1)

Notes for Paper 54, Part 3.

Covariant Euler-Lagrange Equation

It is seen by inspection that the covariant Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial q^{\sim}_a} = D^\mu \left(\frac{\partial \mathcal{L}}{\partial (D^\mu q^{\sim}_a)} \right) \quad -(1)$$

with the Lagrangian density:

$$\mathcal{L} = c^2 T + D_\mu q^a_\sim D^\mu q^\sim_a \quad -(2)$$

gives

$$D^\mu \left(D_\mu q^a_\sim \right) = 0. \quad -(3)$$

This result is consistent with the tetrad postulate:

$$D_\mu q^a_\sim = 0. \quad -(4)$$

Using the fundamental definition:

$$q^a_\mu q^\mu_a = 1 \quad -(5)$$

the Leibnitz theorem is applied to give:

$$D_\nu \left(q^a_\mu q^\mu_a \right) = 0 \quad -(6)$$

2)

and

$$q^a_\mu \left(D_\nu q^\mu_a \right) + \left(D_\nu q^a_\mu \right) q^\mu_a = 0 . \quad -(7)$$

Using eq (4) in eq. (7) gives:

$$D_\nu q^\mu_a = 0 . \quad -(8)$$

Therefore eq. (2) is:

$$\mathcal{L} = c^2 T \rightarrow \frac{mc^2}{V} \quad -(9)$$

Therefore in analogy with gauge theory the second term in eq. (2) can always be added to the lagrangian density (9).

Eq (3) gives the ECE Lemma:

$$\Box q^a_\nu = R q^a_\nu \quad -(10)$$

and ECE wave equation:

$$\left(\Box + kT \right) q^a_\nu = 0 . \quad -(11)$$

As in Appendix J. 3 of Generally Covariant Unified Field Theory (Abramis 2005), the ECE Lemma follows from the tetrad postulate as follows:

$$D_\mu q^a_\lambda = \partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda - \Gamma^\nu_{\mu\lambda} q^a_\nu = 0 \quad -(12)$$

Therefore:

$$D^\mu \left(D_\mu q^a_\lambda \right) = \partial^\mu \left(D_\mu q^a_\lambda \right) = 0 \quad -(13)$$

i.e.

$$\partial^\mu \left(\partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda - \Gamma^\nu_{\mu\lambda} q^a_\nu \right) = 0 \quad -(14)$$

or

$$\Box q^a_\lambda = \partial^\mu \left(\Gamma^\nu_{\mu\lambda} q^a_\nu \right) - \partial^\mu \left(\omega^a_{\mu b} q^b_\lambda \right). \quad -(15)$$

Now define:

$$R q^a_\lambda := \partial^\mu \left(\Gamma^\nu_{\mu\lambda} q^a_\nu \right) - \partial^\mu \left(\omega^a_{\mu b} q^b_\lambda \right) \quad -(16)$$

and use eq (5) to obtain:

$$R := q^\lambda_a \left(\partial^\mu \left(\Gamma^\nu_{\mu\lambda} q^a_\nu \right) - \partial^\mu \left(\omega^a_{\mu b} q^b_\lambda \right) \right)$$

and:

$$\Box q^a_\lambda = R q^a_\lambda . \quad -(17)$$

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Therefore eq. (16) is the most fundamental definition of scalar curvature:

$$R = -kT = q^{\lambda}_{a}\left(\partial^{\mu}\left(\Gamma^{\nu}_{\mu\lambda} q^{a}_{\nu}\right) - \partial^{\mu}\left(\omega^{a}_{\mu b}\, q^{b}_{\lambda}\right)\right)$$

$$\rightarrow -\frac{mk}{V} \quad -(18)$$

in the rest mass limit where:

$$T \rightarrow \frac{m}{V}. \quad -(19)$$

So this is the lagrangian derivation of the ECE Lemma and wave equation. It is fully self-consistent and is based on Hamilton's principle of least action. As usual in lagrangian methods there is a freedom in the choice of lagrangian. It should therefore be possible to obtain the Cartan structure equations from a lagrangian. This is analogous to the way Hilbert derived the EH field equation in 1915.