar{V}^b \quad -(12)$$

$$D_\mu \bar{V}^\nu = \partial_\mu \bar{V}^\nu + \Gamma^\nu_{\mu\lambda} \bar{V}^\lambda. \quad -(13)$$

The metric compatibility condition, eqn. (11) is a fundamental feature of Riemann geometry. It is used to derive the relation between the Christoffel symbol and the metric in the Einstein theory of gravitation.

This is done by using:

$$D_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0 \quad -(14)$$

$$D_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0 \quad -(15)$$

$$D_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0 \quad -(16)$$

4) Now subtract eqns. (15) and (16) from eqn. (14) and assume:

$$\Gamma^{\lambda}_{\rho\mu} = \Gamma^{\lambda}_{\mu\rho} \quad -(17)$$

and so on. Note carefully that eqn (17) means that the torsion tensor is zero:

$$T^{\lambda}_{\rho\mu} = \Gamma^{\lambda}_{\rho\mu} - \Gamma^{\lambda}_{\mu\rho} = 0. \quad -(18)$$

If this is assumed then:

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} = 0 \quad -(19)$$

from which:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \quad -(20)$$

In general relativity eqn (20) is true to one part in 100,000 (NASA Cassini expt. 2002) for central gravitation. This shows that ~~the~~ the tetrad postulate is true to 1 : 100,000 in gravitational general relativity, giving a very accurate foundation for its use in unified field theory.

5) The metric can be expressed as:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad — (21)$$

where $\eta_{ab}$ is the Minkowski metric, for which:

$$D_\mu \eta_{ab} = 0. \quad — (22)$$

Using the Leibniz theorem in eqn (21):

$$D_\rho g_{\mu\nu} = 0 = D_\rho \left( q^a_\mu q^b_\nu \eta_{ab} \right)$$

$$= \eta_{ab} \left( q^b_\nu D_\rho q^a_\mu + q^a_\mu D_\rho q^b_\nu \right)$$

$$= 0 \quad — (23)$$

Since:

$$\eta_{ab} \neq 0 \quad — (24)$$

$$\Rightarrow \quad q^b_\nu D_\rho q^a_\mu + q^a_\mu D_\rho q^b_\nu = 0 \quad — (25)$$

a solution of which is:

$$D_\rho q^a_\mu = D_\rho q^b_\nu = 0 \quad — (26)$$

which is the tetrad postulate, eqn. (9).

It may be shown as in the following two pages that eqn. (26) is the unique solution to eqn. (25), thus proving the tetrad postulate without any assumption of a torsion free connection.

# PROOF OF THE TETRAD POSTULATE FROM FIRST PRINCIPLES

Consider the following basic properties of the tetrad:

$$q^b_\nu q^\nu_b = 1 \quad -(1)$$

$$q^a_\mu q^\mu_a = 1 \quad -(2)$$

$$q^\mu_a q^a_\nu = \delta^\mu_\nu \quad -(3)$$

$$q^a_\mu q^\mu_b = \delta^a_b \quad -(4)$$

where $\delta^\mu_\nu$ and $\delta^a_b$ are Kronecker deltas. We now differentiate eqns (1) to (4) covariantly using the Leibnitz Theorem:

$$q^\nu_b D_\rho q^b_\nu + q^b_\nu D_\rho q^\nu_b = 0 \quad -(5)$$

$$q^a_\mu D_\rho q^\mu_a + q^\mu_a D_\rho q^a_\mu = 0 \quad -(6)$$

$$q^\mu_a D_\rho q^a_\nu + q^a_\nu D_\rho q^\mu_a = 0 \quad -(7)$$

$$q^a_\mu D_\rho q^\mu_b + q^\mu_b D_\rho q^a_\mu = 0. \quad -(8)$$

Rearranging dummy indices in eqn (5) $(a \rightarrow b, \mu \rightarrow \nu)$

$$q^\mu_a D_\rho q^a_\mu + q^b_\nu D_\rho q^\nu_b = 0. \quad -(9)$$

Rearranging dummy indices in eqn (8) $(\mu \rightarrow \nu)$:

$$q^\mu_b D_\rho q^a_\mu + q^a_\nu D_\rho q^\nu_b = 0. \quad -(10)$$

Multiply eqn (9) by $q^a_\mu$:

$$D_\rho q^a_\mu + q^a_\mu q^b_\nu D_\rho q^\nu_b = 0 \quad -(11)$$

7) multiply eqn (10) by $q_{\mu}^{b}$:

$$D_{\rho} q_{\mu}^{a} + q_{\mu}^{b} q^{a}_{\nu} D_{\rho} q^{\nu}_{b} = 0 \quad -(12)$$

It is seen that eqn (11) is of the form:

$$x + ay = 0 \quad -(13)$$

and eqn (12) is of the form:

$$x + by = 0 \quad -(14)$$

where:

$$a \neq b. \quad -(15)$$

The only possible solution is:

$$x = 0 \quad -(16)$$

$$y = 0. \quad -(17)$$

This means:

$$\boxed{D_{\rho} q_{\mu}^{a} = 0} \quad -(18)$$

$$D_{\rho} q^{\nu}_{b} = 0. \quad -(19)$$

Eqn (18) is the tetrad postulate of Cartan.
It is true for all connections because no restriction on the connection is used in deriving it.

8) Finally the Evans Lemma is the identity:

$$D^\rho (D_\rho q^a_\mu) := 0 \quad -(27)$$

which can be written as:

$$\Box q^a_\mu = R q^a_\mu \quad -(28)$$

where $R$ is a well defined scalar curvature. The Einstein equation:

$$R = -kT \quad -(29)$$

gives the Evans wave equation:

$$\boxed{(\Box + kT)\, q^a_\mu = 0}, \quad -(30)$$

So this is also accurate to 1 : 100,000, for symmetric type of connection, for which the theory has been tested, but is valid for any connection.