391(8): Limiting Integral of the Einstein Theory

The relevant Binet equation is:

$$\frac{d^2u}{d\phi^2} + u = \frac{1}{\alpha} + \frac{3mG}{c^2}u^2 \quad -(1)$$

where

$$u = \frac{1}{r}. \quad -(2)$$

If α is very large, where:

$$\alpha = \frac{L^2}{m^2 mG} \quad -(3)$$

then eq. (1) becomes:

$$\frac{d^2u}{d\phi^2} + u \rightarrow Au^2 \quad -(4)$$

where:

$$A := 3mG/c^2 \quad -(5)$$

So:

$$\frac{d^2u}{d\phi^2} = Au^2 - u \quad -(6)$$

$$:= f(u)$$

This is an autonomous equation of the theory of differential equations. Its solution is:

$$C_2 \pm \phi = \int \left(C_1 + 2\int f(u)\,du\right)^{-1/2} du$$

$$= \int \frac{du}{\left(C_1 + 2\int f(u)\,du\right)^{1/2}} \quad -(7)$$

where

$$f(u) = Au^2 - u \quad -(8)$$

So:

$$C_2 \pm \phi = \int \frac{du}{\left(C_1 + 2\left(\frac{Au^3}{3} - \frac{u^2}{2}\right)\right)^{1/2}} \quad -(9)$$

The Wolfram integrator gives a very complicated function for the integral (9), but in the approximation:

$$C_2 \pm \phi \sim \int \frac{du}{(C_1 - u^2)^{1/2}} \quad -(10)$$

$$-(11)$$

the Wolfram integrator gives:

$$C_2 \pm \phi = \frac{1}{\sqrt{C_1}} \tan^{-1}\left(\frac{u}{(C_1 - u^2)^{1/2}}\right) + C_3$$

So:

$$\pm \phi = \frac{1}{\sqrt{C_1}} \tan^{-1}\left(\frac{\frac{1}{r}}{\left(C_1 - \frac{1}{r^2}\right)^{1/2}}\right) + C \quad -(12)$$

where C is a constant:

$$C = C_3 - C_2 \quad -(13)$$

In the limit

$$r \to 0 \quad -(14)$$

the function becomes indeterminate. For the sake of argument, assume

$$\tan^{-1} \frac{1}{r}\left(C_1 - \frac{1}{r^2}\right)^{1/2} = \sqrt{C_1}\,\phi - C \quad -(15)$$

then

$$\frac{1}{r}\left(C_1 - \frac{1}{r^2}\right)^{1/2} = \tan\left(\sqrt{C_1}\,\phi - C\right) \quad -(16)$$

These functions are ill behaved and not observed experimentally.
