1)

Notes for Paper 55, Part Two

Generally Covariant Equations of Rotational Motion

In analogy with papers 52 and 53 equations of motion can be constructed for the torque. In paper 52 the electromagnetic field tensor was defined as:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{bmatrix} \quad -(1)$$

and the electric and magnetic fields as:

$$\underline{E}^a = -\frac{\partial \underline{A}^a}{\partial t} - c\underline{\nabla} A^{0a} - c\omega^{0a}{}_b \underline{A}^b + c\underline{\omega}^a{}_b A^{0b}, \quad -(2)$$

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b. \quad -(3)$$

The Cartan torsion tensor is therefore:

$$T^{\mu\nu} = \begin{bmatrix} 0 & -T_L^1/c & -T_L^2/c & -T_L^3/c \\ T_L^1/c & 0 & -T_S^3 & T_S^2 \\ T_L^2/c & T_S^3 & 0 & -T_S^1 \\ T_L^3/c & -T_S^2 & T_S^1 & 0 \end{bmatrix} \quad -(4)$$

where

T_L = orbital Cartan torsion,

T_S = spin Cartan torsion.

2)

Therefore:

$$\underset{\sim}{T}^a_L = -\frac{\partial \underline{q}^a}{\partial t} - c\,\underline{\nabla} q^{oa} - c\,\omega^{oa}{}_b\,\underline{q}^b + c\,\underline{\omega}^a{}_b\,q^{ob},$$

— (5)

$$\underset{\sim}{T}^a_s = \underline{\nabla} \times \underline{q}^a - \underline{\omega}^a{}_b \times \underline{q}^b. \quad \text{— (6)}$$

The equation of rotational motion is obtained from:

$$\underline{J}^a = J^{(o)}\,\underline{q}^a, \quad \text{— (7)}$$

$$\underline{N}^a = J^{(o)}\,\underline{T}^a, \quad \text{— (8)}$$

where $\underline{J}^a$ is the angular momentum and $\underline{N}^a$ is the torque. So the orbital torque is:

$$\underline{N}^a_L = -\frac{\partial \underline{J}^a}{\partial t} - c\,\underline{\nabla} J^{oa} - c\,\omega^{oa}{}_b\,\underline{J}^b + c\,\underline{\omega}^a{}_b\,J^{ob}$$

— (9)

and the spin torque is:

$$\underline{N}^a_s = c\left(\underline{\nabla} \times \underline{J}^a - \underline{\omega}^a{}_b \times \underline{J}^b\right) \quad \text{— (10)}$$

The Euler equation of motion in the non-relativistic limit is:

3)

$$\underline{N} = \left(\frac{d\underline{J}}{dt}\right)_{fixed} \quad -(11)$$

which is obtained from curvature $R^a{}_b$ in Cartan geometry and not from torsion T^a. Eq. (11) is derived in an inertial frame. The Euler equation is derived by transforming into a rotating frame:

$$\left(\frac{d\underline{J}}{dt}\right)_{fixed} = \left(\frac{d\underline{J}}{dt}\right)_{rotating} + \underline{\omega} \times \underline{J} \quad -(12)$$

where $\underline{\omega}$ is the angular velocity of rotation. Eq (12) is not the same as eq. (11) in form, so covariant. the classical description is not generally covariant. Eqs (9) and (10) are generally covariant, i.e. retain their form under any coordinate transformation. The Newtonian description (11) is for the Newtonian which define $\underline{J}$ and $\underline{N}$ as momentum and force, follows:

$$\underline{J} = \underline{r} \times \underline{p}, \quad -(13)$$

$$\underline{N} = \underline{r} \times \underline{F}. \quad -(14)$$

This again is not generally covariant because it is non-relativistic.

The relativistically correct rotational motion equations are (9) & (10), derived from torsion.

4) The Eulerian limit is obtained if:

$$\underline{N}_L^{\,a} \rightarrow -\frac{\partial \underline{J}^{\,a}}{\partial t}, \quad -(15)$$

$$\underline{N}_S^{\,a} \rightarrow -c\,\underline{\omega}^a{}_b \times \underline{J}^{\,b}, \quad -(16)$$

assuming that:

$$\underline{\nabla} \times \underline{J}^{\,a} \rightarrow \underline{0} \quad -(17)$$

The total torque is:

$$\underline{N}^{\,a} = \underline{N}_L^{\,a} + \underline{N}_S^{\,a}$$

$$= -\frac{\partial \underline{J}^{\,a}}{\partial t} - c\,\underline{\omega}^a{}_b \times \underline{J}^{\,b} \quad -(18)$$

This has the same form as eqn. (12). However eq. (18) comes from the Cartan torsion and ~~the~~ first Cartan structure equation, i.e. ~~the~~ spinning of spacetime, while eqn. (12) comes from Newtonian dynamics, i.e. non-relativistic dynamics, and is an incomplete description.